“DRIVING FORCES” ON MOVING DEFECTS:

DISLOCATIONS and INCLUSION BOUNDARIES with transformation strain

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Energy-release rates (Driving forces) for moving defects: dislocations and inclusions with inertia effects

Dislocation: to create a new slip area

(-1 singularity)

Inclusion boundary: to create a new volume of eigenstrain

(jump discontinuity)

EXPANDING ESHELBY INCLUSIONS

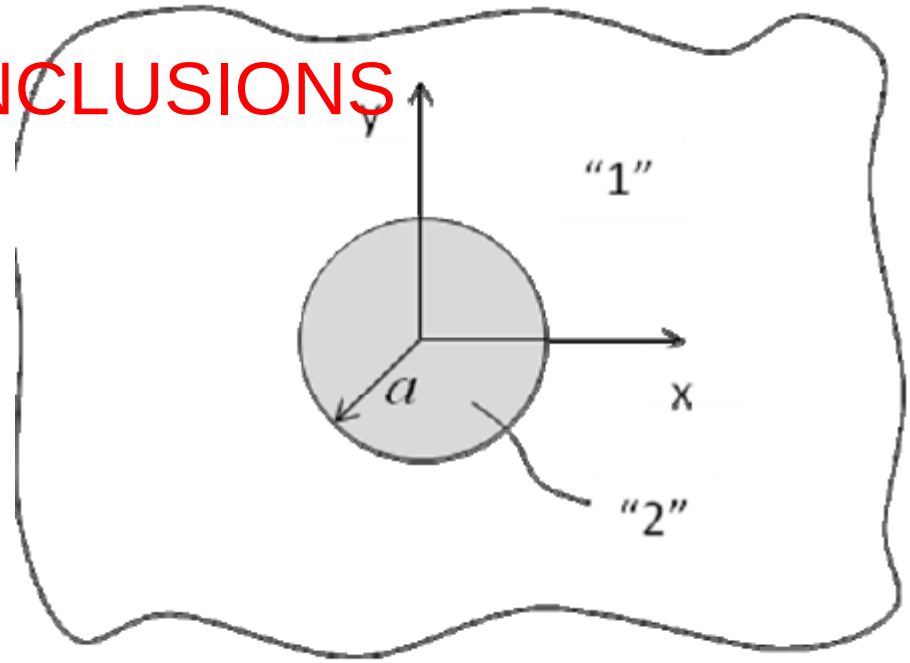
$$\sigma_{ij} = C_{ijkm}\left(\frac{\partial u_k}{\partial x_m} - \epsilon^*_{km}\right)$$

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_{ij}\left(\frac{\partial u_k}{\partial x_m} - \varepsilon^*_{km}\right)$$

Dynamic Hadamard jump conditions

$$n_j[[\sigma_{ij}]] = -\dot{l}(t)[[\dot{u}_i]],$$

$$[[\dot{u}_i]] = -\dot{l}(t)[[\frac{\partial u_i}{\partial n}]]$$

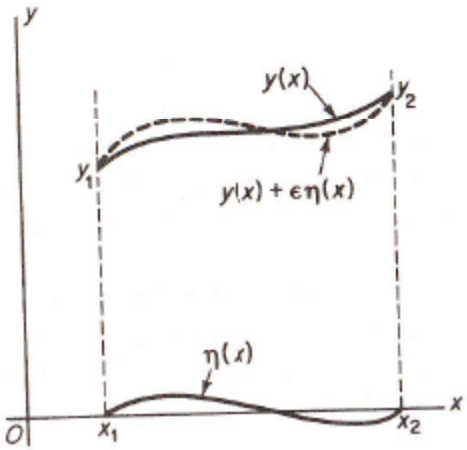


Dynamic fields: Spherically expanding inclusion with dilatational eigenstrain in general motion

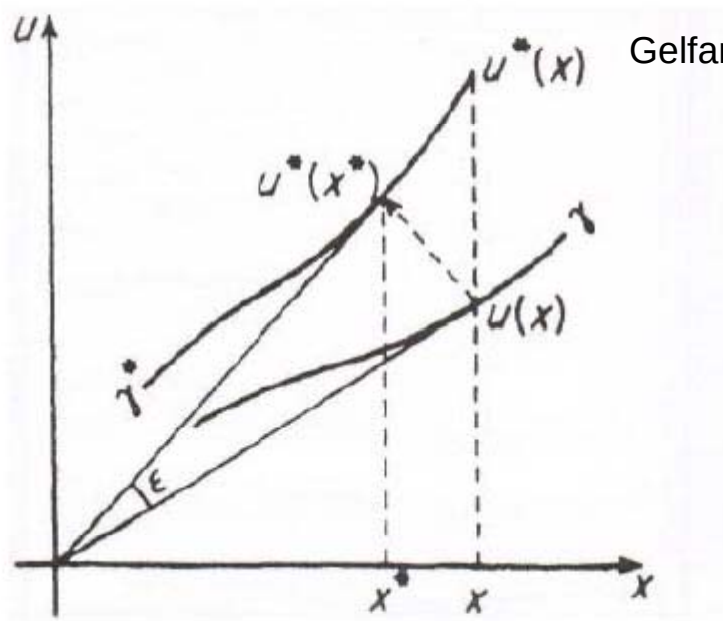
Plane half-space constrained inclusion/ inhomogeneity in general motion (general eigenstrain)

Energy release-rate (driving force) to create incremental region of eigenstrain

Variation of a functional defined on a variable region



$$I(\epsilon) = \int_{x_1}^{x_2} F(x, y + \epsilon\eta, y' + \epsilon\eta')\, dx,$$



Gelfand and Fomin

$$J[u] = \int \cdots \int_R F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n})\, dx_1 \cdots dx_n,$$

$$J[u^*(x^*)] = \int_{R^*} F(x^*, u^*, \nabla^* u^*)\, dx^* \qquad J[u^*(x^*)] - J[u(x)].$$

Invariance for infinitesimal transformations, Noether, 1918

An interpretation of the configurational force

Noether

$$\Pi(u_i, u_{i,j}) = \int_\Omega W(x_i, u_i, u_{i,j}) dV$$

new independent variables $\quad y_i = \hat{y}_i(x_i, \epsilon_i) \qquad u_i = \hat{u}_i(x_i, 0)$

new dependent variables $\quad v_i(y_i) = \hat{v}_i(x_i, \epsilon_i) \quad u_i = \hat{u}_i(x_i, 0)$

$$y_i = x_i + \phi_i + O(\epsilon^2) \qquad v_i = u_i + \psi_i + O(\epsilon^2)$$

$$\delta\Pi = \int_\Omega \frac{d}{dx_i}\left(\frac{\partial W}{\partial u_{k,i}}\bar{\psi}_k + W\phi_i\right) dV + \int_\Omega \Psi_i \bar{\psi}_i dV$$

$$\bar{\psi}_k = \psi_k - u_{k,j}\phi_j \qquad \Psi_i = \frac{\partial W}{\partial u_i} - \frac{d}{dx_j}\frac{\partial W}{\partial u_{i,j}}$$

$\phi_i = \epsilon_i$ translation

$$\delta_x \Pi = \int_{\partial\Omega} (W\delta_{ij} - u_{k,j}p_{ki})\epsilon_j n_i dA + \int_{\Omega} p_{ik,k} u_{i,j} \epsilon_j dV$$

$$F = \int_{\partial\Omega} E_{ij}\epsilon_j n_i dA \qquad E_{ij} = (W\delta_{ij} - u_{k,j}p_{ki})$$

Energy-momentum tensor

conjugate to ϵ_j *configurational force,*

$\delta_x \Pi$ is given by the flux F

J integral

if and only if the equilibrium condition is satisfied.

homogeneous and smooth $\frac{\partial W}{\partial x_i} = 0$ everywhere in Ω

$$\begin{aligned}\delta_x \Pi &= \int_{\Omega} (W\delta_{ij} - u_{k,j}p_{ki})_{,i}\epsilon_j dV + \int_{\Omega} p_{ik,k} u_{i,j}\epsilon_j dV \\ &= \int_{\Omega} (p_{ki}u_{k,ij} - u_{k,ji}p_{ki} - u_{k,j}p_{ki,i})\epsilon_j dV + \int_{\Omega} p_{ik,k} u_{i,j}\epsilon_j dV \\ &= 0\end{aligned}$$

the body remains infinitesimally close to equilibrium even after a small perturbation of the inhomogeneity position.

Gupta and Markenscoff.
Comptes Rendus, 2008

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“Conservation Laws” for Elastodynamics

Noether’s theorem
For Lagrangean
W-T,
Fletcher (1976):

$$\frac{\partial}{\partial x_j}\left[(W-T)\delta_{lj}-\sigma_{ij}u_{i,l}\right]+\frac{\partial}{\partial t}(\rho\dot{u}_i u_{i,l})=0$$

$$E_{ij}=\left((W-T)\delta_{ij}-u_{k,j}\frac{\partial W}{\partial u_{k,i}}\right)$$

Energy-momentum tensor

$$\Pi_L(u_{i,j},\dot{u}_i)=\int_{t_1}^{t_2}\int_{\Omega}(W(x_i,u_{i,j})-T(\dot{u}_i))dVdt,$$

$$y_\alpha=x_\alpha+\phi_\alpha+O(\varepsilon^2)\text{ and }v_i=u_i+\psi_i+O(\varepsilon^2)$$

Dynamic *J* integral

$$\begin{aligned}\delta_x\Pi_L = \quad & \varepsilon_j\int_{t_1}^{t_2}\int_{\partial\Omega}E_{ij}n_i dAdt+\varepsilon_j\int_{t_1}^{t_2}\int_{\Omega}\frac{d}{dt}(\rho u_{k,j}\dot{u}_k)dVdt\\ & -\ \varepsilon_k\int_{t_1}^{t_2}\int_{\Omega}(-\sigma_{ij,j}+\rho\ddot{u}_i)u_{i,k}dVdt\end{aligned}$$

with Anurag Gupta

The equation of motion of a dislocation

Eshelby, 1953

$$F_x = b p_{zy}{}^A = (1-\dot{\xi}^2/c^2)^{-\frac{3}{2}}(\rho b^2/4\pi)\{\ln f(t)\}\partial^2\xi/\partial t^2 + g(t)$$

Peach-Koehler force

Self-force (inertia)

the accelerating dislocation is continually catching up the radiation it has already emitted.

Eshelby: "The dislocation is haunted by its past"

Markenscoff & Ni, *JMPS*, 2008

Energy-release rate for a moving phase boundary

Dynamic *J* integral equivalent to energy-release-rate for jump discontinuity

Moving Cracks:
Atkinson & Eshelby, 1968
Freund, 1972
Stolz,2004

$$\dot{\mathcal{E}} = \lim_{\epsilon \to 0} \int_{S_\epsilon} [n_j \sigma_{ij} \dot{u}_i + v_n (W + T)] dS$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (n_j [[\sigma_{ij} \dot{u}_i]] + \dot{l} [[W]] + \dot{l} [[T]]) dx_2 dx_3,$$

$$n_j [[\sigma_{ij}]] = -\dot{l}(t) [[\dot{u}_i]],$$

$$[[\dot{u}_i]] = -\dot{l}(t) [[\frac{\partial u_i}{\partial n}]]$$

$$\dot{\mathcal{E}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -\dot{l}(t) ([[W]] - < \sigma_{ij} > [[\frac{\partial u_i}{\partial x_j}]]) dx_2 dx_3.$$

$$f = \mathcal{E} / \dot{l}(t) = [[W]] - < \sigma_{ij} > [[\frac{\partial u_i}{\partial x_j}]]$$

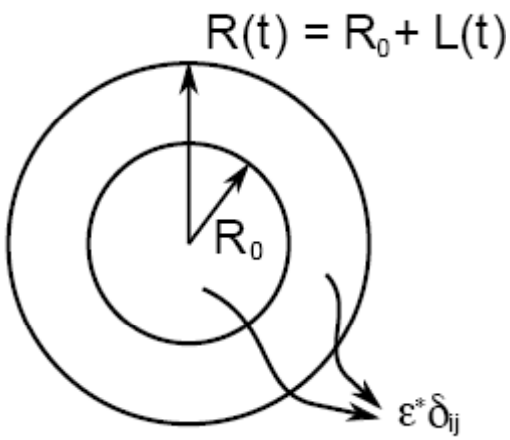
Eshelby, 1970

A dynamically expanding spherical Eshelby Inclusion

$$u_i(\boldsymbol{x},t) = \int_{-\infty}^{\infty} dt' \int_{S(t)} c_{jklm}\delta_{lm}\epsilon^* n_k(\boldsymbol{x}')G_{ij}(\boldsymbol{x}-\boldsymbol{x}',t-t')dS,$$

Willis, 1965

$$G_{km}(\bar{r}_i,\bar{t}) = \frac{1}{4\pi\rho}\left\{\frac{\bar{t}}{\bar{r}^2}[\frac{3\bar{r}_k\bar{r}_m}{\bar{r}^3} - \frac{\delta_{km}}{\bar{r}}]H(\bar{t}-\frac{\bar{r}}{a})H(\frac{\bar{r}}{b}-\bar{t}) + \frac{\bar{r}_k\bar{r}_m}{\bar{r}^3}[\frac{\delta(\bar{t}-\bar{r}/a)}{a^2} - \frac{\delta(\bar{t}-\bar{r}/b)}{b^2}] + \frac{\delta_{km}\delta(\bar{t}-\bar{r}/b)}{b^2}\right\},$$



$$u_r(r,t) = \frac{3\lambda+2\mu}{2\rho a}\left[\int_{-\infty}^{0} \frac{R_0}{r}\epsilon^*\cos\theta_0(t-t')f_0(t-t')dt' + \int_{0}^{\infty} \frac{R(t')}{r}\epsilon^*\cos\theta_{t'}(t-t')f_{t'}(t-t')dt'\right],$$

$$f_s(t-t') = \begin{cases} 0, & a(t-t') < |r-R(s)| \\ 1, & |r-R(s)| \le a(t-t') \le r+R(s) \\ 0, & r+R(s) < a(t-t'). \end{cases}$$

$$\cos\theta_s(t-t') = \begin{cases} 1, & \\ \frac{r^2+R^2(s)-a^2(t-t')^2}{2rR(s)}, & |r-R(s)| \le a(t-t') \le r+R(s), \ (2 \\ -1, & r+R(s) < a(t-t'), \end{cases}$$

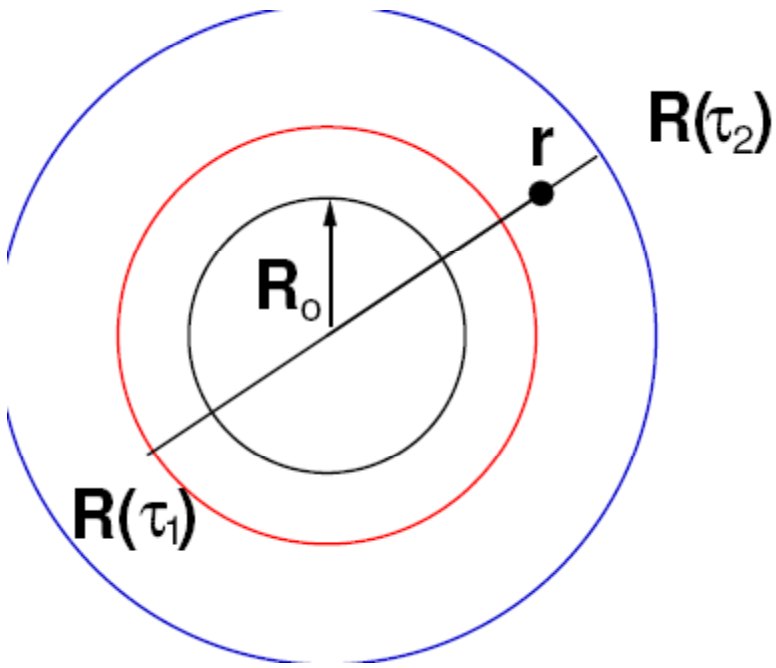
$$u_r(r,t) = -\frac{R_0^3(3\lambda+2\mu)\epsilon^*}{3r(\lambda_2\mu)}H(r-R_0-at) + \frac{r(3\lambda+2\mu)\epsilon^*}{3(\lambda+2\mu)}H(R_0-r-at)$$
$$+u_r^{II}(r,t)H(at-|r-R_0|)H(r+R_0-at)$$
$$+u_r^{III}(r,t)H(at-r-R_0), \qquad (2.5)$$

where u_r^{II} and u_r^{III} are given by

Hadamard jumps

$$u_r(r,t) = \frac{a(3\lambda+2\mu)\epsilon^*}{4(\lambda+2\mu)r^2}\int_t^{\frac{r+R_0}{a}}(r^2+R_0^2-a^2s^2)ds$$
$$+\frac{a(3\lambda+2\mu)\epsilon^*}{4(\lambda+2\mu)r^2}\int_0^{\tau_2}[r^2+R^2(t')-a^2(t-t')^2]dt'$$
$$=\frac{(3\lambda+2\mu)\epsilon^*}{4\rho a^2r^2}\left\{a(r^2+R_0^2)(\tau_2-t)+\frac{1}{3}[2(r^3+R_0^3)+a^3(t-\tau_2)^3]\right.$$
$$\left.+\int_0^{\tau_2}a[2R_0l(t')+l^2(t')]dt'\right\},$$

$$u_r^{III}(r,t) = \frac{(3\lambda+2\mu)\epsilon^*}{4\rho ar^2}\int_{\tau_1}^{\tau_2}[r^2+R^2(t')-a^2(t-t')^2]dt'$$
$$=\frac{(3\lambda+2\mu)\epsilon^*}{4\rho ar^2}\{(r^2+R_0^2)(\tau_2-\tau_1)$$
$$+\frac{a^2}{3}[(t-\tau_2)^3-(t-\tau_1)^3]+\int_{\tau_1}^{\tau_2}[2R_0l(t')+l^2(t')]dt'\}$$



Self-Force on a Dynamically Expanding Spherical Inclusion

$$f = [[W]] - < \sigma_{ij} > [[\partial u_i/\partial x_j]],$$

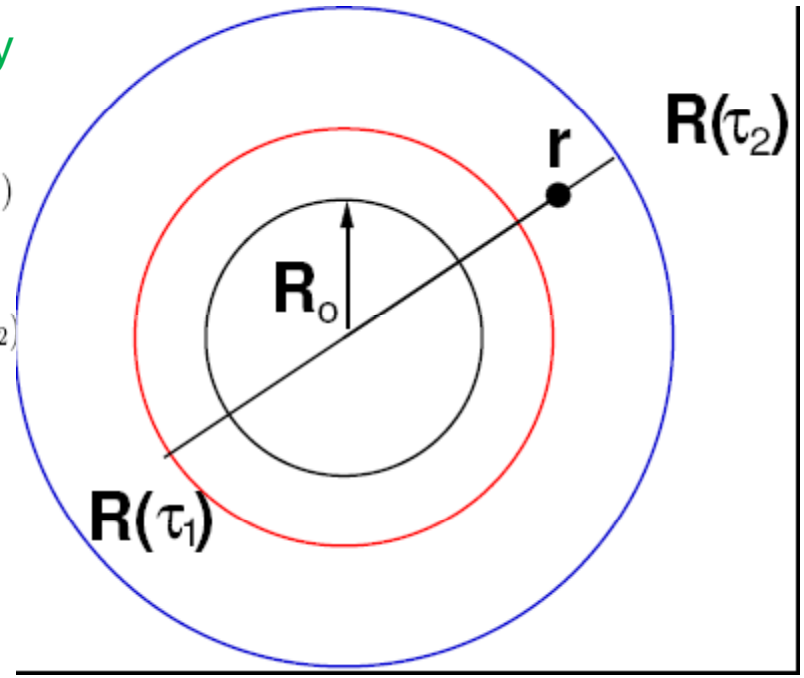
$$= -\frac{2\mu(3\lambda+2\mu)^2\epsilon^{*2}}{(\lambda+2\mu)} - \frac{(3\lambda+2\mu)^2\epsilon^{*2}}{2(\lambda+2\mu)}[\frac{a\dot{l}(t)}{(a^2-\dot{l}^2(t))}]$$

$$+H(at-(R(t)+R_0))\frac{\epsilon^{*2}(3\lambda+2\mu)^2}{2(\lambda+2\mu)}[\frac{R(\tau_1)}{R(\tau_2)}(\frac{\dot{l}(\tau_1)}{a+\dot{l}(\tau_1)})]|_{r=R(t)}$$

Contribution from the back strain discontinuity

$$r + R(\tau_1) = a(t-\tau_1)$$

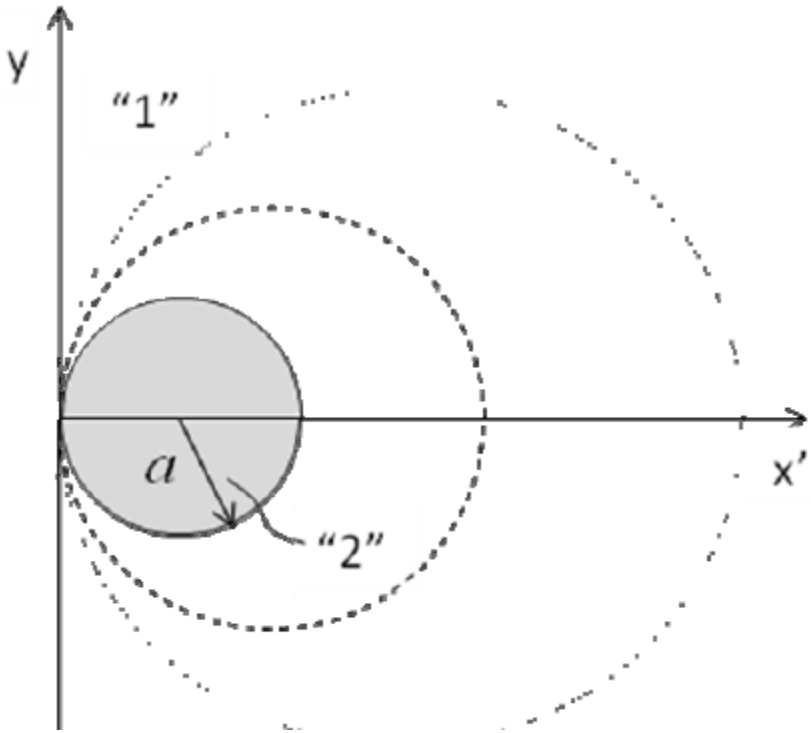
$$a(t-\tau_2) = |r - R(\tau_2)$$



Static term coincides with Gavazza (1977), Eshelby (1977)

Dundurs & XM, *IJSS*, 2009

Plane boundary as a limit of a circular/spherical inclusion/inhomogeneity



$$x' \to x - a$$

$$y \to y$$

$$r^2 \to (x-a)^2 + y^2$$

$$\sigma_{xx}^{(1)} = -2\mathrm{K}e\left\{-\frac{a^2}{(x-a)^2+y^2} + \frac{2a^2(x-a)^2}{\left[(x-a)^2+y^2\right]^2}\right\}$$

$$\sigma_{xy}^{(1)} = -2\mathrm{K}e\frac{2xya^2}{\left[(x-a)^2+y^2\right]^2}$$

$$\sigma_{yy}^{(1)} = -2\mathrm{K}e\left\{\frac{a^2}{(x-a)^2+y^2} - \frac{2a^2(x-a)^2}{\left[(x-a)^2+y^2\right]^2}\right\}$$

$$\sigma_{xx}^{(2)} = \sigma_{yy}^{(2)} = -2\mathrm{K}e, \quad \sigma_{xy}^{(2)} = 0$$

Limit $a \to \infty$, $\quad \dfrac{a^2}{(x-a)^2+y^2} \to 1, \quad \dfrac{a^2(x-a)^2}{\left[(x-a)^2+y^2\right]^2} \to 1$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = -2\mathrm{K}e$$

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} = 0$$

$$\sigma_{yy}^{(1)} = -\sigma_{yy}^{(2)} = 2\mathrm{K}e$$

Eshelby solution + Hill jump conditions

$$[\sigma_{ij}] \equiv \sigma_{ij}(\mathrm{out}) - \sigma_{ij}(\mathrm{in}) = C_{ijkl}\{[u_{k,l}] - [\epsilon^*_{kl}]\}$$

$$= C_{ijkl}\left\{-C_{pqmn}\epsilon^*_{mn}n_q n_l N_{kp}(\boldsymbol{n})/D(\boldsymbol{n}) + \epsilon^*_{kl}\right\}$$

Superposed tractions at infinity

Pure shear eigenstrain

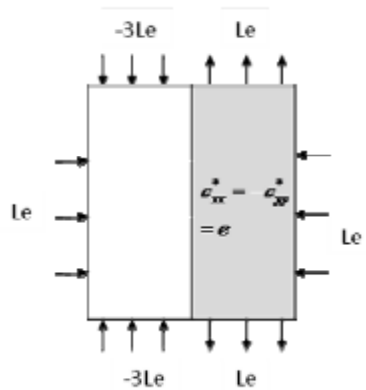
(satisfy equil and compatibility of interface

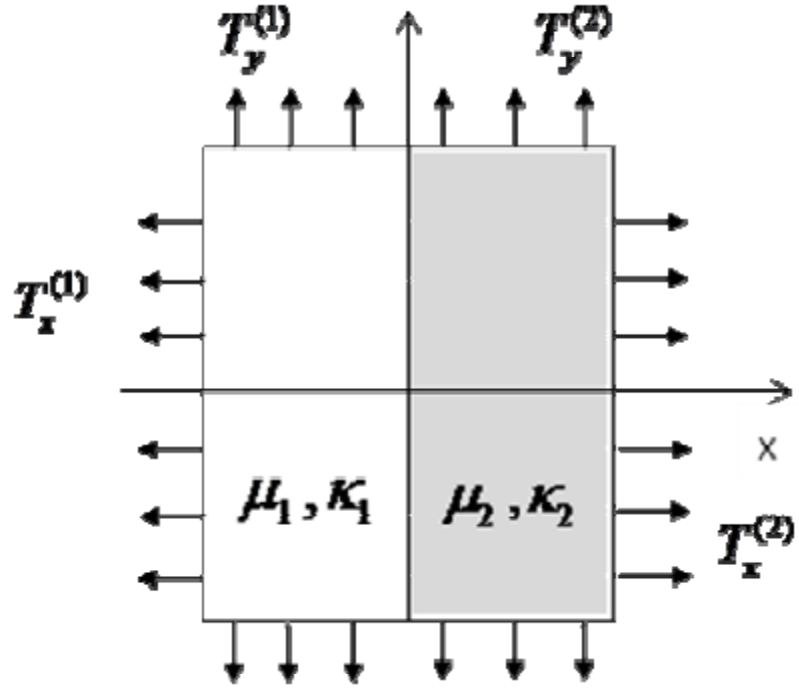
$$(1-\alpha)T_y^{(2)} = (1+\alpha)T_y^{(1)} - 2(\alpha - 2\beta)T_x$$

$$\sigma_{xx}^{(1)} = \sigma_{xx}^{(2)} = -Le$$

$$\sigma_{xy}^{(1)} = \sigma_{xy}^{(2)} = 0$$

$$\sigma_{yy}^{(1)} = -3Le,\ \sigma_{yy}^{(2)} = Le \qquad \text{with} \quad L = \frac{2\mu_2}{\Gamma\kappa_1 + 1}$$





$$W^* = \tfrac{1}{2}\int_D \sigma_{ij}^0 u_{i,j}^0 \, \mathrm{d}D - \tfrac{1}{2}\int_\Omega \sigma_{ij}\epsilon_{ij}^* \, \mathrm{d}D.$$

$$F_x = \frac{1}{2}(\sigma_{xx}^{(1)} + \sigma_{xx}^{(2)})\varepsilon_{xx}^* + \frac{1}{2}(\sigma_{yy}^{(1)} + \sigma_{yy}^{(2)})\varepsilon_{yy}^*$$

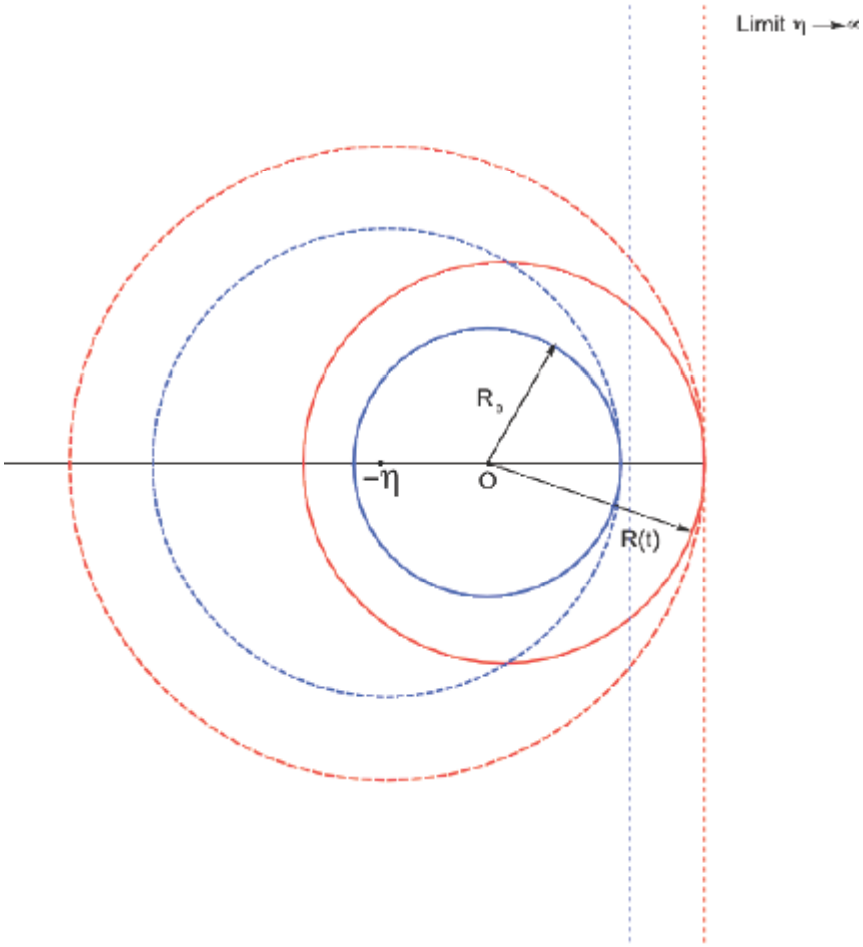
$$= \frac{1}{2}(-2Le)e + \frac{1}{2}(-3Le + Le)(-e) = -Le^2 + Le^2 = 0$$

Mura

Increase the energy

Rogue states!

Limiting procedure to obtain plane moving phase boundary



$$Q_s(R_0, R(t), r; t) \Longrightarrow Q_s(R_0 + \eta, R(t) + \eta, r + \eta; t).$$

$$Q_p(R_0, R(t), x_1; t) = \lim_{\eta \to \infty} Q_s(R_0 + \eta, R(t) + \eta, x_1 + \eta; t).$$

Limiting fields: plane phase boundary

$$a(t-\tau_2) = |x_1 - R_0 - l(\tau_2)|.$$

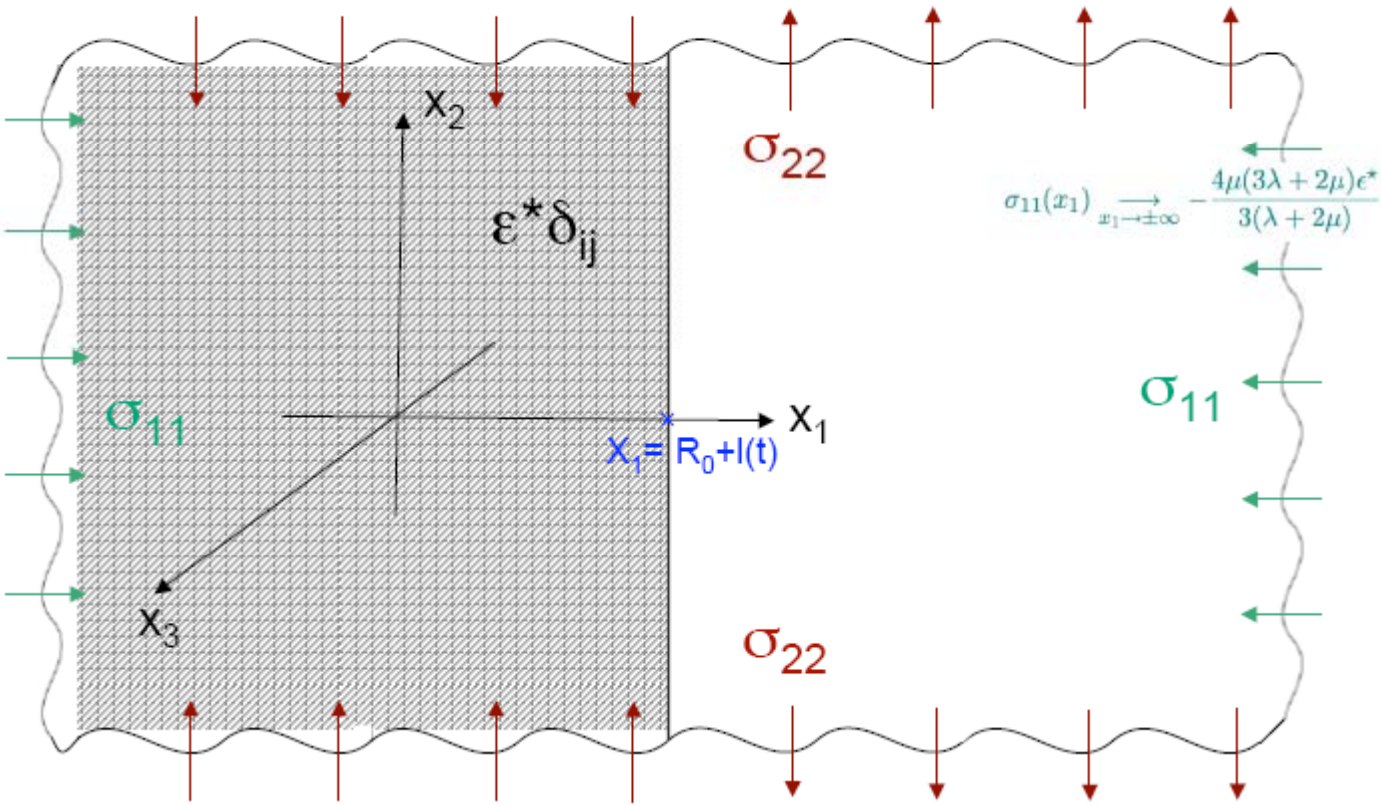
$$(\sigma_{11})_p = -\frac{4\mu(3\lambda+2\mu)\epsilon^*}{3(\lambda+2\mu)}$$
$$-\frac{(3\lambda+2\mu)\epsilon^*}{2}\left(\frac{\dot{l}(\tau_2)}{a+\dot{l}(\tau_2)}\right)H(at-|x_1-R_0|))H(R_0+l(t)-x_1)$$
$$-\frac{(3\lambda+2\mu)\epsilon^*}{2}\left(\frac{\dot{l}(\tau_2)}{a-\dot{l}(\tau_2)}\right)H(at-|x_1-R_0|)H(x_1-(R_0+l(t))),$$

$$(\sigma_{22})_p = (\sigma_{33})_p = -\frac{4\mu(3\lambda+2\mu)\epsilon^*}{3(\lambda+2\mu)}H(R_0+l(t)-x_1)$$
$$+\frac{2\mu(3\lambda+2\mu)\epsilon^*}{3(\lambda+2\mu)}H(x_1-(R_0+l(t)))$$
$$-\frac{\lambda(3\lambda+2\mu)\epsilon^*}{2(\lambda+2\mu)}\left(\frac{\dot{l}(\tau_2)}{a+\dot{l}(\tau_2)}\right)H(at-|x_1-R_0|)H(R_0+l(t)-x_1)$$
$$-\frac{\lambda(3\lambda+2\mu)\epsilon^*}{2(\lambda+2\mu)}\left(\frac{\dot{l}(\tau_2)}{a-\dot{l}(\tau_2)}\right)H(at-|x_1-R_0|)H(x_1-(R_0+l(t))).$$

Jumps across moving phase bndry

Static: Eshelby inside+ Hill jump conditions

Moving plane phase boundary with dilatational eigenstrain



$$(\sigma_{22})_P = (\sigma_{33})_P = -H(R_0 + l(t) - x_1)\left[\frac{4\mu(3\lambda+2\mu)\epsilon^\star}{3(\lambda+2\mu)} + \frac{\lambda(3\lambda+2\mu)\epsilon^\star}{2(\lambda+2\mu)}\frac{\dot{l}(\tau_2)}{a+\dot{l}(\tau_2)}H(at - |x_1 - R_0|)\right]$$

$$+H(x_1 - (R_0 + l(t)))\left[\frac{2\mu(3\lambda+2\mu)\epsilon^\star}{3(\lambda+2\mu)} - \frac{\lambda(3\lambda+2\mu)\epsilon^\star}{2(\lambda+2\mu)}\frac{\dot{l}(\tau_2)}{a-\dot{l}(\tau_2)}H(at - |x_1 - R_0|)\right]$$

SELF- FORCE ON MOVING PLANE BOUNDARY

(limit from the sphere, with dialatational eigenstrain)

$$\dot{\mathcal{E}} = \lim_{S_d \to 0} \int_{S_d} [n_j \sigma_{ij} \dot{u}_i + v_n (W + T)] dS.$$

$$f = \mathcal{E}/\dot{l}(t) = [[W]] - < \sigma_{ij} > [[\frac{\partial u_i}{\partial x_j}]]$$

$$f = - < \sigma_{km} > [[\epsilon^*_{km}]],$$

$$f_p = (< \sigma^0_{11} + \sigma^0_{22} + \sigma^0_{33} >)_p \epsilon^* - \frac{3}{2}(3\lambda + 2\mu)\epsilon^{*2}$$

$$= \lim_{\eta \to \infty} (< \sigma^0_{11} + \sigma^0_{22} + \sigma^0_{33} >)_S|_{(R_0+\eta, R(t)+\eta)} \epsilon^* - \frac{3}{2}(3\lambda + 2\mu)\epsilon^{*2}$$

$$= \lim_{\eta \to \infty} f_s(R(t) + \eta)$$

$$= -\frac{2\mu(3\lambda + 2\mu)\epsilon^{*2}}{(\lambda + 2\mu)} - \frac{(3\lambda + 2\mu)^2 \epsilon^{*2}}{2(\lambda + 2\mu)} [\frac{a\dot{l}(t)}{a^2 - \dot{l}^2(t)}].$$

Self-force

inertia

Static, Eshelby, Gavazza,1977

Markenscoff and Ni, *J.M.P.S.*, 2010

Radiated fields of a plane boundary of a half-space inclusion with general eigenstrain

$$u_\ell(\mathbf{x},t) = \int_{-\infty}^{+\infty} dt' \int_D C_{jk\ell m} \varepsilon^*_{\ell m}(\mathbf{x},t) \frac{\partial}{\partial x_k} G_{ij}(\mathbf{x}-\mathbf{x}', t-t') dV'$$

Willis, 1965

$$\varepsilon^*_{\ell m}(\mathbf{x},t) = \varepsilon^*_{\ell m} H(R_o + \ell(t) - x_1)$$

Initial condition:

Three-dimensional fields of the Eshelby limiting half-space inclusion
(Unique minimum energy solution)
Superposed dynamic 1-D problem (unique)

$$\sigma_{12} := \sigma^0_{12} - \left[\mu\varepsilon^*_{12} \frac{\ell(\tau_2)}{\dot{\ell}(\tau_2) + c_2} H(R(t) - x_1) - \mu\varepsilon^*_{12} \frac{\dot{\ell}(\tau_2)}{\dot{\ell}(\tau_2) - c_2} H(x_1 - R(t)) \right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right)$$

$$c_2(t-\tau) - |x_1 - R_0 - \ell(\tau)| = 0$$

: X.M. & L.Ni,*Q.A.M.*,2010

Propagation of the rotation for a plane boundary with shear eigenstrain

$$\omega_{lm} = \frac{1}{2}\left(\frac{\partial u_l}{\partial x_m} - \frac{\partial u_m}{\partial x_l}\right) = \frac{1}{4\pi}\left(\epsilon^*_{mk}\phi_{,kl} - \epsilon^*_{lk}\phi_{,km}\right)$$

Rotation of static inclusion
Eshelby, 1961

$$\phi = \begin{cases} 2\pi R_0^2 - \frac{2\pi}{3}r^2, & r < R_0 \\ \frac{4}{3}\pi R_0^3/r. & r < R_0 \end{cases}$$

Half-space inclusion

$$\omega^0_{12} = \varepsilon^*_{21} H(x_1 - R_0).$$

$$\omega_{12} = \varepsilon^*_{21} H(x_1 - R(t)) + \frac{\varepsilon^*_{21}}{2}\left[\frac{\ell(\tau_2)}{\dot{\ell}(\tau_2) + c_2} H(R(t) - x_1) + \frac{\dot{\ell}(\tau_2)}{c_2 - \dot{\ell}(\tau_2)} H(x_1 - R(t))\right] H\left(t - \frac{|x_1 - R_0|}{c_2}\right)$$

$$[[\omega_{12}]] = \varepsilon^*_{12}\frac{c_2^2 + \dot{\ell}^2(t)}{c_2^2 - \dot{\ell}^2(t)}$$

Driving force on a plane boundary of a half-space inclusion with *general* eigenstrain

Self-force

$$f = f_o - \frac{1}{2}\frac{\left[(\lambda+2\mu)\varepsilon_{11}^{*} + \lambda(\varepsilon_{22}^{*} + \varepsilon_{33}^{*})\right]^2}{(\lambda+2\mu)}\frac{c_1\dot{\ell}(t)}{c_1^2 - \dot{\ell}^2(t)} - \frac{2\mu c_2\dot{\ell}(t)}{c_2^2 - \dot{\ell}^2(t)}\left[(\varepsilon_{12}^{*})^2 + (\varepsilon_{13}^{*})^2\right]$$

$$\begin{aligned}
f_0 &= \varepsilon_{11}^{\star}\sigma_{11}^{0} + \frac{1}{2}\varepsilon_{22}^{\star}(\sigma_{22}^{0\ (\mathrm{in})} + \sigma_{22}^{0\ (\mathrm{ex})}) + \frac{1}{2}\varepsilon_{33}^{\star}(\sigma_{33}^{0\ (\mathrm{in})} + \sigma_{33}^{0\ (\mathrm{ex})}) \\
&\quad + 2\sigma_{12}^{0}\varepsilon_{12}^{\star} + 2\varepsilon_{13}^{\star}\sigma_{13}^{0} + \varepsilon_{23}^{\star}(\sigma_{23}^{0\ (\mathrm{in})} + \sigma_{23}^{0\ (\mathrm{ex})}) \\
&= -\frac{32\mu(\lambda+\mu)}{15(\lambda+2\mu)}(\varepsilon_{11}^{\star})^2 - \frac{2\mu(\lambda+\mu)}{15(\lambda+2\mu)}\left((\varepsilon_{22}^{\star})^2 + (\varepsilon_{33}^{\star})^2\right) - \frac{4\mu(7\lambda+2\mu)}{15(\lambda+2\mu)}\varepsilon_{11}^{\star}(\varepsilon_{22}^{\star} + \varepsilon_{33}^{\star}) \\
&\quad + \frac{2\mu(\lambda-4\mu)}{15(\lambda+2\mu)}\varepsilon_{22}^{\star}\varepsilon_{33}^{\star} - \frac{4\mu(9\lambda+14\mu)}{15(\lambda+2\mu)}\left((\varepsilon_{12}^{\star})^2 + (\varepsilon_{13}^{\star})^2\right) - \frac{2\mu(3\lambda-2\mu)}{15(\lambda+2\mu)}(\varepsilon_{23}^{\star})^2 \qquad (75)
\end{aligned}$$

EQUATION OF MOTION OF AN INCLUSION BOUNDARY ("kinetic relation")

Superpose *external loading*---all interaction terms in driving force

Noether's theorem (N&S, Also in non-linear elasticity)

total $\mathcal{J} = 0$ (*dynamic*) results in $< \sigma_{k\ell}^{appl} > [[\varepsilon_{k\ell}^{*}(\mathbf{x},t)]]$ (Peach-Koehler force)

A plane boundary with dilatational eigenstrain and superposed applied tension

$$-\frac{2\mu(3\lambda+2\mu)\varepsilon^{*2}}{(\lambda+2\mu)} - \frac{(3\lambda+2\mu)^2\varepsilon^{*2}}{2(\lambda+2\mu)}\left[\frac{a\dot{\ell}(t)}{a^2-\dot{\ell}^2(t)}\right] + \sigma_{11}^{a}\varepsilon^{*} = \begin{cases} 0 \\ F(\dot{l}(t)) \quad \text{(dissipation)} \end{cases}$$

Self-force *Peach-Koehler*

$$\frac{\partial\sigma_{11}^{a}}{\partial\dot{l}} = \frac{(3\lambda+2\mu)^2\epsilon^{*}}{2(\lambda+2\mu)}\left[\frac{a(a^2+\dot{l}^2(t))}{(a^2-\dot{l}^2(t))^2}\right].$$

XM, *Int. J. Fr, 2010*

Equilibrium position of phase boundary

"Eshelby principle"

$$F = [W] - \mathbf{T} \cdot \left[\frac{\partial \mathbf{u}}{\partial n}\right] \tag{47}$$

per unit area of interface, where $\mathbf{T}$ is the surface traction at the interface.

Equation (45) can be used to find the equilibrium position of phase and twin boundaries in the presence of stresses produced by the transformation itself, or applied externally, or both. Since Eq. (45) must be zero for any small $\delta\xi_l$ the boundary must take up a shape for which Eq. (47) is zero all along it. In the case of a stress-free cavity Eq.

Eshelby, 1970

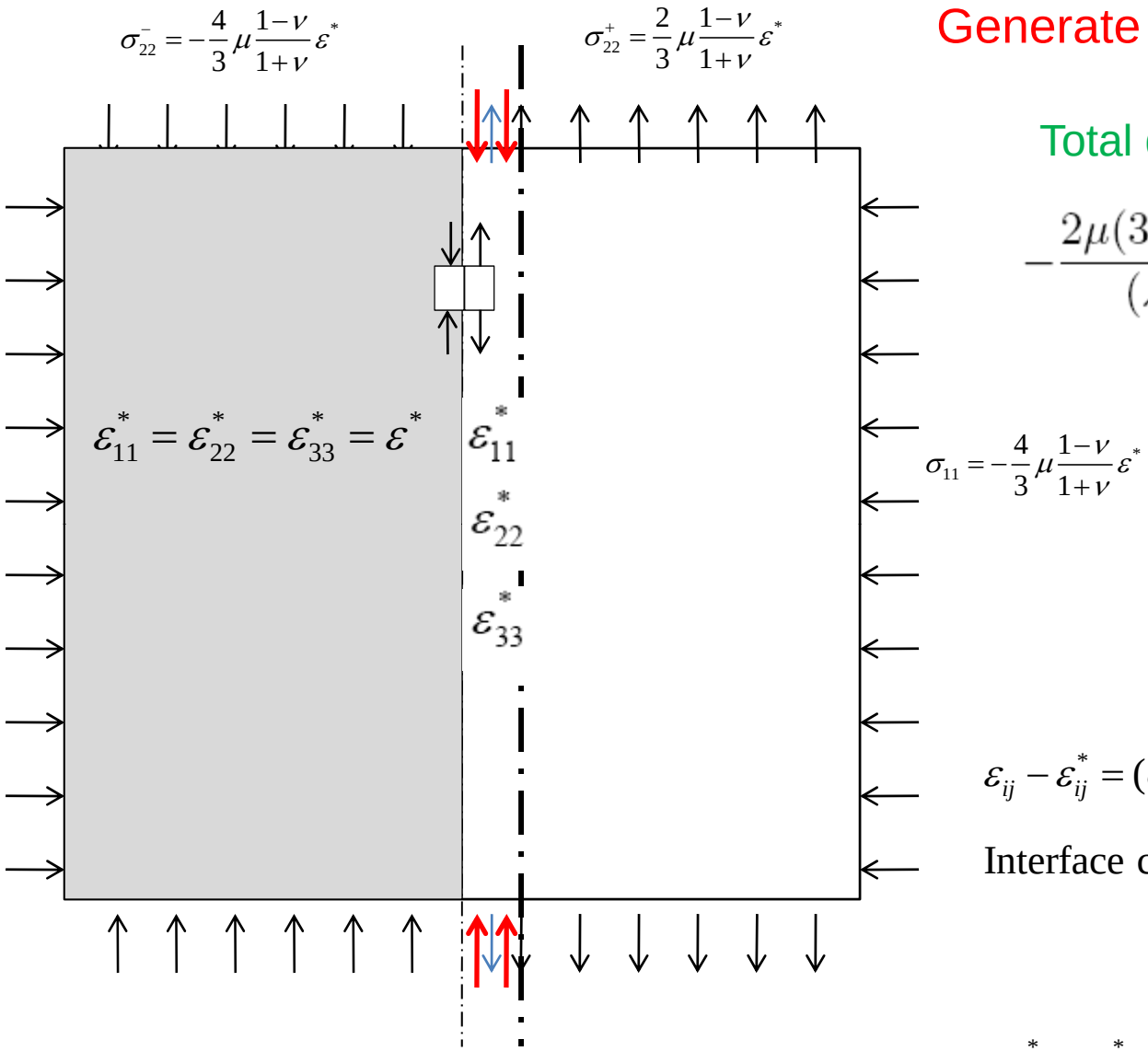
In the physical cases described, and in others, the boundary may be capable of migrating through the material (growth of martensitic plates, change in the form of a cavity by volume or surface diffusion, and so on). In Fig. 3 migration has made S develop a shallow blister, changing it to S'. The migration may be specified by erecting a small vector $\delta\xi_i$ at each point of S. It is then a sensible question to ask what is the change in the total energy of the system as a result of the migration. The answer is that

This expression only gives the change in the elastic energy and the energy of the loading mechanism. If it is to be applied to phase changes we must also include the "chemical" energy, the work required to transform the material which disappeared from A into the material which appeared in B, say

$$\int \delta\xi_l (W_0^B - W_0^A)\, dS_l \tag{44}$$

where $W_0^B - W_0^A$ is the work required to transform a mass of unstressed A into an equal mass of unstressed B, the mass occupying

$W^A(0) - W^B(0) \doteq W_o^A - W_o^B$. The addition of a constant to W does not, of course, alter Eq. (18), because the extra term is proportional to $\int dS_l$ and so is zero for a closed surface being, by Gauss' theorem, the volume integral of the gradient of unity. It appears that the value



Generate new region of eigenstrain

Total driving force=0

$$-\frac{2\mu(3\lambda+2\mu)\epsilon^{*2}}{(\lambda+2\mu)} + \sigma_{11}^{a}\epsilon^{*} = 0,$$

$$\sigma_{11}^{a} = -\frac{2\mu(3\lambda+2\mu)\epsilon^{*}}{(\lambda+2\mu)}$$

determines $\varepsilon_{11}^{*} = \varepsilon^{*}$

$$\varepsilon_{ij} - \varepsilon_{ij}^{*} = (\sigma_{ij} - \delta_{ij}\sigma_{kk}\frac{\nu}{1+\nu})/2\mu$$

Interface condition : $\sigma_{11}^{-} = \sigma_{11}^{+}$

$$\varepsilon_{22}^{-} = \varepsilon_{22}^{+}$$

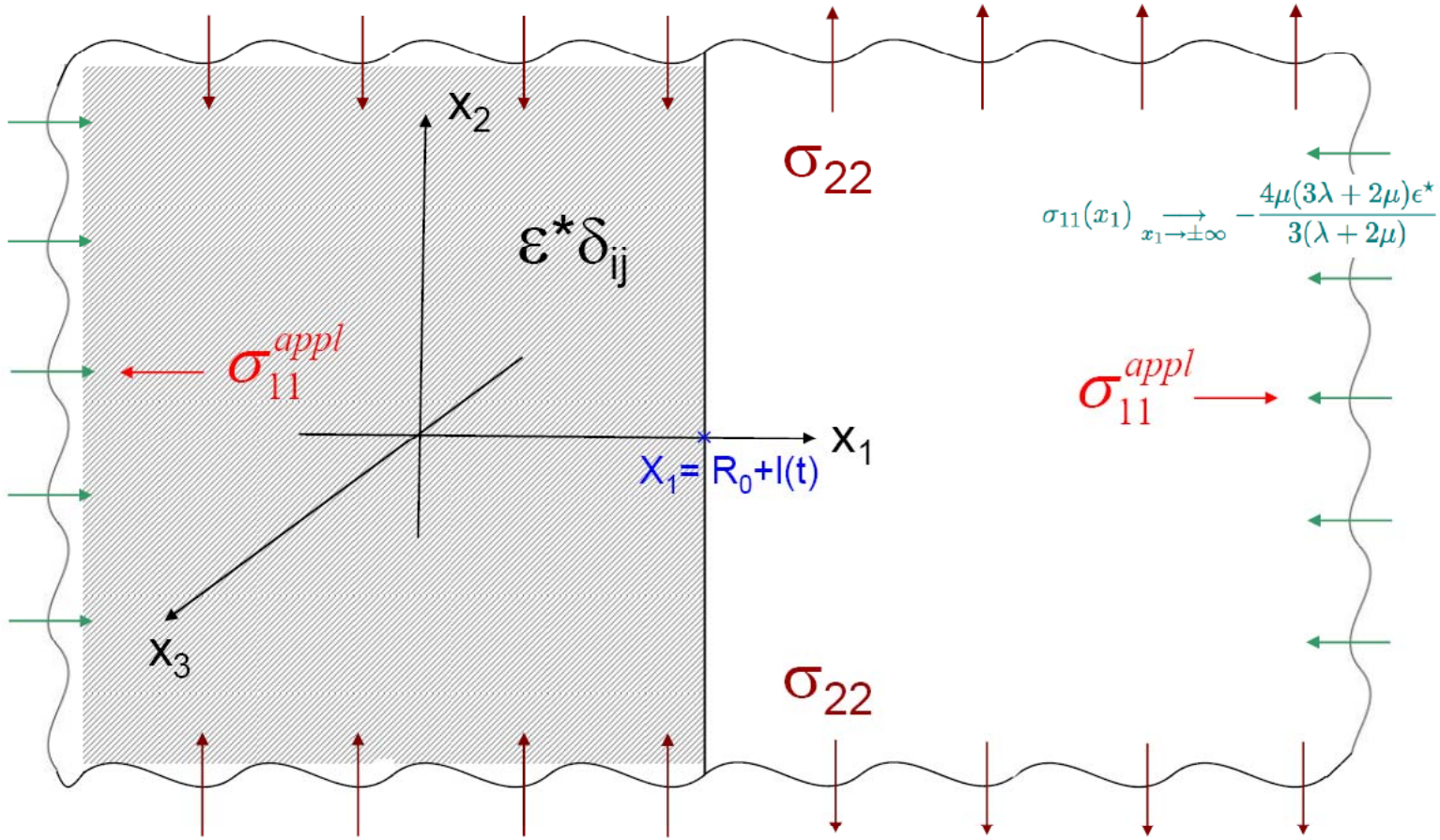
$$\varepsilon_{33}^{-} = \varepsilon_{33}^{+}$$

$\varepsilon_{33}^{*} = \varepsilon_{22}^{*} = \varepsilon^{*}$, ε_{11}^{*} undertermined

Limit of Spherical inclusion

Eshelby inside+ Hill jump conditions outside

XM, Int. J. Fr, 2010



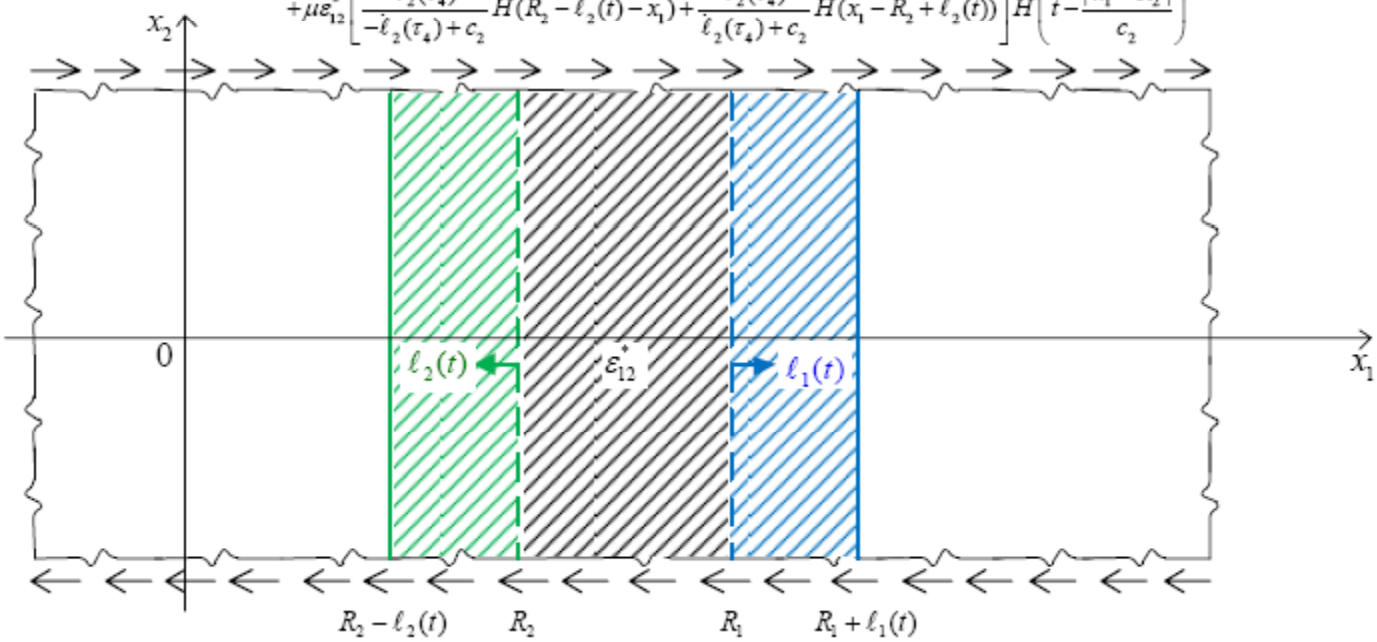
$$(\sigma_{22})_P = (\sigma_{33})_P = -H(R_0 + l(t) - x_1)\left[\frac{4\mu(3\lambda+2\mu)\epsilon^\star}{3(\lambda+2\mu)} + \frac{\lambda(3\lambda+2\mu)\epsilon^\star}{2(\lambda+2\mu)}\frac{\dot{l}(\tau_2)}{a+\dot{l}(\tau_2)}H(at - |x_1 - R_0|)\right]$$

$$+H(x_1 - (R_0 + l(t)))\left[\frac{2\mu(3\lambda+2\mu)\epsilon^\star}{3(\lambda+2\mu)} - \frac{\lambda(3\lambda+2\mu)\epsilon^\star}{2(\lambda+2\mu)}\frac{\dot{l}(\tau_2)}{a-\dot{l}(\tau_2)}H(at - |x_1 - R_0|)\right]$$

XM & L.Ni, QAM, 2010

Expanding/shrinking strip with shear eigenstrain

$$\sigma_{21}=\sigma_{12}=-\mu\varepsilon_{12}^{*}\left[\frac{\dot{\ell}_1(\tau_3)}{\dot{\ell}_1(\tau_3)+c_2}H(R_1+\ell_1(t)-x_1)-\frac{\dot{\ell}_1(\tau_3)}{\dot{\ell}_1(\tau_3)-c_2}H(x_1-R_1-\ell_1(t))\right]H\left(t-\frac{|x_1-R_1|}{c_2}\right)$$
$$+\mu\varepsilon_{12}^{*}\left[\frac{\dot{\ell}_2(\tau_4)}{-\dot{\ell}_2(\tau_4)+c_2}H(R_2-\ell_2(t)-x_1)+\frac{\dot{\ell}_2(\tau_4)}{\dot{\ell}_2(\tau_4)+c_2}H(x_1-R_2+\ell_2(t))\right]H\left(t-\frac{|x_1-R_2|}{c_2}\right)$$



$$f=-<\sigma_{12}>\left[\left[\varepsilon_{12}^{*}(\mathbf{x},t)\right]\right]=-\frac{2\mu c_2\dot{\ell}_1(t)\varepsilon_{12}^{*\,2}}{c_2^2-\dot{\ell}_1^2(t)}$$

$$f=-\frac{2\mu c_2\dot{\ell}_1(t)\varepsilon_{12}^{*\,2}}{c_2^2-\dot{\ell}_1^2(t)}+\frac{\mu\dot{\ell}_2(0)\varepsilon_{12}^{*\,2}}{\dot{\ell}_2(0)+c_2}$$

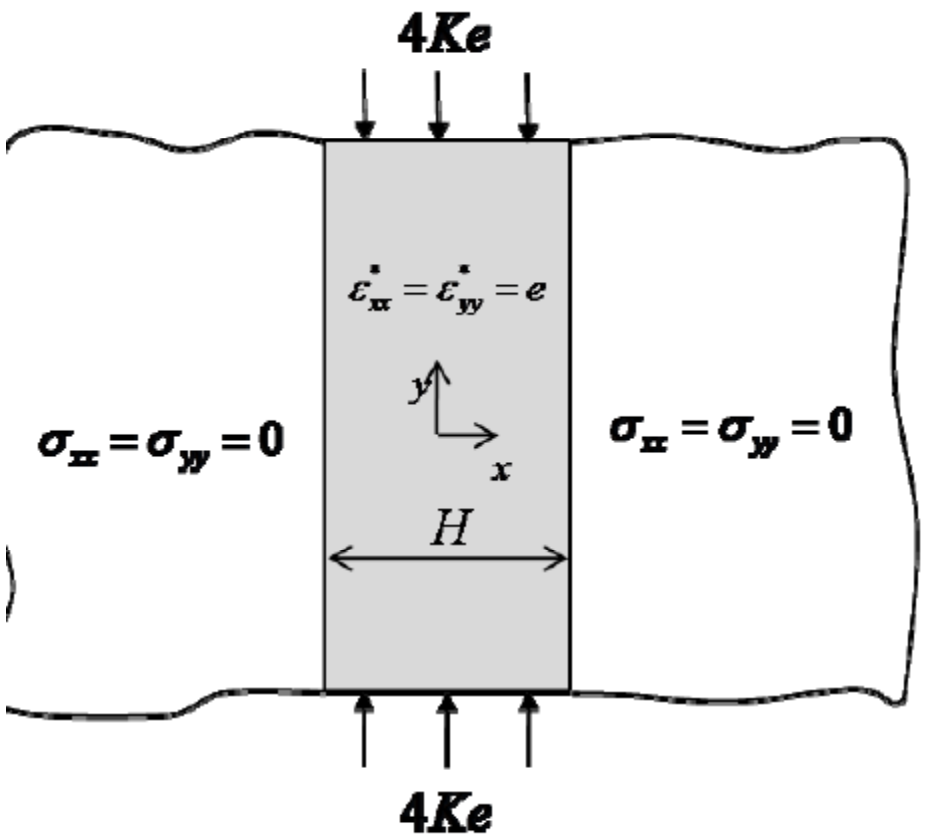
$$|R_1-R_2+\ell_1(t)+\ell_2(\tau^*)|=c_2(t-\tau^*)$$

$$f=-\frac{2\mu c_2\dot{\ell}_1(t)\varepsilon_{12}^{*\,2}}{c_2^2-\dot{\ell}_1^2(t)}+\frac{2\mu\dot{\ell}_2(\tau^*)\varepsilon_{12}^{*\,2}}{\dot{\ell}_2(\tau^*)+c_2}$$

$$<\sigma_{k\ell}^{appl}>[[\varepsilon_{k\ell}^{*}(\mathbf{x},t)]]$$

Strips with eigenstrain meeting a free surface

Cancel tractions by Boussisnesq/Cerruti



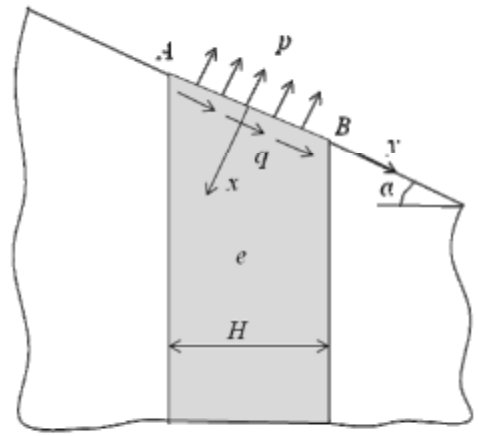
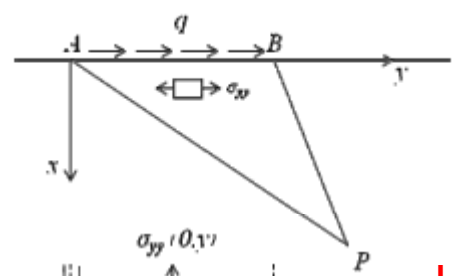
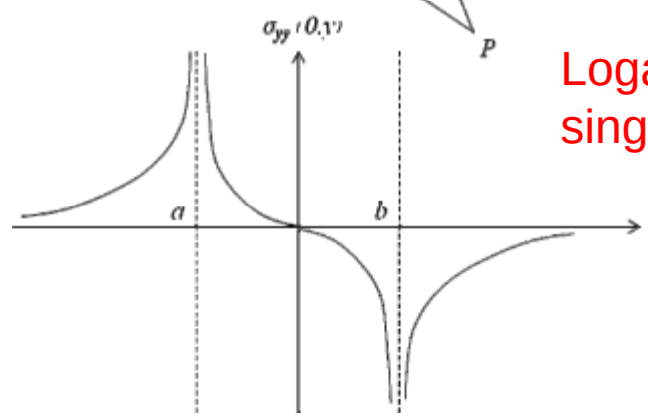


Figure 6. Strip meeting a free surface





Logarithmic singularity

Equilibrium position of inhomogeneity strip

Equivalent eigenstrain, Eshelby

material 2 by material 1 with initial eigenstrain "e"

$$\varepsilon_{equivalent} = \frac{\mu_2\kappa_1 - \mu_2 - \mu_1\kappa_2 + \mu_1}{2\mu_2 + \mu_1\kappa_2 - \mu_1} e$$

(Kun Zhou)

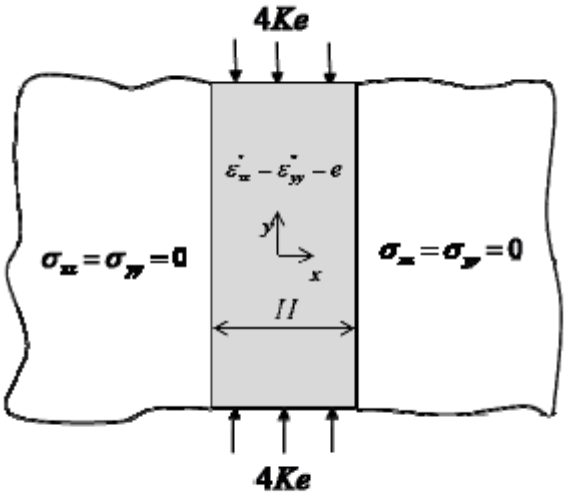


Figure 5(a) Strip with volumetric eigenstrain in an infinite solid

Tractions at infinity with equivalent eigenstrain

Equilibrium position with equivalent eigenstrain in Eshelby force.

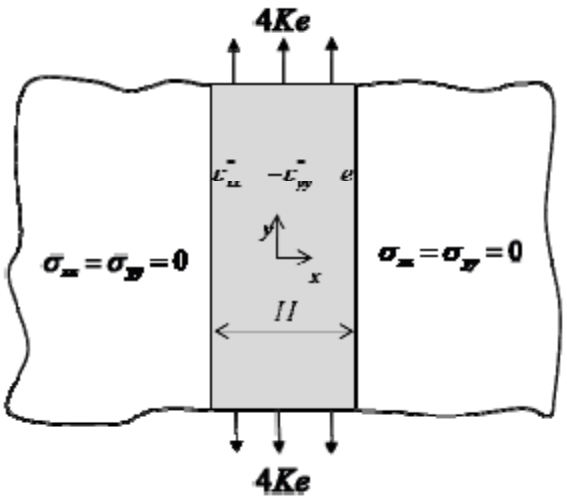


Figure 5(b) Strip with pure shear eigenstrain in an infinite solid

COMPUTE NUMERICALLY ANY FINITE # of STRIPS