Identification of planar crack and volumic defects in dynamic viscoelasticity

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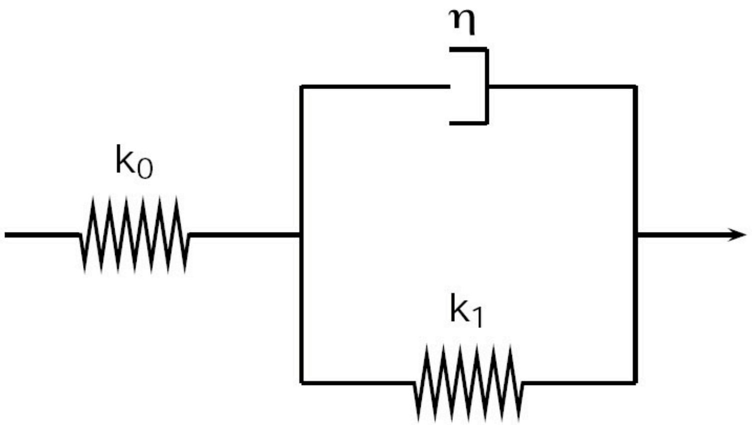
Zener viscoelasticity

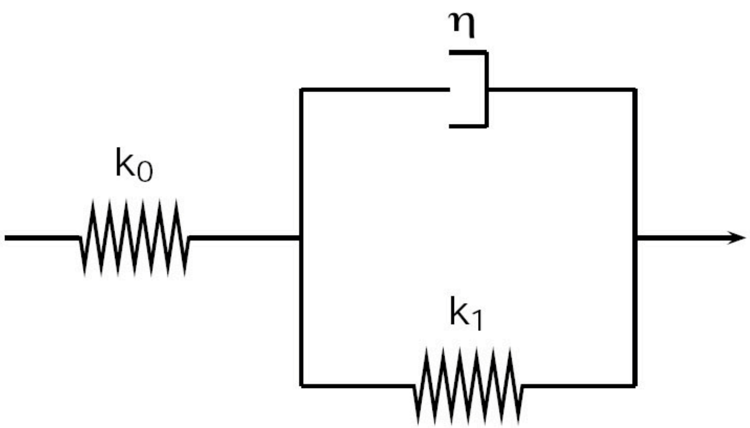


$$\frac{1}{\eta}\sigma + \dot{\sigma} = \Lambda_{\infty} : (\frac{1}{\gamma}\varepsilon + \dot{\varepsilon})$$

$$\sigma + \beta\,\dot{\sigma} = \Lambda : (\varepsilon + \alpha\,\dot{\varepsilon})$$

Λ_{∞}	Instantaneous moduli tensor
Λ	Delayed moduli tensor

Zener viscoelasticity



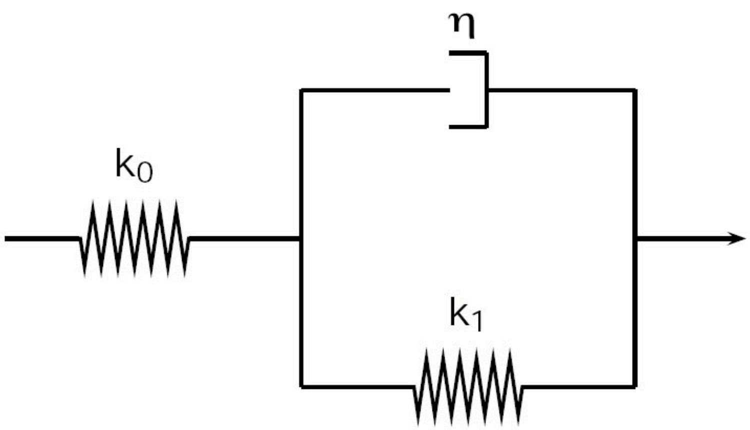
$$\sigma + \beta\,\dot{\sigma} = \Lambda : (\varepsilon + \alpha\,\dot{\varepsilon})$$

$$\alpha = \frac{\eta}{k_1} > \beta = \frac{\eta}{k_0 + k_1}$$

Generally β is much less than α(creep time)

	α (sec)	ω^0 (Hz)
Rock at laboratory time	10^(-3) to 10^(-2)	100 to 1000
Asphalt, rubber	10^(-1) to 10	10^(-1) to 10
Ceramic	10^(-7) to 10^(-5)	10^(5) to 10^(7)

Zener viscoelasticity



$$\frac{1}{\eta}\sigma + \dot{\sigma} = \Lambda_\infty : (\frac{1}{\gamma}\varepsilon + \dot{\varepsilon})$$

$$\sigma + \beta\,\dot{\sigma} = \Lambda : (\varepsilon + \alpha\,\dot{\varepsilon})$$

$$\alpha = \frac{\eta}{k_1} > \beta = \frac{\eta}{k_0 + k_1}$$

Associated displacement, strain and stress (Goriacheva, 1973)

$$\mathbf{u}^* = \mathbf{u} + \alpha\,\dot{\mathbf{u}}$$

$$\varepsilon^* = \varepsilon + \alpha\,\dot{\varepsilon}$$

$$\sigma^* = \sigma + \beta\,\dot{\sigma}$$

$$\sigma^* = \Lambda : \varepsilon^*$$

Associated fields (I. Goriacheva, 1973)

$$\mathbf{u}^* = \mathbf{u} + \alpha\,\dot{\mathbf{u}}$$
$$\varepsilon^* = \varepsilon + \alpha\,\dot{\varepsilon}$$
$$\sigma^* = \sigma + \beta\,\dot{\sigma}$$

$$\boxed{\sigma^* = \Lambda : \varepsilon^*}$$

Time harmonic loading

$$\mathbf{u}(\mathbf{x},t) = \mathbf{v}(\mathbf{x})\cos\omega t$$
$$\sigma(\mathbf{x},t) = w(\mathbf{x})\cos(\omega t + \theta)$$

Equation of motion

$$div\,\sigma - \rho\,\ddot{\mathbf{u}} = 0 \quad \Leftrightarrow \quad \sigma \text{ and } \mathbf{u} \text{ in phase} \quad \Rightarrow \quad \boxed{\theta \text{ small}}$$

Associated displacement, strain and stress

$$\sigma^* = \Lambda : \varepsilon^*$$

$$\mathbf{u}^*(\mathbf{x},t) = \mathbf{v}(\mathbf{x})\frac{1}{\cos(\psi)}\cos(\omega t + \psi)$$

$$\tan\psi = \alpha\omega \; (0 \le \psi < \pi/2)$$

$$\sigma^*(\mathbf{x},t) = w(\mathbf{x})\frac{1}{\cos(\psi-\theta)}\cos(\omega t + \psi)$$

$$\tan(\psi-\theta) = \beta\omega \; (0 \le \psi-\theta < \pi/2)$$

$$div\,\sigma^* + \rho\omega^2\mathbf{u}^* \square \; \rho(\beta-\alpha)\omega^3\mathbf{v}$$

$\rho\omega^2\mathbf{u}^* \square \; \rho(\beta-\alpha)\omega^3\mathbf{v}$?

$\Updownarrow$

$div\,\sigma^* + \rho\omega^2\mathbf{u}^* \square\; 0$ $\Leftarrow$ small $\theta \square$ $(\alpha-\beta)\omega \square\; 1$

Associated displacement, strain and stress

Low frequency $\omega \ll \omega_0 = \dfrac{1}{\alpha - \beta}$

$$\sigma^* = \Lambda : \varepsilon^*$$

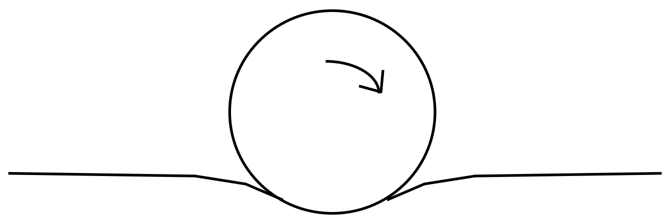
$$div\,\sigma^* + \rho\omega^2 \mathbf{u}^* \approx 0$$

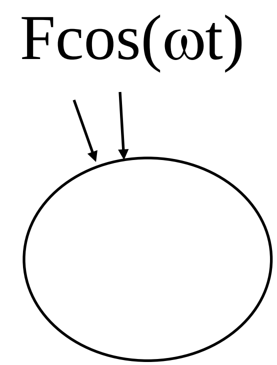
The associated fields u*, ε*, σ* satisfy the elastodynamic equations for low frequency.

Solutions in elastodynamics provide the corresponding solutions in viscoelasticity for small ω. Even for inverse problems

Applications to direct problems

1. Rolling contact on viscoelastic plane, in quasistatics (Goriacheva 1973). Solution for **u***, then solution of the differential equation $\mathbf{u}^* = \mathbf{u} + \alpha\,\dot{\mathbf{u}}$





2. Fatigue of viscoelastic structures Determination of stress fields σ* using elastic solutions

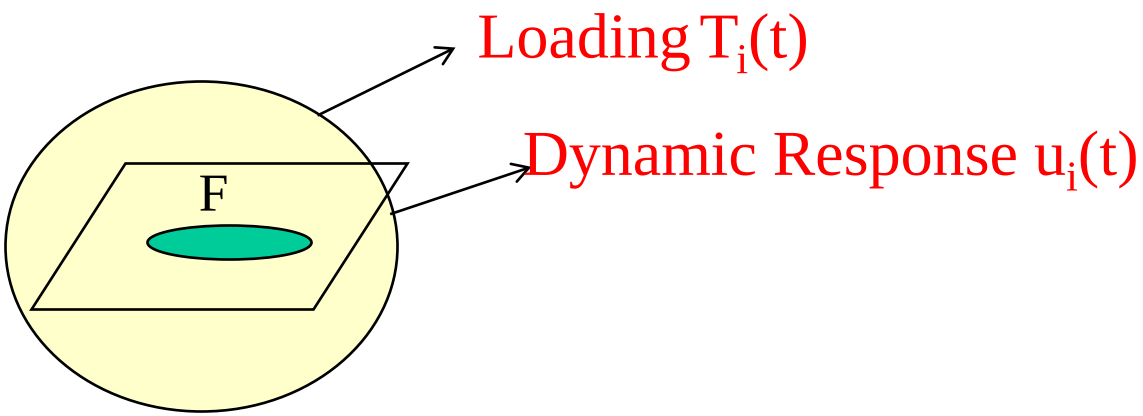
Applications to inverse problems

1. Identification of a planar crack in a 3D bounded solid from surface data u* and T*

2. Identification of a defect (tumor, damaged zone) in a bounded solid from surface data u* and T*

- General approach to Inverse problems in Viscoelasticity with relaxation functions

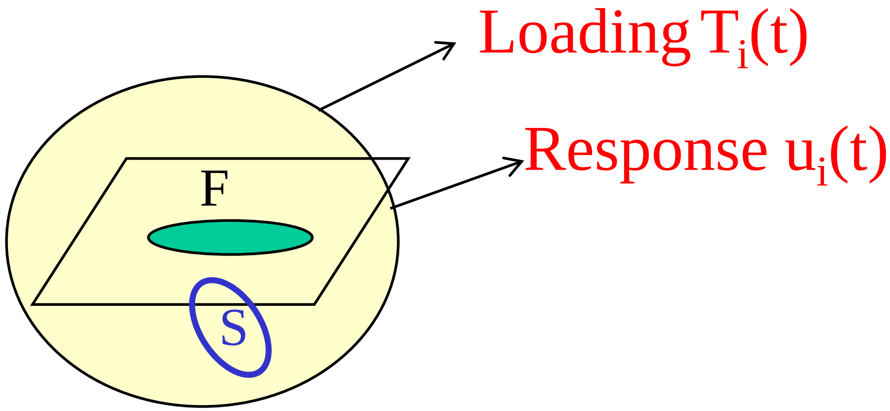
$$\sigma_{ij}(t) = \int_0^t \lambda(t-\tau)\delta_{ij}\dot{\varepsilon}_{kk}(\tau)d\tau + \int_0^t 2\mu(t-\tau)\dot{\varepsilon}_{ij}(\tau)d\tau$$



- Find the crack F under dynamic loadings from surface data measurements

- General method of solution
- Optimization in a four-dimensional space !
- $R^3 \times [0,T]$ for the static or dynamic case

Optimization or Control theory in $\square^{4}$



$$T_i(t) \;\Rightarrow\; u(t, S) \;\Rightarrow\; \left\| u(S) - u \right\|^2 = \int_{\partial\Omega} dA \int_0^T \left\| u(t, S) - u(t) \right\|^2 dt$$

Minimize the residual $\quad \underset{S}{Min} \left\| u(S) - u \right\|^2$

$$F(T) = Arg \;\; \underset{S}{Min} \left\| u(S) - u \right\|^2$$

Very ill-posed problems

<u>Control theory in □ ^{4} for elastodynamics</u>

Earthquake inverse problems to determine
the fault process (moving fault F(t))

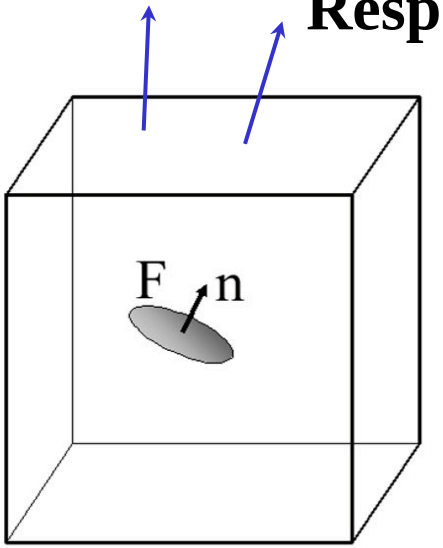
$$F(t) = Arg\,\underset{S}{Min}\left\| u(S) - u \right\|^2$$

(Das and Suhadolc, 1996) Oxford University

"**even if the fitting of data seems to be quite good, the faulting process is <u>poorly reproduced</u>, so that in the real case, it would be difficult to know when one has obtained the correct solution**”.

Inverse crack problem of the real vectorial Helmholtz equation

Load $\mathbf{T}^d$ **Response $\mathbf{u}^d$**



(We only consider a stationary planar crack)

Bui et al, *Annals of Solid and Structural Mech* (2010)

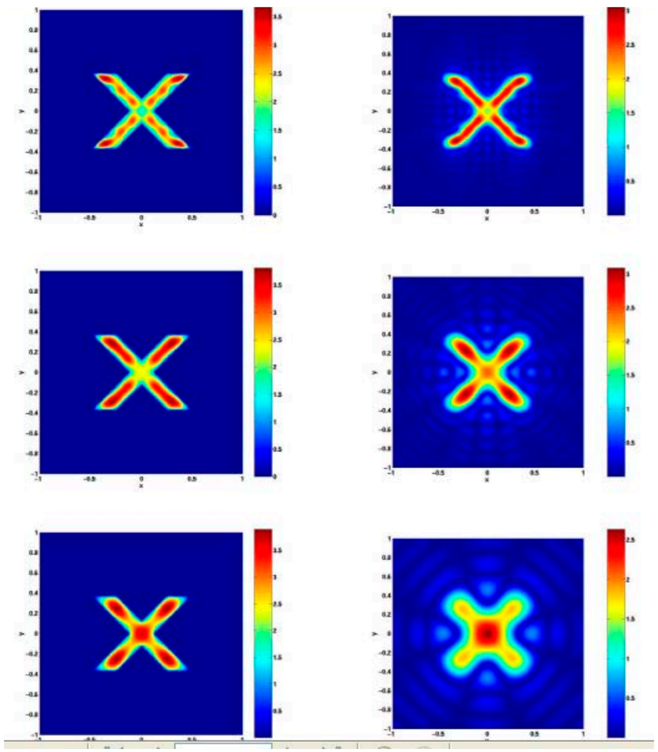
Inverse crack problem of the real vectorial Helmholtz equation

The approach is originated from Calderon’s method (1980) for determining an inclusion in the harmonic case,

then extended to crack problems in elasticity (Andrieux et al, 1999),
to transient elastodynamics (Bui et al, 2005), with applications
to the earthquake inverse problem (Bui, 2006),
to the scalar Helmholtz equation (Ben Abda et al, 2005) and
to the vectorial Helmholtz equation (Bui et al, 2010)

The method used is based on the **reciprocity gap functional**

<u>Inverse crack problem of the real scalar Helmholtz equation in 3D</u>



k=40

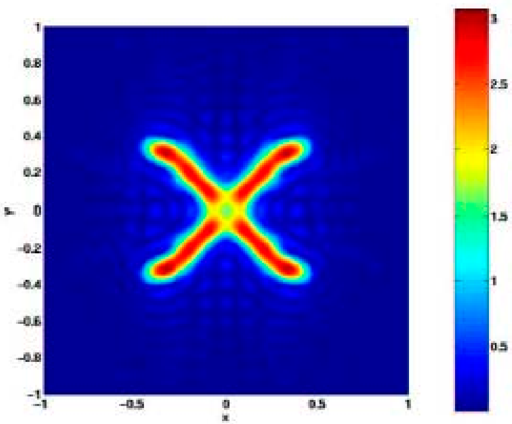
k=25

k=15

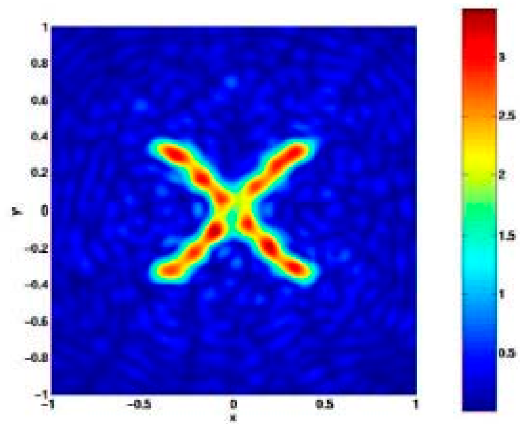
Exact   Reconstructed (no noise)

(Ben Abda et al, 2005)

Inverse crack problem of the real scalar Helmholtz equation



No noise



Reconstructed
with 5% noise on data

k=40

(Ben Abda et al, 2005)

The nonlinear variational equation for a crack in 3D elasticity

Find **F** such that the displacement discontinuity [**u**] satisfies

$$\int_F [\mathbf{u}^*(\mathbf{F})].\sigma(\mathbf{w}).\mathbf{n}\,dS = \int_{S_{ext}} \{\mathbf{w}.\sigma(\mathbf{u}^*).\mathbf{n} - \mathbf{u}^*.\sigma(\mathbf{w}).\mathbf{n}\}\,dS$$

$\forall\, \mathbf{w}$ (Adjoint field satisfying the Helmholtz in Ω)

div grad u* +$\rho\omega^2$ **u*** =0 in Ω\F
$\sigma(\mathbf{u}^*).\mathbf{n} = \mathbf{0}$ on F

div grad **w** +$\rho\omega^2$ **w*** =0 in Ω
No boundary condition

R, linear in **w** is called **reciprocity gap functional**

$$R(\mathbf{w}) \square \int_{S_{ext}} \{\mathbf{w}.\sigma(\mathbf{u}^*).\mathbf{n} - \mathbf{u}^*.\sigma(\mathbf{w}).\mathbf{n}\}\,dS$$

1.The non linear variational equation

Find F such that:

$$\int_F [\mathbf{u}^*(F)].\sigma(\mathbf{w}).\mathbf{n}\,dS = R(\mathbf{w};\mathbf{T}^*,\mathbf{u}^*)$$

$\forall\, \mathbf{w}$ (Adjoint field satisfying the Helmholtz in Ω)

$$R(\mathbf{w};\mathbf{T}^*,\mathbf{u}^*) \square \int_{S_{ext}} \{\mathbf{w}.\sigma(\mathbf{u}^*).\mathbf{n} - \mathbf{u}^*.\sigma(\mathbf{w}).\mathbf{n}\}\,dS$$

We take advantage of the arbitrariness of adjoint functions to **parameterize w** by a set of parameters p $\Rightarrow$ R(p), with dim{p}=dim{unknowns}

1.The non linear variational equation

Find F such that:

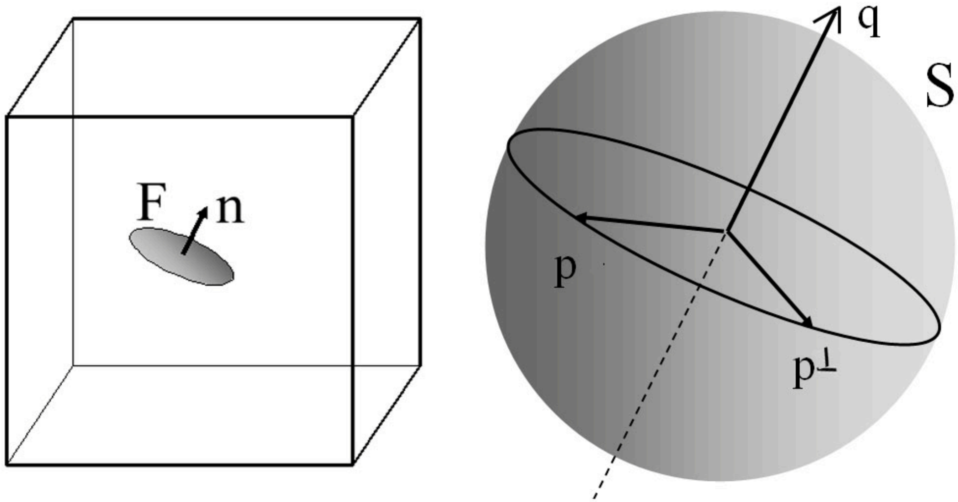
$$\int_F [\mathbf{u}^*(F)].\sigma(\mathbf{w}).\mathbf{n}dS = R(\mathbf{w};\mathbf{T}^*,\mathbf{u}^*)$$

$\forall$ **w** (Adjoint field satisfying the Helmholtz in Ω)

- If [**u***]=0 : R=0 (Reciprocity theorem) No defect
  R≠0 is a defect indicator.

- The crack identification consists in searching the zeros of R(p), with **dim{p}=dim{unknowns}.**
  It is a **Zero crossing method**.

Solution for the normal n



Take paramerters $\mathbf{p}, \mathbf{p}^{\perp}$ on the sphere of radius $k/\sqrt{\mu}$

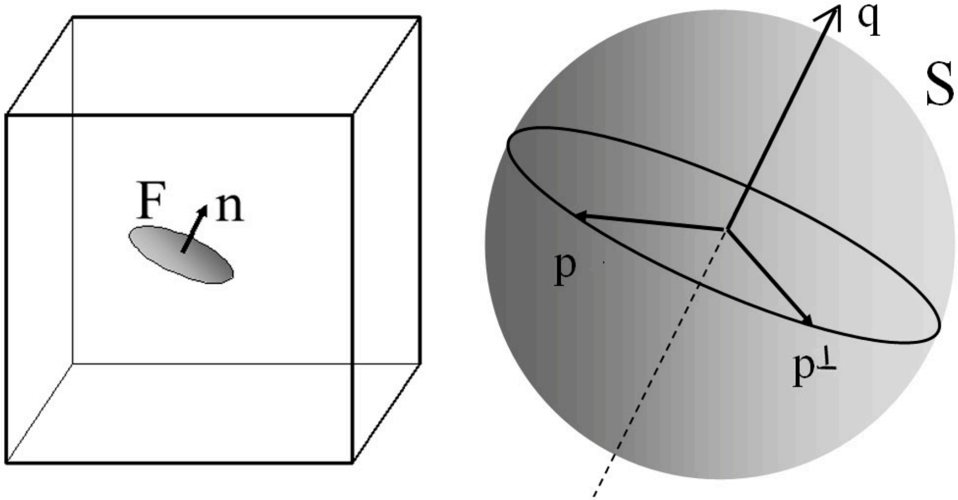
$$k^2 = \rho\omega^2, \quad \mathbf{q} = \mathbf{p} \times \mathbf{p}^{\perp} / \|\mathbf{p}\|$$

$$\mathbf{w} = \mathbf{p}\sin(\mathbf{x.p}^{\perp}),$$

$$\mathrm{R} \propto (\mathbf{n.p} + \mathbf{n.p}^{\perp}) \propto \mathrm{R}(\mathbf{q})$$

The solution n corresponds to the North or South poles q for which R(q)=0, p and p$^{\perp}$ in the equatorial plane

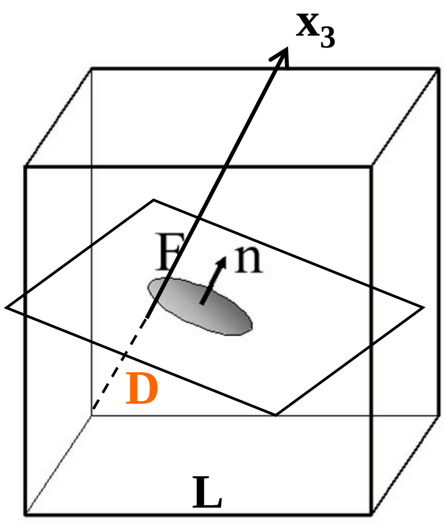
Solution for the normal n

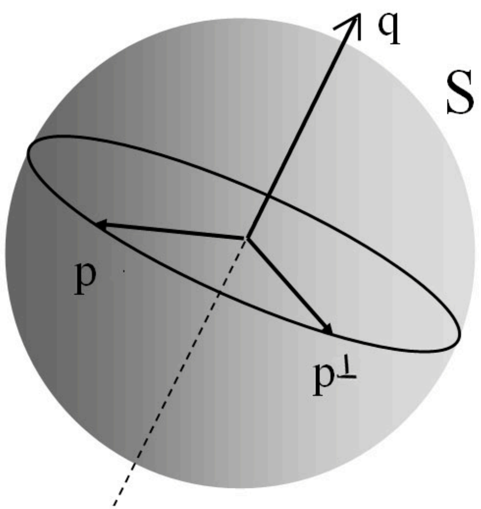


The solution n corresponds to the North or South poles q such that R(w)=0

A plot of the function $R(q=p \times p^{\perp})$ on the sphere reveals the zeros or the normal direction n

Solution for the crack plane $x_3 - D = 0$





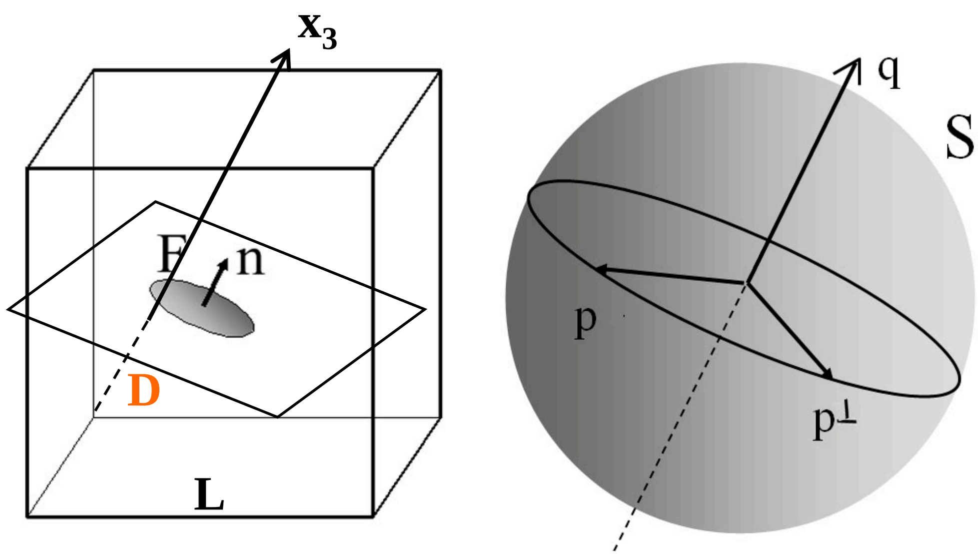
$$q = \frac{k}{\sqrt{\lambda + 2\mu}}$$

choose k or q such that $\frac{2\pi}{q} > \mathrm{L}$

$$\mathbf{w}(\kappa) = \mathbf{e}^3 \cos(q(x_3 - \kappa)) \quad \Rightarrow \quad R(\kappa) \propto \sin(q(D - \kappa))$$

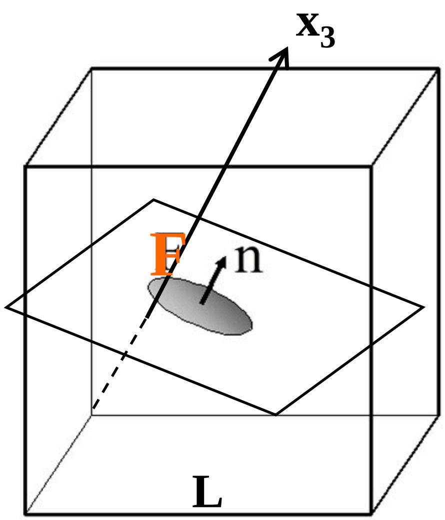
We find that D is the unique zero R(κ=D)=0

Solution for the crack plane



The crack plane is determined by the zero D of function R(κ).

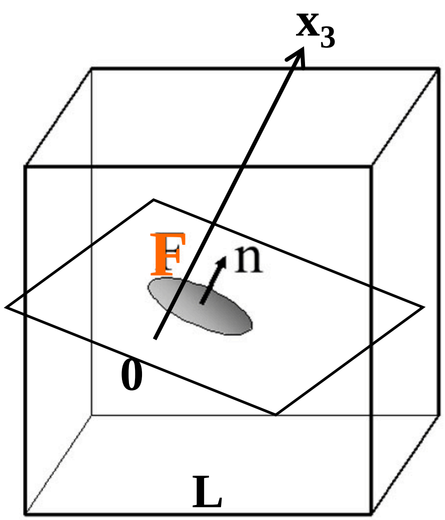
Solution for the crack geometry F



The method consists in the determination of [u]
For k=0, the inverse crack problem in elasticity have been solved by Andrieux et al (1999).

We adapt this solution to the Helmholtz equation !

Solution for the crack geometry F



Take the crack plane as Ox_1x_2

Parameters $\mathbf{p}=(p_1,p_2,0)$, $\mathbf{p}^{\perp}=(-p_2,p_1,0)$

parallel to the crack plane

$$Z(\mathbf{p})^{\pm} = \mathbf{p} \pm i\gamma \|\mathbf{p}\| \mathbf{e}^3$$

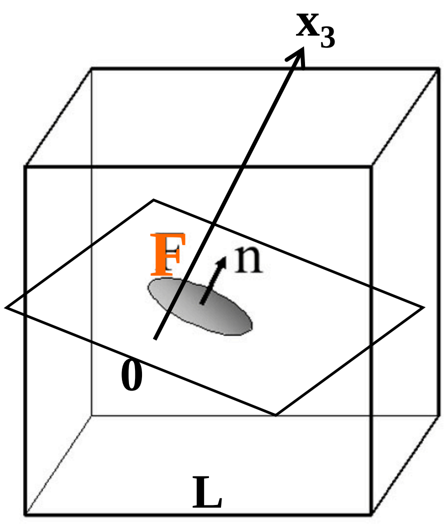
$$\gamma^2 = 1 - \frac{k^2}{(\lambda+2\mu)\|\mathbf{p}\|^2}$$

$$\mathbf{w}(\mathbf{x};\mathbf{p})^{\pm} = \nabla_x \exp\{-iZ(\mathbf{p})^{\pm}.\mathbf{x}\}$$

Adjoint vector field $\quad \mathbf{w}(\mathbf{x};\mathbf{p}) = \mathbf{w}^{+} + \mathbf{w}^{-}$

satisfies the wave equation

Solution for the crack geometry F



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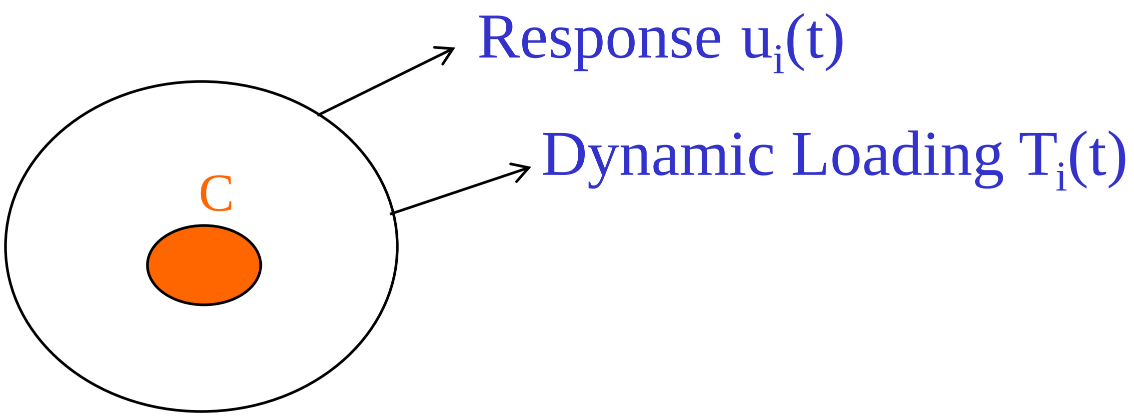
$$\gamma^2 = 1 - \frac{k^2}{(\lambda + 2\mu)\|\mathbf{p}\|^2}$$

$$\mathbf{w}(\mathbf{x};\mathbf{p})^{\pm} = \nabla_x \exp\{-iZ(\mathbf{p})^{\pm}.\mathbf{x}\}, \qquad \mathbf{w}(\mathbf{x};\mathbf{p}) = \mathbf{w}^+ + \mathbf{w}^-,$$

$$[u_3(\mathbf{x})] = \frac{1}{(2\pi)^2} \int_{p_3=0} \frac{\exp(i\mathbf{p}.\mathbf{x}) R(\mathbf{p};\mathbf{u}^d,\mathbf{T}^d) dp_1 dp_2}{2\left\{\lambda(\gamma^2 - 1) + 2\mu\gamma^2\right\}\|\mathbf{p}\|^2}$$

The defect identification (2D scalar problem)



- Find the inclusion **C** under dynamic loadings from surface data $T_i(t)$, $u_i(t)$

The 2D scalar problem

$$div\{(1+h(\mathbf{x}))grad\, u^*(x;h)\}+k^2u^*=0 \quad \text{in } \Omega$$

$$\left.\begin{aligned} &u^* = f \\ &\frac{\partial}{\partial n}u^* = g \end{aligned}\right\} \quad \text{Data on } \partial\Omega$$

By linearization

Calderon (1980) solved the inverse problem for small value of h(x) and $k=0$ (statics)

The 2D scalar problem

Normalized shear modulus, perturbation h(x)

$$div\{(1+h(\mathbf{x}))grad\, u^*(x;h)\}+k^2u^*=0 \qquad \text{in } \Omega$$

$$\left.\begin{aligned} u^* &= f \\ \frac{\partial}{\partial n}u^* &= g \end{aligned}\right\} \qquad \text{on } \partial\Omega$$

- Calderon's linearization (1980) in the static case k=0,for small h, upon replacing grad u*(x,h) by grad u*(x,0)

- The linearized solution for k≠0 is straightforwards

The nonlinear case

Find $h(\mathbf{x})$ (we drop the * symbol)

$$\boxed{\int_{\mathbf{C}} h(\mathbf{x}) \, grad \, u(x;h) \, grad \, \varphi(x) \, d^2x = R(\varphi; f, g), \quad \forall \varphi}$$

$$div \, grad \, \varphi + k^2 \varphi = 0 \qquad \text{in } \Omega$$

$$R(\varphi; f, g) = \int_{\partial\Omega} (\varphi g - \frac{\partial \varphi}{\partial n} f) ds \qquad \text{(Data linear form in } \varphi)$$

Nonlinear variational equation for h and C

Reduction to two linear problems

$$div\{(1+h(\mathbf{x}))grad\,u(x;h)\}+k^2u=0$$

$$\Updownarrow$$

♦ Problem I (source inverse problem): Solution denoted by U(x)

$$div\{grad\,u\}+k^2u+S(\mathbf{x})=0 \quad \text{in} \quad \Omega$$

+ The same two boundary data f, g and

$$S(\mathbf{x})=div\{h(\mathbf{x})grad\,u(\mathbf{x};h)\}$$

♦ Problem II $\int_{Supp(S)} h(\mathbf{x})grad\,U(\mathbf{x})grad\,\varphi(\mathbf{x})d^2x = R(\varphi;f,g) \quad \forall\varphi$

where $U(x)=u(x;h)$ is the solution of Problem I

♦Problem I (source inverse problem): Solution U(x)

$$div\{grad\, u\} + k^2 u + S(\mathbf{x}) = 0 \quad \text{in} \quad \Omega$$

+ The same two boundary data

$$S(\mathbf{x}) = div\{h(\mathbf{x}) grad\, u(\mathbf{x}; h)\}$$

- The solution of problem I provides the source S(x) and $U(x) := u(x; h)$

♦Problem I (source inverse problem): Solution U(x)

$$div\{grad\, u^*\} + k^2 u^* + S(\mathbf{x}) = 0 \quad \text{in} \quad \Omega$$

+ The same two boundary data

$$S(\mathbf{x}) = div\{h(\mathbf{x})\, grad\, u^*(\mathbf{x}; h)\}$$

- The solution of problem I provides the source S(x) and $U(x) := u^*(x; h)$
- The support of function S(x) is the same of supp(**h**) i.e. the geometry of the inclusion is determined by a source inverse problem

♦<u>Problem I</u> (source inverse problem): Solution U(x)

$$div\{grad\, u\} + k^2 u + S(\mathbf{x}) = 0 \quad \text{in} \quad \Omega$$

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$$S(\mathbf{x}) = div\{h(\mathbf{x})grad\, u(\mathbf{x}; h)\}$$

- The solution of problem I provides the source S(x) and $U(x) := u(x; h)$
- The support of function S(x) is the same of supp(**h**) i.e. the geometry of the inclusion is determined by a source inverse problem
- Source inverse problem is solved by Alves and Ha Duong (1997)

- <u>Problem II</u> $\int_{Supp(S)} h(\mathbf{x}) grad\, U(\mathbf{x}) grad\, \varphi(\mathbf{x}\boldsymbol{\xi})\, d^2x = R(\varphi; f, g) \quad \forall \varphi$

  where $\quad U(x) = u(x; h) \quad$ is the solution of Problem I

  $\varphi(\mathbf{x}\boldsymbol{\xi})$ is the adjoint field depending on <u>parameter</u> $\xi \in \mathbb{R}^2$

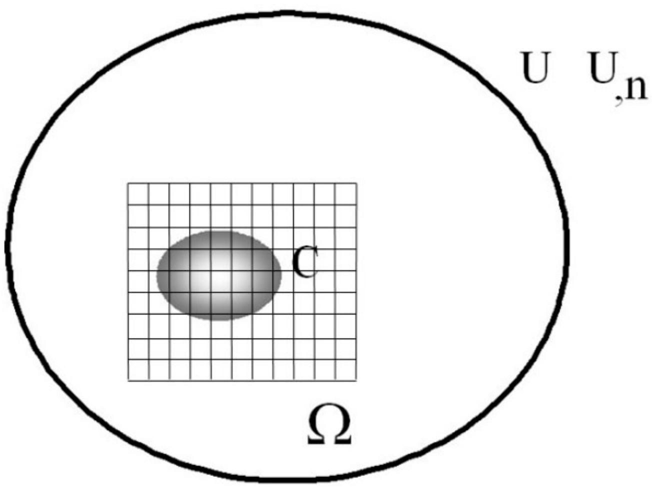
Since C=supp(S)

Problem II provides a **linear** Volterra integral eq. for h(x) with kernel $grad\, U(\mathbf{x}) grad\, \varphi(\mathbf{x}\boldsymbol{\xi})$

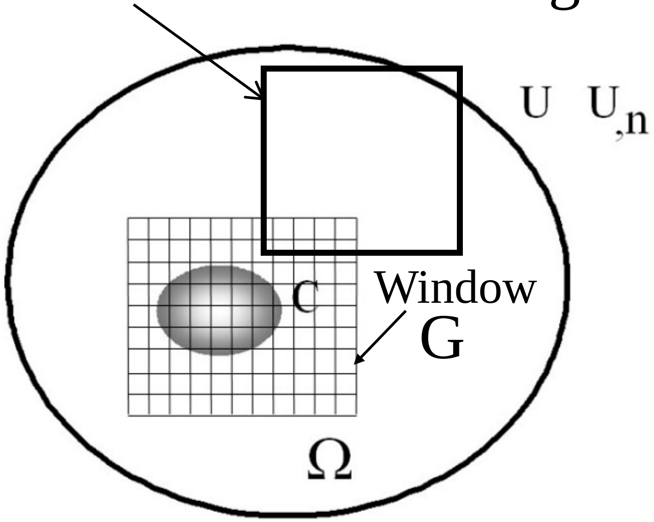
Thus the nonlinear Calderon's inverse problem reduces to two linear problems !

A numerical approach to the source inverse problem

- Difficult problem when the sources domain is small because S(x)=0 for x outside the inclusion C.
- Proposed method: a **moving windows.**
- Numerical solution in statics k=0



Bad window G : wrong solution

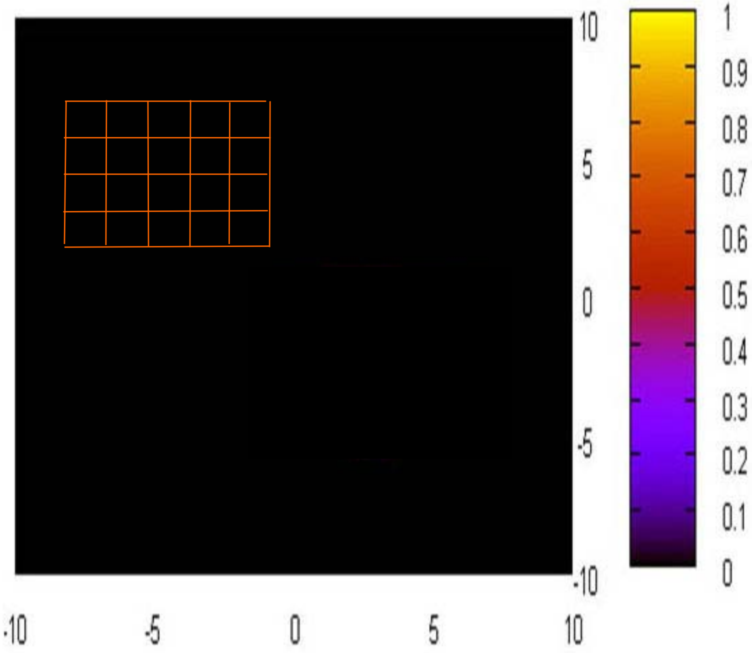


- Medical imaging by echography
  Selection of the scanned zone
  Discretization of the selected window

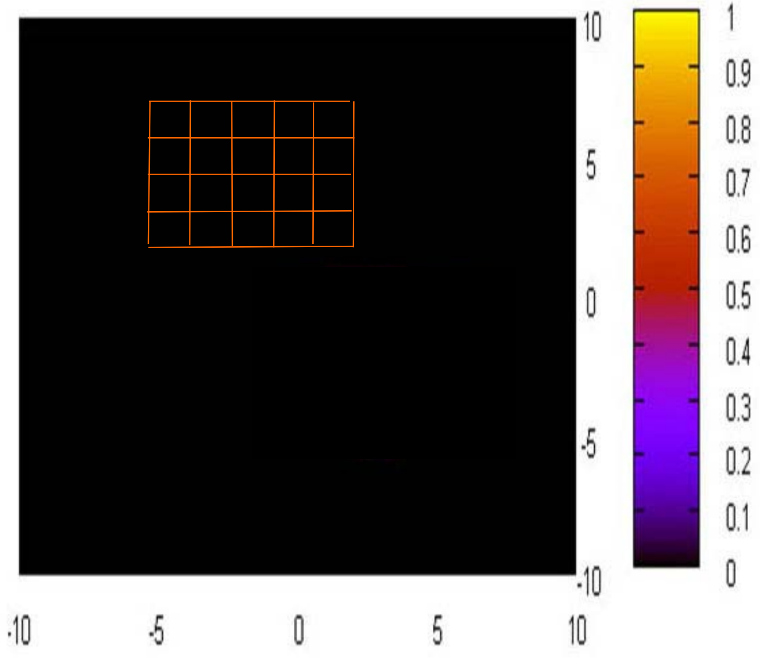
Sharp image for the right position
of the transducer

Detection of a tumor

Discretization of a selected windows

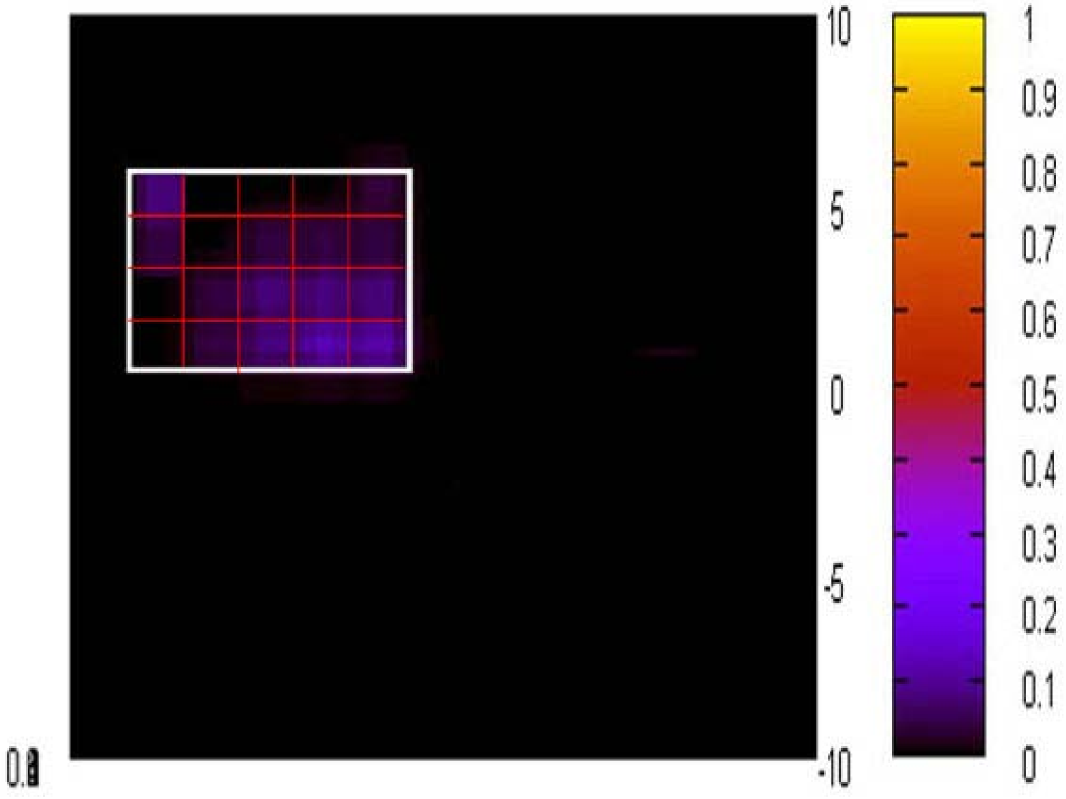


Detection of a tumor
Discretization of a selected windows



Detection of a tumor

Bad windows, wrong solution, blurred image



Detection of a tumor

Bad windows, wrong solution, blurred image

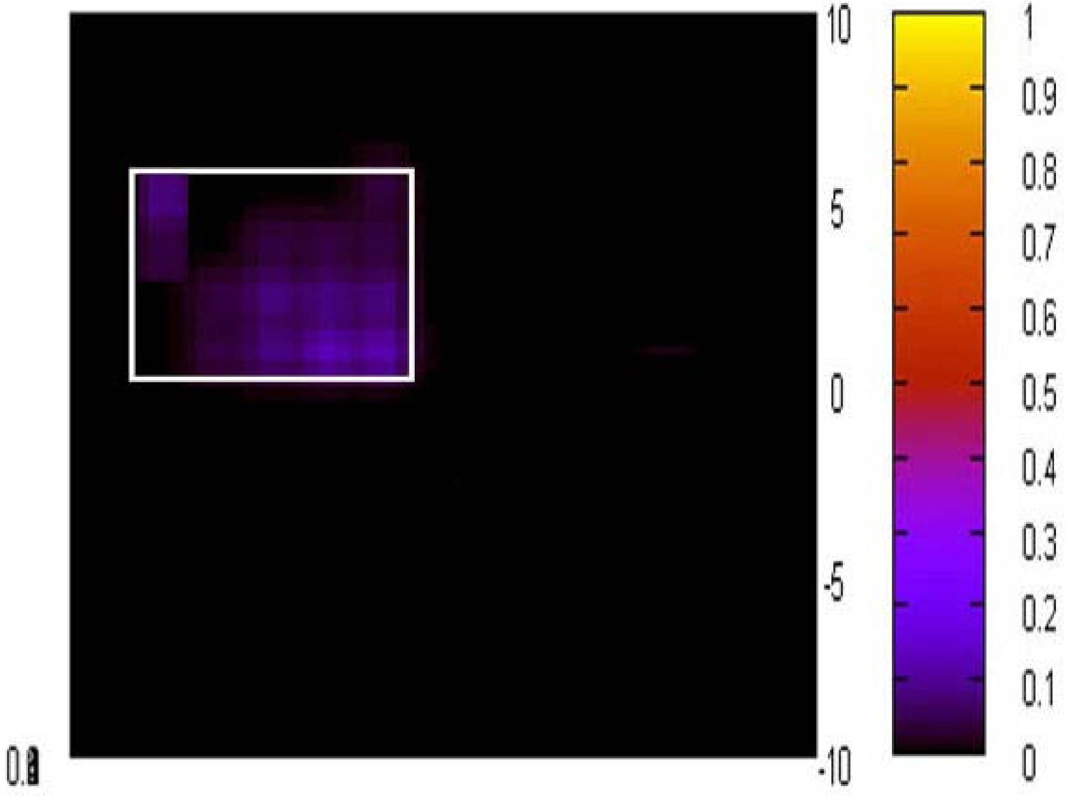
$$\underset{Z}{Min}\left\|u(Z)-u\right\|^2 \ : \ \text{large error} \quad \Rightarrow \quad \text{Wrong solution}$$

$$\underset{Z}{Min}\left\|u(Z)-u\right\|^2 \ : \ \text{small error} \quad \Rightarrow \quad \text{"Good" solution}$$

$$\underset{Z}{Min}\left\|u(Z)-u\right\|^2 = 0 \quad \Rightarrow \quad \text{Exact solution}$$

Detection of a tumor

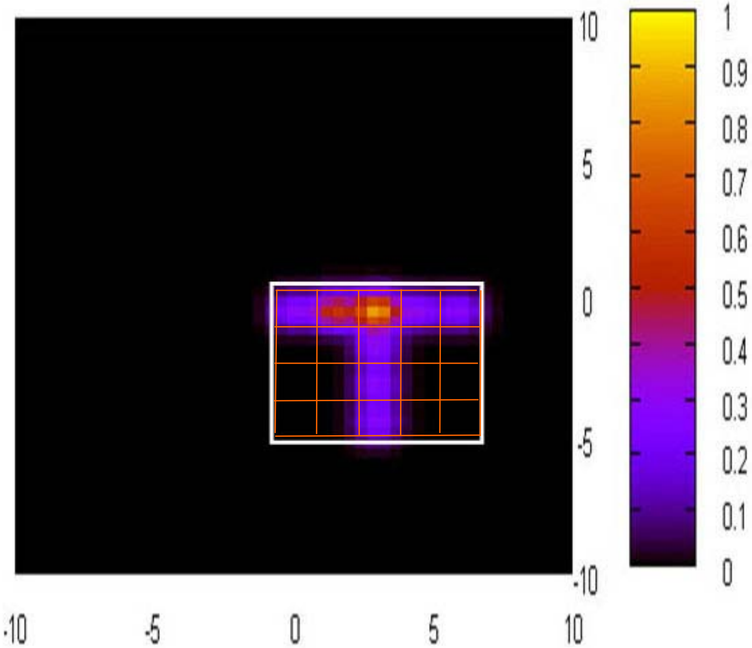
Bad windows, wrong solution, blurred image



Detection of a tumor

Correct windows, good/exact solution,

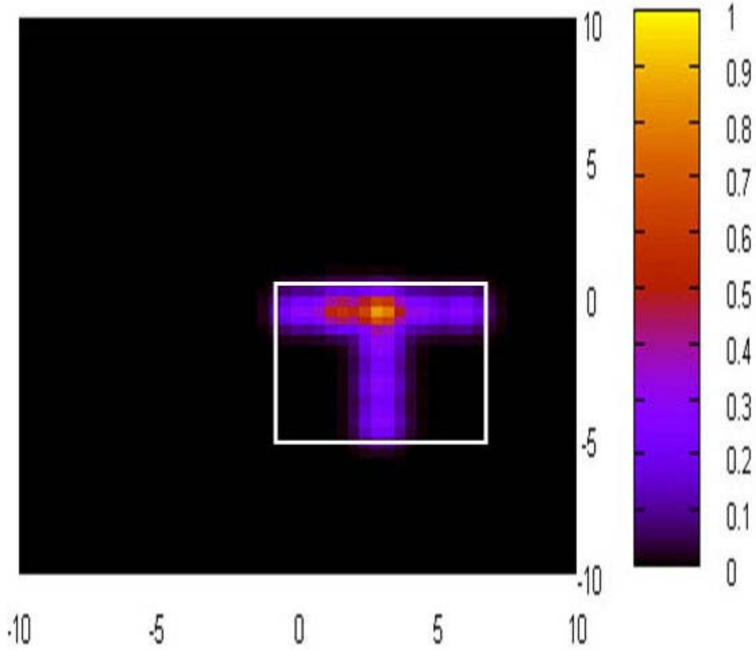
Sharper image



Detection of a tumor

Correct windows, good/exact solution,

Sharper image



CONCLUSIONS

- The Zener viscoelasticity law enables us to establish an *elastic-viscoelastic correspondence* for low frequency
- The use of Goriacheva's *associated fields* leads to the dynamic equation in elasticity for low ω
- Inverse problems in viscoelasticity to determine a *planar crack* and a *volumic defect* are solved by the reciprocity gap functional method.

Thank you

and

Congratulations to Leon Keer

Problem I (source inverse problem)

$$div\{grad\,U(x)\} + k^2 U + S(\mathbf{x}) = 0 \quad \text{in} \quad \Omega$$

u and $\dfrac{\partial u}{\partial n}$ are known on $\partial\Omega$

Solution given by Alves and Ha Duong (1997) for N point sources

If S(x) is a finite sum of point sources S_i of intensity q_i the solution exists and is unique.

- Discretize the expected windows containing C

$$S(\mathbf{x}) = \sum_{i=1}^{N} q_i \delta(\mathbf{x} - \mathbf{a}_i)$$ 2N unknowns a_i, q_i

Non linear systems of equations obtained with 2N adjoint fields