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The self-force and effective mass of a generally accelerating dislocation I: Screw dislocation

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The authors would like to dedicate this paper to Professor John Dundurs on the occasion of his 85th birthday (Riga, Latvia; September 13, 1922).

Abstract

The self-force and effective mass of a moving dislocation in a generally accelerating motion are explicitly obtained on the basis of a surface-independent dynamic *J*-integral. Logarithmic singularities due to non-zero acceleration result in divergent integrals in the dynamic *J*-integral, which are treated by smearing out the dislocation core (ramp-core) and by regularizing in the sense of distributions, both coinciding in the leading terms.

Keywords: Dislocations; Dynamics; Asymptotic analysis; Conservation integrals; Self-force

1. Introduction

Macroscopic dynamic plastic deformation is a consequence of the motion of dislocations at the microscopic level, and to understand the relation of the physical processes at different scales, macro and micro, the dislocation dynamics with inertial effects needs to be investigated. The determination of the self-force and the effective mass of a generally accelerating dislocation is a fundamental problem in dynamic plasticity. When a defect is moving with respect to the material, the associated "force" on the defect is a configurational force (when the stress field is solely created by the existence of the defect itself, the configurational force on the defect is called self-force), which, in the static case, is Eshelby's configurational force on an elastic singularity given by a path-independent integral (Eshelby, 1951), later called *J*-integral (Rice, 1968). The path independence of the *J*-integral is due to the conservation laws derived from Noether's theory (Noether, 1918) for elastostatics (Günther, 1962). Eshelby (1951, 1970, 1975) showed that the path-independent integral is equal to minus the change of the total energy of the system under an infinitesimal translation of the singularity, and defined it as *the force on the singularity*. More recently, Eshelby's theory has been developed

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into an active branch of engineering science as the configurational mechanics of materials (see Maugin, 1993; Kienzler and Herrmann, 2000; Gurtin, 2000; Kienzler and Maugin, 2001).

In the dynamic case, from the dynamic energy momentum tensor and the conservation laws in elastodynamics, Eshelby (1970) derived a surface-independent expression for the force on a moving crack tip. Rogula (1977), Maugin and Trimarco (1992), Dascalu and Maugin (1993), and Maugin (1993, 1994) derived the configurational force on a moving inhomogeneity and moving defect in the linear and nonlinear elasticity framework. In this paper, starting from the same surface-independent dynamic J-integral as in the above references for the force on a moving inhomogeneity, we give a brief summary of the force on moving defects, and focus on those aspects relevant to our calculation. On the basis of a limiting process using Friedrichs' mollifier (Friedrichs, 1953; Schechter, 1977; Yosida, 1980), the dynamic J-integral can be extended to the cases where the Lagrangian and the fields are discontinuous and non-singular. As usually in mathematical physics, we construct a diminishingly infinitesimal neighborhood around the defect and define the force on a moving defect by the limit of the forces on the diminishingly infinitesimal neighborhood (considered as an inhomogeneity), as it shrinks upon the defect (a point, a line, or a surface singularity).

In the calculation of the self-force and effective mass for a generally accelerating dislocation, an essential step is to have obtained the near-field solutions of an accelerating dislocation. To this effect we prove important theorems of asymptotic analysis (Section 3) and obtain the complete evaluation of all the near fields of a generally accelerating screw dislocation.

Eshelby (1970) made the first passage (later Freund, 1990; Maugin, 1993, 1994) from the Lagrangian to the Hamiltonian, and noted that while this is possible for a moving crack, in the case of an accelerating dislocation there is divergence due to the logarithmic singularity associated with the acceleration, a fact known to Eshelby (1953), and he concluded that "an atomic treatment is really necessary" (Eshelby, 1970, p. 109) in order to treat this singular defect. In Section 3, for a generally accelerating (screw) dislocation by using the explicit near-field solutions we show that the energy release rate (as defined by Atkinson and Eshelby, 1968; Freund, 1972) is logarithmically divergent, and hence not contour-independent. In contrast, the energy release rate is proved to be contour-independent as the contour shrinks to zero, for a moving crack in general motion (Eshelby, 1970; Bui, 1977; Atluri and Nishioka, 1983; Atluri, 1986; Freund, 1990; Maugin, 1993) and for dislocations moving with a constant velocity jumping from rest (Clifton and Markenscoff, 1981).

The presence of the logarithmic singularity associated with acceleration of the moving dislocation (Eshelby, 1953; Callias and Markenscoff, 1988; Ni and Markenscoff, 2003) is the counterpart of the logarithmic singularity due to the curvature of a dislocation loop (Gavazza and Barnett, 1976), the former resulting from a second order derivative with respect to time and the latter with respect to a space variable. In the case of an arbitrarily expanding dislocation loop, Markenscoff and Ni (1990) showed that the logarithmic singularity depends on the current value of the acceleration of the tangent to the loop, the rotation of the tangent, and the current radius of the osculating circle to the loop. In steady-state motion of dislocations, there is no self-force (Clifton and Markenscoff, 1981), and the energy release rate is zero, since as much the energy flux as is emitted from the core of dislocation is received from the previously radiated energy during the motion of the dislocation from time $t \to -\infty$. Those two energy fluxes are not balanced in accelerating motion, which Eshelby described as "the dislocation is haunted by its past" (Eshelby, 1951, p. 111). For a constant acceleration motion of a screw dislocation, Eshelby (1953) gave an expression for the self-force and the effective mass of a moving Peierls dislocation containing a logarithmic singularity associated with the acceleration. Here the objective is to obtain the expression for the self-force and the effective mass for a generally accelerating motion.

In the surface-independent dynamic J-integral, the volume integral is in general an improper integral. For moving cracks, the volume integral converges in the sense of a Cauchy principal value (CPV), which has been used in fracture mechanics and in calculating the energy of elastic defects (Dascalu and Maugin, 1994). For an accelerating dislocation, the improper volume integral in the dynamic J-integral does not converge in the sense of CPV, and diverges to order $\ln \varepsilon$. Two ways are proposed and developed here to treat this divergence:

(1) Smearing the singularity (ramp-core) as in Eshelby (1977), by using a delta sequence to approximate the delta function in the Volterra model. Various smearing techniques for the dislocation core are classical in

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the literature, e.g., Peierls–Nabarro model (see e.g., Hirth and Lothe, 1968), and Eshelby (1951), Al'shitz et al. (1971), Eshelby (1977), Weertman and Weertman (1980), and, recently, Markenscoff and Ni (2001a,b), and Lubarda and Markenscoff (2006, 2007).
(2) Regularizing the involved divergent integrals based on the theory of distributions (other than CPV), e.g., Gel'fand and Shilov (1964), and Kanwal (1998).

The self-force is evaluated by both methods which yield the same result to the leading order. It is obtained as a function of the current value of the acceleration, velocity, history of the motion, and a constant cut-off radius of the core. The cut-off core radius that is undetermined here will have to be determined by matching this self-force to the one from a lattice scale model (e.g., Kresse and Truskinovsky, 2003).

Several versions of the effective mass of a moving dislocation exist in the literature, which are based on solutions of uniform motion (Sakamoto, 1991; Hirth et al., 1998). Eshelby (1953) derived the logarithmic singularity of the self-stress associated with the acceleration, and defined the effective mass as the coefficient of the acceleration in this self-force, which exhibits a negative $\frac{3}{2}$ exponent of $(1 - v^2/c_2^2)$ in the leading term. In this paper, we define the effective mass m_e analogously to Newton's law: the inertial part of the self-force to be the time derivative of the linear momentum $m_e v$. By inertial part of the self-force we mean the part of the self-force depending on the acceleration. This definition results in a negative $\frac{1}{2}$ power of $(1 - v^2/c_2^2)$ in the leading term of the effective mass. For a screw dislocation moving in a generally accelerating motion, the self-force and the effective mass are explicitly evaluated based on a smearing (ramp-core) method in Section 4, and based on the theory of distributions in Section 5. Because of the length of the presentation, the evaluation of the self-force and effective mass for a generally accelerating edge dislocation will be presented in part II (in preparation).

2. The configurational force on a moving defect

2.1. The configurational force on a moving inhomogeneity

Eshelby (1970) derived the path-independent integral expression for the force on a moving crack tip

$$F_1 = \int_V \frac{\partial P_{1\alpha}}{\partial x_\alpha}\, \mathrm{d}V = \int_S P_{1j}\, \mathrm{d}S_j + \frac{\mathrm{d}}{\mathrm{d}t}\int_V P_{10}\, \mathrm{d}V, \tag{2.1}$$

where repeated indices are summed for $j = 1, 2, 3$ and $\alpha = 0, 1, 2, 3$, x_j for $j = 1, 2, 3$, are spatial independent variables, $x_0 = t$ is the time variable, $P_{\alpha\beta}$ are 4×4 dynamic energy momentum tensor for $\alpha, \beta = 0, 1, 2, 3$, with

$$P_{l0} = \rho \dot{u}_i u_{i,l} \tag{2.2}$$

is the pseudo-momentum vector, and

$$P_{lj} = (W - T)\delta_{lj} - u_{i,l}\sigma_{ij} \tag{2.3}$$

is the spatial components of the energy momentum tensor for $l, j = 1, 2, 3$, where ρ is the mass density, u_i are the displacement components, $\dot{u}_i$ and $u_{i,l}$ are the time and space derivatives, respectively, σ_{ij} are the stress components, and $W = \frac{1}{2}\sigma_{jk}\, u_{j,k}$ and $T = \frac{1}{2}\dot{u}_i\dot{u}_i$ are the elastic potential energy density and kinetic energy density, respectively.

Rogula (1977) showed that the force on a moving inhomogeneity is given by

$$F_l = -\int_V \left(\frac{\partial L}{\partial x_l}\right)_{\exp} \mathrm{d}V, \tag{2.4}$$

where $l = 1, 2, 3$ is fixed, $L = L(x_\alpha, u_i, u_{i,\alpha})$ is the Lagrangian density, the explicit partial differentiation with respect to x_l means the partial derivative with respect to x_l provided that $\dot{u}_i$, $u_{i,j}$, and x_β, $\beta \neq l$, are fixed.

Note that the explicit partial derivative of the Lagrangian density is in fact corresponding to the translation of the coordinate in Neother's theorem (Noether, 1918; Gel'fand and Fomin, 1963; Fletcher, 1976) and related

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to the energy–momentum tensor (Eshelby, 1970), i.e.,

$$-\left(\frac{\partial L}{\partial x_l}\right)_{\exp} = \frac{\partial P_{l\beta}}{\partial x_\beta} = \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}] + \frac{\partial}{\partial x_j}[(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]. \tag{2.5}$$

Then, the force is written as

$$F_l = \int_V \frac{\partial P_{l\beta}}{\partial x_\beta}\,\mathrm{d}V. \tag{2.6}$$

From the last equation and Gauss divergence theorem, it follows the dynamic J-integral for the configurational force on a moving inhomogeneity

$$F_l = \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j, \tag{2.7}$$

where V is a volume containing the inhomogeneity, $S = \partial V$, and $\mathrm{d}S_j = n_j \mathrm{d}S$ with n_j the outer normal of S. Maugin and Trimarco (1992), Maugin (1993, 1994), and Dascalu and Maugin (1993) gave the forces on moving inhomogeneities and singularities both in differential form (force density) and integral form in the nonlinear elasticity framework.

The dynamic J-integral is surface-independent, which means that for any two volumes V_1 and V_2 such that $V_2 \supset V_1$, both containing the same elastic inhomogeneity,

$$\begin{aligned}\int_{V_2} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_{\partial V_2} [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j \\ = \int_{V_1} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_{\partial V_1} [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j.\end{aligned} \tag{2.8}$$

That is evident by using the conservation laws (Noether, 1918; Gel'fand and Fomin, 1963; Fletcher, 1976)

$$\frac{\partial P_{l\beta}}{\partial x_\beta} = \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}] + \frac{\partial}{\partial x_j}[(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}] = 0, \tag{2.9}$$

and Gauss divergence theorem on the homogeneous region $V_2 \backslash V_1$.

So far the field variables and the Lagrangian are assumed to be sufficiently smooth, and thus continuous over the boundary of the inhomogeneity. However, in general, the elastic field may not be continuous over the boundary of the inhomogeneity. When the field variables are discontinuous over the boundary but not singular, we may use a convolution with Friedrichs' mollifier (Friedrichs, 1953; Schechter, 1977; Yosida, 1980) to smooth the discontinuous field to an infinitely differentiable field. Namely, e.g., the infinitely differentiable displacement field is constructed as

$$(u_i)_a = (u_i)\star\theta_a = \int_\Omega u_i(x-y)\theta_a(y)\,\mathrm{d}y^3 = \int_\Omega u_i(y)\theta_a(x-y)\,\mathrm{d}y^3, \tag{2.10}$$

where $a > 0$, $\star$ is the symbol of convolution, for simplicity, here x, y denote three-dimensional vectors, and Friedrichs' mollifier θ_a is defined by (Friedrichs, 1953; Schechter, 1977; Yosida, 1980)

$$\begin{aligned}&\theta_a(x) = h_a^{-1}\exp((|x|^2/a^2 - 1)^{-1}) \quad \text{for } |x| = \sqrt{\Sigma x_i^2} < a, \\ &\theta_a(x) = 0 \quad \text{for } |x| = \sqrt{\Sigma x_i^2} \geqslant a,\end{aligned} \tag{2.11}$$

with h_a a normalization constant such that

$$\int_{R^3} \theta_a(x)\,\mathrm{d}x^3 = 1. \tag{2.12}$$

Note that θ_a is a three-dimensional δ-sequence (see e.g., Kanwal, 1998). For such regularized fields, the configurational force is well-defined and expressed by the dynamic J-integral. In view of the facts that all the original field variables are non-singular and the volumes involved are bounded (the unbounded volume is

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considered as a limiting case), then the Lebesque convergence theorem[1] (see e.g., EDM, 221c, p. 841, 1993) applies when taking limit as $a \to 0$. Hence, by a limiting process, it follows that, for the discontinuous case, the dynamic configurational force on the moving inhomogeneity is expressed by the dynamic J-integral (2.7) as well.

2.2. The configurational force on a moving defect

An elastic defect may be a point singularity, a line, or surface singularity. At the point or line singularity, the stress or displacement may become infinite; on the surface singularity, the stress or displacement becomes infinite, or, discontinuous (see Eshelby, 1951). As usually in mathematical physics, we construct a diminishingly infinitesimal, symmetric or asymmetric, neighborhood N_ε of the singularity, such as a small circle, a narrow cylinder, or other infinitesimal body. The region of the small neighborhood N_ε of the singularity is then considered as a non-singular inhomogeneity. On the boundary of N_ε, there exists a jump discontinuity, the dynamic configurational force on such inhomogeneity is well-defined as discussed earlier, and given by

$$F_l^\varepsilon = \int_{V\setminus N_\varepsilon} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j, \tag{2.13}$$

where $S = \partial V$.

The configurational force on the singularity is then defined as the limit of the force F_l^ε when N_ε shrinks upon the singularity, namely,

$$F_l = \lim_{\varepsilon\to 0}\left\{\left[\int_{V\setminus N_\varepsilon}\right] \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j\right\}. \tag{2.14}$$

Mathematically, the limit

$$\lim_{\varepsilon\to 0}\int_{V\setminus N_\varepsilon} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V, \tag{2.15}$$

is exactly the definition of the improper volume integral.

Therefore, the configurational force on a moving defect is given by the surface-independent dynamic J-integral

$$F_l = \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j, \tag{2.16}$$

where again the volume integral is in general an improper integral.

The improper volume integral (2.15) may exist in following senses:

(1) *As a usual improper integral*: If limit (2.15) exists for arbitrary small neighborhood N_ε of the singularity x_0 as ε approaches to zero, where ε is the maximum distance between x and x_0 for all x in the neighborhood, then the limit approaches to a convergent improper integral in the usual sense.

(2) *As an integral of the Cauchy type*: If limit (2.15) does not exist for arbitrary neighborhood N_ε, and, however, exists when

$$N_\varepsilon = B_\varepsilon \equiv \{x \| x - x_0| \leqslant \varepsilon\}, \tag{2.17}$$

where the norm $|\cdot|$ in R^n is defined by $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, for $n \geqslant 2$, then the limit approaches to an integral of the Cauchy type,

$$\lim_{\varepsilon\to 0}\int_{V\setminus B_\varepsilon} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V = \fint_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V. \tag{2.18}$$

[1]If $\lim_{n\to\infty} f_n(x)$ exists almost everywhere on E, and there exists a $\phi(x)$ such that $|f(x)| \leqslant \phi(x)$ and $\int_e \phi < \infty$ (for example, if $\mu(E) < \infty$ and $|f_n(x)| < M$) then

$$\lim_{n\to\infty}\int_e f_n = \int_e \left(\lim_{n\to\infty} f_n\right).$$

Applying the conservation laws (2.9) in the homogeneous region $V\backslash N_\varepsilon$, we have

$$\int_{V\backslash N_\varepsilon} \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j = \int_{S_\varepsilon} [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j, \tag{2.19}$$

where $S_\varepsilon = \partial N_\varepsilon$. Therefore

$$F_l = \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]\,\mathrm{d}V + \int_S [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j$$
$$= \lim_{\varepsilon\to 0} \int_{S_\varepsilon} [(W-T)\delta_{lj} - u_{i,l}\sigma_{ij}]\,\mathrm{d}S_j. \tag{2.20}$$

When N_ε is an arbitrary ε-neighborhood and limit in Eq. (2.20) exists, the volume integral converges as usual improper integral; when N_ε is a symmetric ε-ball, the volume integral converges as an integral of Cauchy type. Otherwise, when the limit in Eq. (2.20) does not exist, then the configurational force is divergent.

2.3. The effective mass of a moving defect

Several versions of the effective mass of a moving dislocation exist in the literature, which are based on solutions of uniform motion (Sakamoto, 1991; Hirth et al., 1998). Eshelby (1953) derived the logarithmic singularity of the self-stress associated with the acceleration, and defined the effective mass as the coefficient of the acceleration in this self-force, which exhibits a negative $\frac{3}{2}$ exponent of $(1 - v^2/c_2^2)$ in the leading term. In this paper, we define the effective mass m_e analogously to Newton's law: the inertial part of the self-force to be the time derivative of the linear momentum $m_\mathrm{e}v$. By inertial part of the self-force we mean the part of the self-force depending on the acceleration. This definition results in a negative $\frac{1}{2}$ power of $(1 - v^2/c_2^2)$ in the leading term of the effective mass.

Suppose that the self-force may be decomposed into the non-trivial inertial part and non-inertial part, then the effective mass m_e is defined according to

$$F^{\mathrm{in}} = \frac{\mathrm{d}}{\mathrm{d}t}(m_\mathrm{e} v), \tag{2.21}$$

and given by

$$m_\mathrm{e} \equiv \frac{1}{v}\int_0^t F^{\mathrm{in}}\,\mathrm{d}t, \tag{2.22}$$

for $t>0$, and where $v = \dot{l}(t)$, and F^{in} is the inertial part of the self-force F.

2.4. Example: steadily moving screw dislocation

As an illustration, consider a simple example of a screw dislocation parallel to the z-direction moving in a steady motion along the x-direction relative to an whole-space isotropic elastic material. It is well-known that the self-force on a dislocation moving in a steady motion is zero (e.g., Clifton and Markenscoff, 1981). According to Frank (1949), the non-zero component of the displacement field is

$$u_3 = -\frac{b}{2\pi}\tan^{-1}\left(\frac{x-vt}{\gamma y}\right), \tag{2.23}$$

where $\gamma = \sqrt{1 - v^2/c_2^2}$ with $c_2 = \sqrt{\mu/\rho}$ as the speed of the shear stress wave. And the field solutions are

$$u_{3,1} = -\frac{b}{2\pi}\frac{\gamma y}{(x-vt)^2 + \gamma^2 y^2}, \quad u_{3,2} = \frac{b}{2\pi}\frac{\gamma(x-vt)}{(x-vt)^2 + \gamma^2 y^2}, \tag{2.24}$$

$$\dot{u}_3 = \frac{b}{2\pi}\frac{\gamma y v}{(x-vt)^2 + \gamma^2 y^2}. \tag{2.25}$$

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Choose the integral volume V to be a cylinder of a radius r around the dislocation line, and with a unit height in the z-direction. At time t, the dislocation line is located at the position $x = vt, y = z = 0$. It suffices to calculate the force F_1, the discussion for F_2 is analogous. From Eq. (2.20), F_1 is given by

$$\begin{aligned} F_1 &= \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,1}]\,\mathrm{d}V + \int_S [(W-T)\delta_{1j} - u_{i,1}\sigma_{ij}]\,\mathrm{d}S_j \\ &= \lim_{\varepsilon\to 0}\int_{S_\varepsilon} [(W-T)\delta_{1j} - u_{i,1}\sigma_{ij}]\,\mathrm{d}S_j, \end{aligned} \tag{2.26}$$

where $S_\varepsilon \equiv \partial V_\varepsilon$, and V_ε is a cylinder similar to V with an infinitesimal radius ε. Then the surface integral on S_ε is written as

$$\int_{S_\varepsilon} [(W-T)\delta_{1j} - u_{3,1}\sigma_{3j}]\,\mathrm{d}S_j = \int_{S_\varepsilon} [(W-T) - u_{3,1}\sigma_{31}]\,\mathrm{d}S_1 - \int_{S_\varepsilon} u_{3,1}\sigma_{32}\,\mathrm{d}S_2, \tag{2.27}$$

where $\mathrm{d}S_1 = \varepsilon\cos\theta\,\mathrm{d}\theta$, $\mathrm{d}S_2 = \varepsilon\sin\theta\,\mathrm{d}\theta$, $\theta = \tan^{-1}(y/(x-vt))$ and $0 \leqslant \theta < 2\pi$.

From Eqs. (2.24)–(2.25), we have

$$W = \frac{\mu}{2}[u_{3,1}^2 + u_{3,2}^2] = \frac{\mu b^2}{8\pi^2\varepsilon^2}\frac{\gamma^2}{\cos^2\theta + \gamma^2\sin^2\theta}, \tag{2.28}$$

$$T = \frac{\rho}{2}\dot{u}_3^2 = \frac{\rho b^2}{8\pi^2\varepsilon^2}\frac{\gamma^2 v^2 \sin^2\theta}{[\cos^2\theta + \gamma^2\sin^2\theta]^2}, \tag{2.29}$$

$$u_{3,1}\sigma_{31} = \frac{\mu b^2}{4\pi^2\varepsilon^2}\frac{\gamma^2\sin^2\theta}{[\cos^2\theta + \gamma^2\sin^2\theta]^2} \tag{2.30}$$

and

$$u_{3,1}\sigma_{32} = \frac{\mu b^2}{4\pi^2\varepsilon^2}\frac{\gamma\sin\theta\cos\theta}{[\cos^2\theta + \gamma^2\sin^2\theta]^2}. \tag{2.31}$$

Substituting Eqs. (2.28)–(2.31) into Eq. (2.27), by the symmetry in θ, we derive

$$\int_{S_\varepsilon} [(W-T)\delta_{1j} - u_{3,1}\sigma_{3j}]\,\mathrm{d}S_j = 0, \tag{2.32}$$

which, according to Eq. (2.26), implies that the self-force, so that the effective mass as well, of a steadily moving screw dislocation is zero. As discussed in Section 2.2, Eq. (2.32) also implies that the volume integral in Eq. (2.26) converges in the sense of CPV. Furthermore, it is clear that the validity of Eq. (2.32) is independent of the choice of the radius ε of S_ε. Hence in Eq. (2.26), if we choose S to be a circle, then the volume integral converges and equal to zero in the sense of CPV, which can also be verified by independent evaluation.

3. The self-force on a generally accelerating screw dislocation

Consider that a Volterra screw dislocation is situated on the z-axis at rest for $t \leqslant 0$ in an infinite, homogeneous, isotropic elastic solid. For $t > 0$, it moves according to $x = l(t)$ in the (positive) x-direction, where $l(t)$ is an arbitrarily given smooth function such that

$$0 < \frac{\mathrm{d}l(t)}{\mathrm{d}t} < c_2 = \sqrt{\mu/\rho}, \tag{3.1}$$

and c_2 is the shear wave speed, so that the motion is subsonic and advances in the positive x-direction.

3.1. Definition of the self-force on a moving screw dislocation

Dislocations are defects in the elastic material, and their motion with respect to the material changes the configuration. From Section 2, the configurational force on an elastic defect is defined as the limit of the force

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on the inhomogeneity over the diminishingly infinitesimal neighborhood around the defect when it is shrinking upon the defect. Let us choose V as a cylindrical volume around the screw dislocation line at $(x, y) = (l(t), 0)$ at time t, and with a unit length in the z-direction. The infinitesimal volume V_ε around the dislocation line is chosen to be a cylinder with a radius ε for $0 < \varepsilon \ll 1$. The force on the moving dislocation is defined as the limit of the force on the inhomogeneity V_ε when V_ε is shrinking upon the dislocation. Specifically, the force along the x-direction is written as

$$\begin{aligned} F_1 &= \lim_{\varepsilon \to 0} \left\{ \left[\int_{V \setminus V_\varepsilon} \right] \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] \, \mathrm{d}V + \int_S [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] \, \mathrm{d}S_j \right\} \\ &= \lim_{\varepsilon \to 0} I_\varepsilon \equiv \lim_{\varepsilon \to 0} \int_{S_\varepsilon} [(W - T)\delta_{1j} - \sigma_{3j} u_{3,1}] \, \mathrm{d}S_j, \end{aligned} \tag{3.2}$$

where $S_\varepsilon = \partial V_\varepsilon$.

The surface integral I_ε is given by

$$\begin{aligned} I_\varepsilon &= \int_{S_\varepsilon} [(W - T)\delta_{1j} - \sigma_{3j} u_{3,1}] \, \mathrm{d}S_j \\ &= \int_0^{2\pi} \frac{1}{2} [\mu u_{3,1}^2 + \mu u_{3,2}^2 - \rho \dot{u}_3^2] \cos\theta \varepsilon \, \mathrm{d}\theta - \int_0^{2\pi} \mu u_{3,1} (u_{3,1} \cos\theta + u_{3,2} \sin\theta) \, \varepsilon \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{2} [\mu u_{3,2}^2 - \mu u_{3,1}^2 - \rho \dot{u}_3^2] \cos\theta \, \varepsilon \, \mathrm{d}\theta - \int_0^{2\pi} \mu u_{3,1} u_{3,2} \sin\theta \, \varepsilon \, \mathrm{d}\theta. \end{aligned} \tag{3.3}$$

It is seen that to evaluate the limit of I_ε as $\varepsilon \to 0$, we need to know the near-field behavior of $u_{3,1}$, $u_{3,2}$, and $\dot{u}_3$.

3.2. *The evaluation of the near fields*

We will perform the classical coordinate perturbation and make all length involving variables and parameters dimensionless by using a length scale $L_0 \gg b$. The dimensionless quantities are defined according to

$$\begin{aligned} &\hat{x} = \frac{x}{L_0}, \quad \hat{y} = \frac{y}{L_0}, \quad \hat{u}_3 = \frac{u_3}{L_0}, \quad \hat{l}(t) = \frac{l(t)}{L_0}, \quad \hat{b} = \frac{b}{L_0}, \\ &\hat{c}_2 = \frac{c_2}{L_0}, \quad \hat{\rho} = \rho L_0^3, \quad \hat{\mu} = \mu_3 L_0, \quad \hat{F} = \frac{F}{L_0}, \quad \hat{\sigma}_{3j} = \sigma_{3j} L_0, \end{aligned}$$

but, in the sequel, we will omit the hat symbol for the sake of simplicity wherever the above quantities appear. Other dimensionless length quantities are also indicated as such in the pertinent sections. We may note here, as pointed by G.I. Barenblatt, that the physical problem for constant acceleration dislocation motion has a physical characteristic length scale $c_2^2/\dot{v}$, where $\dot{v}$ is the acceleration of the dislocation. This is in contrast to the steady-state constant velocity motion of dislocation, where no such physical characteristic length scale exists.

The motion of a non-uniformly moving screw dislocation starting from rest satisfies the Navier equation of elastodynamics for $y \neq 0$,

$$\frac{\partial^2 u_3(x, y, t)}{\partial x^2} + \frac{\partial^2 u_3(x, y, t)}{\partial y^2} = \frac{1}{c_2^2} \frac{\partial^2 u_3(x, y, t)}{\partial t^2}, \tag{3.4}$$

and the discontinuity condition at $y = 0$,

$$u_3(x, 0^+, t) - u_3(x, 0^-, t) = -\frac{b}{2} [H(x - l(t)) - H(l(t) - x)], \tag{3.5}$$

where $H(\cdot)$ is the Heaviside step function.

Taking into account of the oddness of the displacement $u_3(x, y, t)$ in y and the symmetric property of the Navier equation, the problem can then be reduced to a mixed initial-boundary-value problem in a half-space with the Navier equation (3.4) for $y>0$, and initial conditions

$$u_3(x,y,0) = u_3^s(x,y) = -\frac{b}{2\pi}\tan^{-1}(x/y) \tag{3.6}$$

and

$$\frac{\partial}{\partial t}[u_3(x,y,0)] = 0, \tag{3.7}$$

where $u_3^s(x,y)$ is the solution of a static screw dislocation at the z-axis in the infinite whole space, and the boundary condition on $y=0$

$$u_3(x,0,t) = -\frac{b}{2}H(x-l(t)). \tag{3.8}$$

The near-field solution is the asymptotic expansion in the dimensionless perturbation parameter ε of the solution at the field point which is in a ε-neighborhood of the dislocation, i.e.,

$$(x-x_0)^2 + (y-y_0)^2 = \varepsilon^2 \tag{3.9}$$

or,

$$x = x_0 + \varepsilon\cos\theta, \quad y = y_0 + \varepsilon\sin\theta, \tag{3.10}$$

for $0\leqslant\theta<2\pi$ and $\varepsilon>0$, where (x,y) is the field point and (x_0,y_0) is the position of the dislocation.

The leading terms of the near-field solutions are the same as the corresponding solutions of the steady-state motion with the instantaneous velocity as the uniform velocity (Clifton and Markenscoff, 1981; Markenscoff and Ni, 1993). The leading terms denoted by $u_{3,i}^0, i=1,2,t$, with $u_{3,t}^0 \equiv \dot{u}_3^0$, are written as

$$u_{3,1}^0 = -\frac{b}{2\pi}\frac{\gamma y}{(x-l(t))^2+\gamma^2y^2} = -\frac{b}{2\pi}\frac{\gamma\sin\theta}{\cos^2\theta+\gamma^2\sin^2\theta}\frac{1}{\varepsilon}, \tag{3.11}$$

$$u_{3,2}^0 = \frac{b}{2\pi}\frac{\gamma(x-l(t))}{(x-l(t))^2+\gamma^2y^2} = \frac{b}{2\pi}\frac{\gamma\cos\theta}{\cos^2\theta+\gamma^2\sin^2\theta}\frac{1}{\varepsilon}, \tag{3.12}$$

$$u_{3,t}^0 = \frac{b}{2\pi}\frac{v(t)\gamma y}{(x-l(t))^2+\gamma^2y^2} = \frac{b}{2\pi}\frac{v(t)\gamma\sin\theta}{\cos^2\theta+\gamma^2\sin^2\theta}\frac{1}{\varepsilon}, \tag{3.13}$$

where $v(t)\equiv\dot{l}(t)$ and $\gamma = \sqrt{1-v^2(t)/c_2^2}$.

For the near-field expansion of $u_{3,2}$, based on a closed form solution of $u_{3,2}$ by Markenscoff (1980), Callias and Markenscoff (1988) showed that

$$u_{3,2} \approx -\frac{b}{2\pi}\frac{\gamma\cos\theta}{\cos^2\theta+\gamma^2\sin^2\theta}\frac{1}{\varepsilon} + f_{32}\ln\varepsilon + g_{32} + \text{h.o.t.}, \tag{3.14}$$

the logarithmic term in ε is given explicitly by

$$f_{32} = -\frac{b}{4\pi}\frac{\dot{v}(t)}{c_2^2\gamma^3}. \tag{3.15}$$

In general, the near-field expansions for $u_{3,i}$ for $i=1,2,t$ are written as

$$u_{3,1} = u_{3,1}^0 + f_{31}(\theta,t)\ln\varepsilon + g_{31}(\theta,t) + \text{h.o.t.}, \tag{3.16}$$

$$u_{3,2} = u_{3,2}^0 + f_{32}(\theta,t)\ln\varepsilon + g_{32}(\theta,t) + \text{h.o.t.}, \tag{3.17}$$

$$u_{3,t} = u_{3,t}^0 + f_{3t}(\theta,t)\ln\varepsilon + g_{3t}(\theta,t) + \text{h.o.t.}, \tag{3.18}$$

where $\varepsilon = \sqrt{(x-l(t))^2+y^2}>0$, $\theta = \tan^{-1}(y/(x-l(t)))$. We call f_{3j} and g_{3j}, $j=1,2,t$, in the above near-field expansions *the near-field coefficients*. Then, four of those nine coefficients, i.e., $u_{3,i}^0$, for $i=1,2,t$, and f_{23} are

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known as shown above, the other five coefficients are unknown. In this section we shall give the explicit solutions of all those five unknowns.

We prove a lemma regarding the global property of $u_{3,1}$ and $u_{3,t}$, which will be useful in proving the forthcoming theorems off asymptotic analysis.

Lemma 1. *For all* x, $u_{3,1}(x,y,t)$ *and* $u_{3,t}(x,y,t)$ *can be expressed as*

$$u_{3,1}(x,y,t) = -\frac{b}{2\pi}\frac{\gamma y}{(x-l(t))^2+\gamma y^2} + G_1(x,y,t), \tag{3.19}$$

$$u_{3,t}(x,y,t) = \frac{\dot{l}(t)\gamma y}{(x-l(t))^2+\gamma y^2} + G_2(x,y,t), \tag{3.20}$$

where $G_i(x,y,t), i=1,2$, *are continuous odd functions in* y, *and satisfy*

$$\lim_{y\to 0} G_i(x,y,t) = 0, \tag{3.21}$$

for every x.

Proof. It suffices to prove the lemma for $u_{3,1}$, the proof for $u_{3,t}$ being analogous. From Eq. (3.5), we have the jump condition for $u_{3,1}$,

$$u_{3,1}(x,0^+,t) - u_{3,1}(x,0^-,t) = -b\delta(x-l(t)), \tag{3.22}$$

where $\delta(\cdot)$ is the Dirac delta function. Note that the leading term (3.11) is in fact a delta series, i.e.,

$$\lim_{y\to 0}\frac{b}{2\pi}\frac{\gamma y}{(x-l(t))^2+\gamma^2 y^2} = \frac{b}{2}\delta(x-l(t)). \tag{3.23}$$

If we define

$$2G_1(x,y,t) \equiv u_{3,1}(x,y,t) - u_{3,1}(x,-y,t) + \frac{b}{\pi}\frac{\gamma y}{(x-l(t))^2+\gamma^2 y^2}, \tag{3.24}$$

then in view of Eqs. (3.22) and (3.23), it follows that for each $x \in (-\infty,\infty)$,

$$\lim_{y\to 0} G_1(x,y,t) = 0. \tag{3.25}$$

On the other hand, from Eq. (3.24) and the oddness of $u_{3,1}$ in y, it follows that $G_1(x,y,t)$ is an odd function in y and continuous in y for $y\neq 0$ and all x. Hence, Eq. (3.25) implies that $G_1(x,y,t)$ is continuous in y at $y=0$ as well, which completes the proof. □

We have the following two important theorems for the near-field coefficients, which are central in evaluating the self-force and effective mass.

Theorem 1. *Let the near-field coefficients* $f_{3j}(\theta,t), j=1,2,t$, *be defined in the near-field expansions* (3.16)–(3.18). *Then the partial differentiations of* f_{3j} *with respect to* θ, $f'_{3j}(\theta,t)$, *satisfy the homogeneous system of linear equations*

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ v\sin\theta & 0 & \sin\theta \\ -\mu\sin\theta & \mu\cos\theta & -\rho v\sin\theta \end{bmatrix}\begin{pmatrix} f'_{31} \\ f'_{32} \\ f'_{3t}\end{pmatrix} = \mathbf{0}. \tag{3.26}$$

So that

$$f'_{31} = f'_{32} = f'_{3t} = 0. \tag{3.27}$$

Furthermore,

$$f_{31}(\theta,t) = f_{3t}(\theta,t) = 0 \tag{3.28}$$

and

$$f_{32}(\theta, t) = f_{32}(t) \tag{3.29}$$

is independent of θ.

Theorem 2. *Let the near-field coefficients* $g_{3j}(\theta, t), j = 1, 2, t$, *be defined in the near-field expansions* (3.16)–(3.18). *Then the partial differentiations of* g_{3j} *with respect to* θ, $g'_{3j}(\theta, t)$, *satisfy the inhomogeneous system of linear equations*

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ v\sin\theta & 0 & \sin\theta \\ -\mu\sin\theta & \mu\cos\theta & -\rho v\sin\theta \end{bmatrix} \begin{pmatrix} g'_{31} \\ g'_{32} \\ g'_{3t} \end{pmatrix} = \begin{pmatrix} f_{32}\cos\theta \\ -U_{31} \\ \rho U_{3t} - \mu f_{32}\sin\theta \end{pmatrix}, \tag{3.30}$$

where

$$U_{3j} = \varepsilon \frac{\partial}{\partial t}[u^0_{3,j}]\bigg|_{\exp}, \tag{3.31}$$

for $j = 1, t$, *and the explicit partial differentiation with respect to t means the partial differentiation with respect to t when* ε *and* θ *are assumed to be fixed.*

Furthermore, when f_{32} *is given by*

$$f_{32} = -\frac{b}{4\pi}\frac{\dot v(t)}{c_2^2\gamma^3}, \tag{3.32}$$

where $\gamma = \sqrt{1 - v^2/c_2^2}$, *then*

$$g'_{32} = \frac{b\dot v\cos\theta\sin\theta}{2\pi c_2^2\gamma}\left[\frac{(2-3\gamma^2)\cos^2\theta + \gamma^2(\gamma^2-2)\sin^2\theta}{(\cos^2\theta + \gamma^2\sin^2\theta)^3}\right] - \frac{b\dot v v^2\cos\theta\sin\theta}{4\pi c_2^4\gamma^3(\cos^2\theta + \gamma^2\sin^2\theta)}, \tag{3.33}$$

$$g'_{3t} = \frac{bv\dot v}{4\pi c_2^2\gamma^3}\left[\frac{\cos^2\theta - \gamma^2\sin^2\theta}{(\cos^2\theta + \gamma^2\sin^2\theta)^2} + \frac{2\gamma^2(3\gamma^2\sin^2\theta - \cos^2\theta)}{(\cos^2\theta + \gamma^2\sin^2\theta)^3}\right], \tag{3.34}$$

$$g'_{31} = f_{32} - g'_{32}\tan\theta. \tag{3.35}$$

Proof of Theorem 1. It suffices to consider $y > 0$. For $y > 0$, $u_{3,i}$ for $i = 1, 2, t$, are continuously differentiable. From the relation

$$\frac{\partial u_{3,1}}{\partial y} = \frac{\partial u_{3,2}}{\partial x}, \tag{3.36}$$

and the expansions (3.16) and (3.17), it follows that

$$\frac{\partial u^0_{3,1}}{\partial y} + f'_{31}\frac{\cos\theta\ln\varepsilon}{\varepsilon} + f_{31}\frac{\sin\theta}{\varepsilon} + g'_{31}\frac{\cos\theta}{\varepsilon} = \frac{\partial u^0_{3,2}}{\partial x} - f'_{32}\frac{\sin\theta\ln\varepsilon}{\varepsilon} + f_{32}\frac{\cos\theta}{\varepsilon} - g'_{32}\frac{\sin\theta}{\varepsilon} + \text{h.o.t.}, \tag{3.37}$$

where the following relations are used:

$$\frac{\partial\varepsilon}{\partial x} = \cos\theta, \quad \frac{\partial\varepsilon}{\partial y} = \sin\theta, \quad \frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{\varepsilon}, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{\varepsilon}.$$

Then in Eq. (3.37), using the relation $\partial u^0_{3,1}/\partial y = \partial u^0_{3,2}/\partial x$ and collecting the like terms of $(\ln\varepsilon)/\varepsilon$ which are the most singular terms, we have

$$f'_{31}\cos\theta + f'_{32}\sin\theta = 0. \tag{3.38}$$

Similarly, from

$$\frac{\partial u_{3,1}}{\partial t}=\frac{\partial \dot{u}_3}{\partial x}, \tag{3.39}$$

and the expansions (3.16) and (3.18), it follows that

$$\begin{aligned}&\frac{\partial u^0_{3,1}}{\partial t}+f'_{31}\frac{v(t)\sin\theta\ln\varepsilon}{\varepsilon}-f_{31}\frac{v(t)\cos\theta}{\varepsilon}+g'_{31}\frac{v(t)\sin\theta}{\varepsilon}\\&\quad=\frac{\partial u^0_{3,t}}{\partial x}-f'_{3t}\frac{\sin\theta\ln\varepsilon}{\varepsilon}+f_{3t}\frac{\cos\theta}{\varepsilon}-g'_{3t}\frac{\sin\theta}{\varepsilon}+\text{h.o.t.},\end{aligned} \tag{3.40}$$

where the following relations are used:

$$\frac{\partial\theta}{\partial t}=\frac{v(t)\sin\theta}{\varepsilon},\quad \frac{\partial\varepsilon}{\partial t}=\frac{-v(t)\cos\theta}{\varepsilon}.$$

From Eqs. (3.11) and (3.13), we have

$$\frac{\partial u^0_{3,1}}{\partial t}-\frac{\partial u^0_{3,t}}{\partial x}=\left(\frac{\partial u^0_{3,1}}{\partial t}\right)_{\text{exp}}, \tag{3.41}$$

here the subscript exp represents the explicit derivative which is defined as the partial derivative with respect to t assuming that ε and θ, or, $x_1-l(t)$ and y, are fixed. Noting that in the expressions of $u^0_{3,i}$, $\gamma=\sqrt{1-v^2(t)/c_2^2}$ depends on t as well, hence the right-hand side of Eq. (3.41) does not necessarily vanish. And it is easy to check that

$$\left(\frac{\partial u^0_{3,1}}{\partial t}\right)_{\text{exp}}\sim O(1/\varepsilon). \tag{3.42}$$

Then collecting the like terms of $(\ln\varepsilon)/\varepsilon$ in (3.40), we have

$$v(t)f'_{31}\sin\theta+f'_{3t}\sin\theta=0. \tag{3.43}$$

To establish the third equation for $f'_{31}, f'_{32}, f'_{3t}$, we use the equation of motion (3.4)

$$\mu\frac{\partial u_{3,1}}{\partial x}+\mu\frac{\partial u_{3,2}}{\partial y}=\rho\frac{\partial u_{3,t}}{\partial t}, \tag{3.44}$$

together with the expansions (3.16)–(3.18). Then, we have

$$\begin{aligned}&\mu\frac{\partial u^0_{3,1}}{\partial x}+\mu\left[-f'_{31}\frac{\sin\theta\ln\varepsilon}{\varepsilon}+f_{31}\frac{\cos\theta}{\varepsilon}-g'_{31}\frac{\sin\theta}{\varepsilon}\right]\\&\quad+\mu\frac{\partial u^0_{3,2}}{\partial y}+\mu\left[f'_{32}\frac{\cos\theta\ln\varepsilon}{\varepsilon}+f_{32}\frac{\sin\theta}{\varepsilon}+g'_{32}\frac{\cos\theta}{\varepsilon}\right]\\&\qquad=\rho\frac{\partial u^0_{3,t}}{\partial t}+\rho\left[f'_{3t}\frac{v(t)\sin\theta\ln\varepsilon}{\varepsilon}-f_{3t}\frac{v(t)\cos\theta}{\varepsilon}+g'_{3t}\frac{v(t)\sin\theta}{\varepsilon}\right]+\text{h.o.t.}\end{aligned} \tag{3.45}$$

From Eqs. (3.11)–(3.13), we have

$$\rho\frac{\partial u^0_{3,t}}{\partial t}-\mu\left[\frac{\partial u^0_{3,1}}{\partial x}+\frac{\partial u^0_{3,2}}{\partial y}\right]=\rho\left(\frac{\partial u^0_{3,t}}{\partial t}\right)_{\text{exp}}. \tag{3.46}$$

It is easy to show that the term on the right-hand side is in the order of $O(1/\varepsilon)$. Hence collecting the like terms of $(\ln\varepsilon)/\varepsilon$ in Eq. (3.45), we have the third equation

$$-\mu f'_{31}\sin\theta+\mu f'_{32}\cos\theta-\rho v f'_{3t}\sin\theta=0. \tag{3.47}$$

Eqs. (3.38), (3.43), and (3.47) form a homogeneous system for the unknowns f'_{3j}, $j = 1, 2, t$,

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ v\sin\theta & 0 & \sin\theta \\ -\mu\sin\theta & \mu\cos\theta & -\rho v\sin\theta \end{bmatrix} \begin{pmatrix} f'_{31} \\ f'_{32} \\ f'_{3t} \end{pmatrix} = \mathbf{0}. \tag{3.48}$$

The determinant of the matrix of the coefficients in Eq. (3.48) is calculated to be

$$\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ v\sin\theta & 0 & \sin\theta \\ -\mu\sin\theta & \mu\cos\theta & -\rho v\sin\theta \end{vmatrix} = (\rho v^2 \sin^2\theta - \mu)\sin\theta \neq 0, \tag{3.49}$$

for $\theta \neq \pi, 0$, since the motion is subsonic $v < c_2 = \sqrt{\mu/\rho}$. Therefore, we conclude that

$$f'_{31} = f'_{32} = f'_{3t} = 0, \tag{3.50}$$

for $\theta \neq \pi, 0$. f_{31}, f_{32} and f_{3t} are then independent of θ for $y > 0$ and $y < 0$, respectively.

By using Lemma 1 in the near field, we know that $u_{3,1}$ and $\dot{u}_3$, thus $f_{3,1}$ and $f_{3,t}$ as well, are continuous across $y = 0$, and odd in y. Hence we have

$$f_{31} = f_{3t} = 0. \tag{3.51}$$

Since $u_{3,2}$ and f_{32} are even in y, Eq. (3.50) implies that

$$f_{32}(\theta, t) = f_{32}(t) \tag{3.52}$$

is independent of θ, Theorem 1 is then proved. □

Proof of Theorem 2. In Eqs. (3.37), (3.40), and (3.45), noting that $f'_{3j} = 0$ and $f_{31} = f_{3t} = 0$, then collecting the like terms of $1/\varepsilon$, we obtain the inhomogeneous system of linear equations

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ v\sin\theta & 0 & \sin\theta \\ -\mu\sin\theta & \mu\cos\theta & -\rho v\sin\theta \end{bmatrix} \begin{pmatrix} g'_{31} \\ g'_{32} \\ g'_{3t} \end{pmatrix} = \begin{pmatrix} f_{32}\cos\theta \\ -U_{31} \\ \rho U_{3t} - \mu f_{32}\sin\theta \end{pmatrix}, \tag{3.53}$$

where

$$U_{3j} = \varepsilon \frac{\partial}{\partial t}[u^0_{3,j}]|_{\exp}, \tag{3.54}$$

for $j = 1, t$.

The matrix of coefficients of the system of linear equations in Eq. (3.53) is exactly the same as that of the system (3.48), so that its determinant is given in Eq. (3.49) and equal to

$$(\rho v^2 \sin^2\theta - \mu)\sin\theta = -(\cos^2\theta + \gamma^2\sin^2\theta)\sin\theta. \tag{3.55}$$

g'_{32} and g'_{3t} are solved to be

$$g'_{32} = \frac{\cos\theta}{c_2^2}\left[\frac{vU_{31} - U_{3t} + v^2 f_{32}\sin^2\theta}{\cos^2\theta + \gamma^2\sin^2\theta}\right], \tag{3.56}$$

$$g'_{3t} = \frac{\rho v U_{3t}\sin^2\theta - \mu U_{31} - \mu v f_{32}\sin\theta}{\mu\sin\theta(\cos^2\theta + \gamma^2\sin^2\theta)}. \tag{3.57}$$

U_{31} and U_{3t} are explicitly rewritten as

$$U_{31} = \frac{bv\dot{v}}{2\pi c_2^2\gamma}\left[\frac{\cos^2\theta - \gamma^2\sin^2\theta}{(\cos^2\theta + \gamma^2\sin^2\theta)^2}\right], \tag{3.58}$$

$$U_{3t} = \frac{b\dot{v}\sin\theta}{2\pi\gamma}\left[\frac{(2\gamma^2-1)\cos^2\theta+\gamma^2\sin^2\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^2}\right]. \tag{3.59}$$

Substitute those expressions into Eqs. (3.33) and (3.57), when f_{32} is given by Eq. (3.32), we obtain that

$$g'_{32} = \frac{b\dot{v}\cos\theta\sin\theta}{2\pi c_2^2\gamma}\left[\frac{(2-3\gamma^2)\cos^2\theta+\gamma^2(\gamma^2-2)\sin^2\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^3}\right] - \frac{b\dot{v}v^2\cos\theta\sin\theta}{4\pi c_2^4\gamma^3(\cos^2\theta+\gamma^2\sin^2\theta)}, \tag{3.60}$$

$$g'_{3t} = \frac{bv\dot{v}}{4\pi c_2^2\gamma^3}\left[\frac{\cos^2\theta-\gamma^2\sin^2\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^2}+\frac{2\gamma^2(3\gamma^2\sin^2\theta-\cos^2\theta)}{(\cos^2\theta+\gamma^2\sin^2\theta)^3}\right]. \tag{3.61}$$

From the first equation in the system (3.53), it follows that

$$g'_{31} = f_{32} - g'_{32}\tan\theta, \tag{3.62}$$

which completes the proof of the theorem. □

The near-field coefficients $g_{3j}(\theta, t)$ can be obtained by integrating g'_{3j} with respect to θ,

$$g_{3j}(\theta, t) = \int_0^\theta g'_{3j}(\theta, t)\,\mathrm{d}\theta + g_{3j}(0, t). \tag{3.63}$$

Again by employing Lemma 1 in the near field, $g_{31}(\theta, t)$ and $g_{3t}(\theta, t)$ are odd and continuous in θ. Hence, $g_{31}(0, t) = g_{3t}(0, t) = 0$. We obtain that

$$g_{31}(\theta, t) = -\frac{b\dot{v}}{4\pi c_2^2\gamma^3}\left[\theta - \frac{1}{\gamma}\tan^{-1}(\gamma\tan(\theta))\right] - \frac{b\dot{v}\sin\theta\cos\theta}{8\pi c_2^2\gamma^3}\left[\frac{2\cos^2\theta+\gamma^2(3-2\gamma^2)\sin^2\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^2}\right]. \tag{3.64}$$

$$g_{32}(\theta, t) = \frac{b\dot{v}}{2\pi c_2^2\gamma}[(2-3\gamma^2)\cos^2\theta-\gamma^4\sin^2\theta] + \frac{b\dot{v}}{8\pi c_2^2\gamma^3}\ln(\cos^2\theta+\gamma^2\sin^2\theta) + g_{32}(0, t), \tag{3.65}$$

$$g_{3t}(\theta, t) = -\frac{bv\dot{v}}{4\pi c_2^2\gamma^3}\left[\frac{((1-2\gamma^2)\cos^2\theta+3\gamma^2\sin^2\theta)\sin\theta\cos\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^2} - \frac{2}{\gamma}\tan^{-1}(\gamma\tan(\theta))\right], \tag{3.66}$$

where the evaluation of $g_{32}(0, t)$ will be given below.

$g_{32}(0, t)$ is calculated by applying a theorem of asymptotic analysis (Callias and Markenscoff, 1988) to the closed form solution $u_{3,2}$ of an accelerating screw dislocation (Markenscoff, 1980). The theorem by Callias and Markenscoff (1988) contains some errors. A corrected theorem is given below, the corrections are important to the evaluation of $g_{32}(0, t)$.

Theorem 3. *Let $f(s, y)$ be such that*

(i) $f \in C^\infty([0, p] \times [0, \infty))$;
(ii) $|\partial_s^k f(s, y)| \leqslant y^k h_k(y)$, *for all* s, y, *and* $k = 0, 1, 2, \ldots$, *where* $\int_0^\chi h_k(1/s)\,\mathrm{d}s < \infty$, *for each* $\chi > 0$.

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Then we have as $\varepsilon \to 0^+$,

$$\int_0^p f(s,\varepsilon/s)\,\mathrm{d}s \sim \int_0^p f(s,0)\,\mathrm{d}s + \sum_{m=1}^{\infty}\varepsilon^m\left\{L_m(f)+Q_m(f;p)+\frac{1}{m!(m-1)!}\partial_s^{m-1}\partial_y^m f(0,0)\sum_{j=1}^{m-1}\frac{1}{j}\right\} - \sum_{m=1}^{\infty}\frac{\varepsilon^m\ln\varepsilon}{m!(m-1)!}\partial_s^{m-1}\partial_y^m f(0,0) - \sum_{m=1}^{\infty}\varepsilon^m\left\{\sum_{j=1}^{m-2}\frac{(j-1)!}{m!(m-1)!}\frac{1}{p^j}\partial_s^{m-1-j}\partial_y^m f(p,0)\right\}+\frac{\ln p}{m!(m-1)!}\partial_s^{m-1}\partial_y^m f(p,0), \tag{3.67}$$

where

$$L_m(f) = -\frac{1}{(m-1)!}\int_0^\infty \ln\zeta\,\partial_\zeta[\zeta^m\partial_x^{m-1}R_{m+1}(0,1/\zeta)]\,\mathrm{d}\zeta, \tag{3.68}$$

$R_{m+1}(x,y)$ *is the remainder of* $f(x,y)$ *in the Taylor expansion about* $y=0$ *after m terms, i.e.*,

$$R_{m+1}(x,y) = f(x,y) - \sum_{k=0}^{m}\frac{1}{k!}\partial_y^k f(x,0)y^k \tag{3.69}$$

and

$$Q_m(f;p) = -\frac{1}{m!(m-1)!}\int_0^p \ln s\,\partial_s^m\partial_y^m f(s,0)\,\mathrm{d}s. \tag{3.70}$$

We apply the theorem to the following solution of the stress σ_{32} for a non-uniformly moving screw dislocation starting from rest given by Markenscoff (1980):

$$\sigma_{32} = \frac{b\mu}{2\pi}\int_0^\infty \frac{(t-\eta(\xi))(x-\xi)^2H(t-\eta(\xi)-r/c)}{r^4[(t-\eta(\xi))^2-r^2/c^2]^{1/2}}\,\mathrm{d}\xi - \frac{b\mu}{2\pi}y^2\frac{\partial}{\partial t}\int_0^\infty\frac{(t-\eta(\xi))^2H(t-\eta(\xi)-r/c)}{r^4[(t-\eta(\xi))^2-r^2/c^2]^{1/2}}\,\mathrm{d}\xi + \frac{b\mu}{2\pi}\frac{x}{x^2+y^2}, \tag{3.71}$$

where $c\equiv c_2=\sqrt{\mu/\rho}$, $r^2=(x-\xi)^2+y^2$, and $\eta(\xi)=\tau$ is the inverse function of $\xi=l(\tau)$.

The term $g_{32}(0,t)$ will be calculated from the near-field expansion of the stress σ_{32} at the positions $(x,y)=(l(t)+\varepsilon,0)$ as $\varepsilon\to 0$. From the right-hand side of Eq. (3.71), letting $y\to 0$, we obtain the expression for σ_{32} at $(x,0)$

$$\frac{b\mu}{2\pi}\int_0^{\xi_0}\frac{t-\eta(\xi)}{(x-\xi)^2[(t-\eta(\xi))^2-(x-\xi)^2/c^2]^{1/2}}\,\mathrm{d}\xi+\frac{b\mu}{2\pi}\frac{1}{x}, \tag{3.72}$$

where $x=x_0+\varepsilon$, and $\xi_0=l(\tau_0)$ is the root of equation

$$c(t-\eta(\xi)) = |x-\xi_0|. \tag{3.73}$$

The physical meaning of ξ_0 is the last position from which the wavelet of the moving screw dislocation can reach the field point $(x,y)=(l(t)+\varepsilon,0)$ at time t.

Apart from the static solution term, the expression in Eq. (3.72) is rewritten as

$$\frac{b\mu}{2\pi}\int_0^{\xi_0}\frac{s\psi(s)+\varepsilon A_0(t,\varepsilon)}{(s+c\varepsilon A_0(t,\varepsilon))^2[s^2(\psi^2-1/c^2)+2\varepsilon sA_0(t,\varepsilon)(\psi-1/c)]^{\frac{1}{2}}}\,\mathrm{d}s \equiv \frac{b\mu}{2\pi}\int_0^{\xi_0}h(s,\varepsilon,x,t)\,\mathrm{d}s \equiv I_\varepsilon, \tag{3.74}$$

where $s = \xi_0 - \xi = l(\tau_0) - l(\tau)$,

$$\psi(s, \xi_0) = \frac{\eta(\xi) - \eta(\xi)}{s} = \frac{\eta(\xi) - \eta(\xi_0 - s)}{s} \tag{3.75}$$

and

$$A_0(t, \varepsilon) = \frac{(t - \tau_0)}{\varepsilon} = \frac{1}{c - v(t)} + O(\varepsilon), \tag{3.76}$$

with $v(t) = \dot{l}(t)$.

It is noted that seeking the $O(1)$ term $g_{32}(0, t)$ of the asymptotic expansion of the integral I_ε defined in Eq. (3.74) as $\varepsilon \to 0$, either by taking the limit under the integral, or by the usual Taylor expansion of the integrand, will lead to divergence. Therefore, the special asymptotic analysis described in Theorem 3 is needed. However, it turns out that for the integral I_ε, the smoothness condition required for the integrand in Theorem 3 is not satisfied over the whole interval of the integration $[0, \xi_0]$, which is $[0, x_0] = [0, l(t)]$ as $\varepsilon \to 0$. That must be decomposed into two subintervals $[0, \omega]$ and $[\omega, x_0]$ for $0 < \omega < x_0$. So that over the subinterval $[\omega, x_0]$, the limit $\varepsilon \to 0$ can be taken under the integral, while over the subinterval $[0, \omega]$, the smoothness condition required in Theorem 3 is satisfied and the special asymptotic analysis can apply. Based on Theorem 3, it is easy to verify that the evaluation of $g_{32}(0, t)$ is invariant under different choices of ω.

After a lengthy calculation, $g_{32}(0, t)$ is evaluated as

$$\begin{aligned} g_{32}(0, t) = \frac{b}{2\pi} \Bigg\{ & \frac{c\dot{v}(t)}{2(c^2 - v^2(t))^{3/2}} \left[1 + \ln\left(\frac{cv}{2(c^2 - v^2)} \right) \right] \\ & + \frac{\dot{v}}{2c(c - v)^4} \left[4c^2(c - 2v) + v^2(2c - v) \right] \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v} \right) + \frac{\dot{v}(c^2 + v^2)}{2v(c - v)^2 \sqrt{c^2 - v^2}} + \frac{1}{l(t)} \\ & + \int_0^{\eta(\omega)} \frac{c(t - \tau) l(\tau)\, \mathrm{d}\tau}{(l(t) - l(\tau))^2 [c^2(t - \tau)^2 - (l(t) - l(\tau))^2]^{1/2}} + \int_{\eta(\omega)}^{t} \frac{\ln(l(t) - l(\tau)) K(t, \tau, v(\tau))\, \mathrm{d}\tau}{[c^2(t - \tau)^2 - (l(t) - l(\tau))^2]^{5/2}} \\ & + \frac{c(t - \eta(\omega))}{(l(t) - \omega)[c^2(t - \eta(\omega))^2 - (l(t) - \omega)^2]^{\frac{1}{2}}} \\ & + \frac{c \ln(l(t) - \omega)[(t - \eta(\omega)) v(\eta(\omega)) - (l(t) - \omega)](l(t) - \omega)}{v(\eta(\omega))[c^2(t - \eta(\omega))^2 - (l(t) - \omega)^2]^{3/2}} \Bigg\}, \end{aligned} \tag{3.77}$$

where $c = c_2$, $v(\tau) = \dot{l}(\tau)$, and $K(t, \tau, v(\tau))$ is defined by

$$\begin{aligned} K(t, \tau, v(\tau)) = & \frac{c}{v^2(\tau)[c^2(t - \tau)^2 - (l(t) - l(\tau))^2]^{5/2}} \\ & \times \{ v^3(\tau)(t - \tau)[c^2(t - \tau)^2 + 2(l(t) - l(\tau))^2] \\ & - 2v^2(\tau)(l(t) - l(\tau))[2c^2(t - \tau)^2 + (l(t) - l(\tau))^2] \\ & - 3c^2 v(\tau)(t - \tau)(l(t) - l(\tau))^2 \\ & - \dot{v}(\tau)(l(t) - l(\tau))^2 [c^2(t - \tau)^2 - (l(t) - l(\tau))^2] \}. \end{aligned} \tag{3.78}$$

In Eq. (3.77), the first three terms have an explicit factor of the acceleration, $\dot{v}$, and they are related to the inertial part of the self-force and contribute to the effective mass of the moving dislocation, while the integrations in the expression depend on the history of the motion.

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3.3. The self-force

By substituting the near-field expansions (3.16)–(3.18) into (3.3), we obtain the expression for the surface integral I_ε,

$$I_\varepsilon = \mu b f_{32} \ln \varepsilon + \frac{\mu b \gamma}{2\pi} \int_0^{2\pi} \frac{c_2^2 g_{32}(\theta, t) - v(t) \sin\theta \cos\theta g_{3t}(\theta, t)}{c_2^2[\cos^2\theta + \gamma^2 \sin^2\theta]} \, \mathrm{d}\theta + O(\varepsilon), \tag{3.79}$$

where again

$$f_{32} = -\frac{b}{4\pi} \frac{\dot{v}(t)}{c_2^2 \gamma^3},$$

and $g_{32}(\theta, t)$ and $g_{3t}(\theta, t)$ are the near-field coefficients. In the integration of Eq. (3.79), by substituting $g_{32}(\theta, t)$ and $g_{3t}(\theta, t)$ with the values obtained in Eqs. (3.65) and (3.66), we have the evaluation of I_ε to order $O(t)$ terms,

$$I_\varepsilon = -\frac{\mu b^2 \dot{v}}{4\pi c_2^2 \gamma^3} \ln \varepsilon + \frac{\mu b^2 \dot{v}}{4\pi c_2^2 \gamma^3} \left[\ln(\gamma(1+\gamma)/2) - \frac{4 + \beta^4 - \beta^2(7 + 2\gamma)}{(1+\gamma)^2} \right] + \mu b g_{32}(0, t), \tag{3.80}$$

where $\gamma = \sqrt{1 - v^2(t)/c_2^2}$, $\beta = v(t)/c_2$, and $g_{32}(0, t)$ is given in Eq. (3.77).

The last equation implies that as $\varepsilon \to 0$, I_ε is divergent. Therefore, according to Eq. (3.2), in the surface-independent dynamic J-integral for the self-force on an accelerating (screw) dislocation, the volume integral does not exist even in the sense of CPV, and as a result, the self-force of an accelerating screw dislocation is divergent and not well-defined. We will deal with the divergence by two different approaches:

(1) On the basis of a smearing method, smooth the singularity at the core of dislocation;
(2) On the basis of the theory of distributions, regularize the divergent volume integral.

Then, the self-force becomes well-defined and can be evaluated by the surface-independent dynamic J-integral.

3.4. Remark on the energy flux for an accelerating dislocation

Clifton and Markenscoff (1981) obtained the force on a moving dislocation which jumps from rest to a constant velocity. Their calculation is based on defining the self-force (which they called "drag force") by

$$F = \dot{E}/v_{\mathrm{d}}, \tag{3.81}$$

where $\dot{E}$ is the energy flux given by

$$\dot{E} = \int_\Sigma \left[\dot{u}_i \sigma_{ij} n_j + \frac{1}{2}(\sigma_{ij} u_{i,j} + \rho \dot{u}_i \dot{u}_i) \mathcal{V}_n \right] \mathrm{d}s, \tag{3.82}$$

where σ and u_i are the stress and displacement field, ρ is the density of the solid material, $\boldsymbol{n}$ is the outer normal of the surface Σ, and $\mathcal{V}_n = (\mathcal{V}, \boldsymbol{n})$ with $\mathcal{V}$ as the velocity of the moving dislocation.

Clifton and Markenscoff pointed out that, for a moving dislocation, which jumps from rest and moves along the x-axis with a constant velocity v_{d}, the integral of the energy flux is independent of the choice of the surface which surrounds the core of the moving dislocation as it shrinks to it. For instance, choose the surface as a circle S_d at the core of the dislocation with an infinitesimal radius d, then one can define uniquely the energy flux

$$\dot{E}_0 = \lim_{S_d \to 0} \dot{E}_{S_d} = \lim_{S_d \to 0} \int_{S_d} [(W + T)\delta_{1j} - \sigma_{3j} u_{3,1}] \, \mathrm{d}S_j. \tag{3.83}$$

The energy flux through the core used by Clifton and Markenscoff (1981) for this particular motion cannot be applied to the case of arbitrarily accelerating motion. As pointed by Freund (1972), "in any case, path independence of" the integral of the energy flux "should be checked before it is used for any particular problem" Here we show that for the case of an accelerating screw dislocation, the path independence does not hold, as the integral surface is shrinking to the core of the dislocation.

It suffices to prove that the energy flux through the core, i.e., the energy release rate, in the case of an accelerating screw dislocation, is *not* path-independent, as S_d shrinks to the core of the dislocation. By using the near-field solutions obtained in previous subsections, $\dot{E}_d$ is calculated and found to be

$$\dot{E}_{\rm d} \sim -\frac{\mu b}{4\pi}\frac{\dot{v}(t)}{c_2^2\gamma^3}\ln d + \frac{\mu b\gamma}{2\pi}\int_0^{2\pi}\frac{c_2^2 g_{32}(\theta,t)+v(t)\sin\theta\cos\theta g_{3t}(\theta,t)}{c_2^2[\cos^2\theta+\gamma^2\sin^2\theta]}\,{\rm d}\theta, \tag{3.84}$$

which is divergent as $d \to 0$. Therefore, the surface integral (3.83) is not path-independent, when $S = S_d$ is a d-circle as $d \to 0$. Because the difference between two such surface integrals goes to zero if and only if the surface integral has a finite limit as $d \to 0$. Consequently, the energy flux cannot be used for defining the self-force on an accelerating dislocation, and the Volterra dislocation is too strong of a singularity in this case.

4. The self-force and effective mass based on a smearing (ramp-core) method

Eshelby (1951) pointed out that "Singularities with infinite self-energy can be regarded as limiting case of singularities with finite self-energy, and when we make the passage to the limit the expression for the force is still valid". In Al'shitz et al. (1971), for deriving the force in a periodic lattice field a smearing method was used ("The divergence of the elastic field on the dislocation axis is eliminated by smearing the nucleus of the dislocation over a region of radius $r_0 \approx a$"). Eshelby (1977) discussed the smearing (ramp-core) technique in the calculation of the configurational force on a moving crack, where the Dirac delta function was replaced by a delta sequence. The smearing is necessary in dislocations, as stated by Weertman and Weertman (1980): "A discrete dislocation cannot exist in a real crystal because a real crystal cannot contain an infinite amount of energy nor can it support infinite stresses"; "If a discrete dislocation is 'smeared-out' over a localized region on its glide or climb plane, the infinite stresses and self-energy can be eliminated". This elimination is effectuated here by a mathematical limiting process with respect to a length scale (denoted below by a); if the singularity of the Volterra dislocation is smeared in a way such that the force on the smeared singularity is well-defined, then this approach (smearing or ramp-core method) is validated. Weertman and Weertman in their paper (Weertman and Weertman, 1980) gave a full reference list in addition to the references in Hirth and Lothe (1968) for the Peierls–Nabarro and related models.

4.1. A smearing (ramp-core) method

Recall that the Navier equation of elastodynamics for a moving screw dislocation, for $y \neq 0$,

$$\frac{\partial^2 u_3(x,y,t)}{\partial x^2}+\frac{\partial^2 u_3(x,y,t)}{\partial y^2}=\frac{1}{c_2^2}\frac{\partial^2 u_3(x,y,t)}{\partial t^2}, \tag{4.1}$$

and the discontinuity condition is

$$u_3(x,0^+,t)-u_3(x,0^-,t)=-\frac{b}{2}[H(x-l(t))-H(l(t)-x)]. \tag{4.2}$$

Note that the discontinuity condition (4.2) is equivalently rewritten as

$$u_3(x,0^+,t)-u_3(x,0^-,t)=-\frac{b}{2}[H(x-l(t))-H(l(t)-x)]\star\delta(x), \tag{4.3}$$

where $\star$ is the symbol of the convolution defined by

$$[f\star h](x)\equiv\int_{-\infty}^{\infty}f(x-\xi)h(\xi)\,{\rm d}\xi=\int_{-\infty}^{\infty}f(\xi)h(x-\xi)\,{\rm d}\xi.$$

Follow Eshelby's smearing (ramp-core) technique (Eshelby, 1977), we replace the delta function in the discontinuity condition (4.3) by a delta sequence

$$g_a(x)\equiv\frac{1}{\pi}\frac{a}{x^2+a^2}, \tag{4.4}$$

where $a>0$ is the dimensionless smearing parameter, and define the smeared field variable by

$$\hat{u}_3(x,y,t)=u_3\star g_a. \tag{4.5}$$

Then the discontinuity condition for the smeared displacement $\hat{u}_3(x,y,t)$ is

$$\begin{aligned}\hat{u}_3(x,0^+,t)-\hat{u}_3(x,0^-,t) &= -\frac{b}{2}[H(x-l(t))-H(l(t)-x)]\star g_a(x)\\ &= -\frac{b}{\pi}\tan^{-1}\left(\frac{x-l(t)}{a}\right).\end{aligned} \tag{4.6}$$

This is exactly the ramp-core model analyzed by Markenscoff and Ni (2001a,b) in obtaining the radiated fields for a dislocation in a general motion as well as one jumping from rest to a constant velocity.

It is shown that $\hat{u}_3(x,y,t)$ is infinitely differentiable with respect to x, y, t, for $y\neq 0$ and all x, t, and bounded in the closed half-space $y\geqslant 0$, e.g.,

$$\hat{u}_{3,1}=\int_{-\infty}^{\infty}u_3(\xi,y,t)\frac{1}{\pi}\frac{\partial}{\partial x}\left[\frac{a}{(x-\xi)^2+a^2}\right]\mathrm{d}\xi. \tag{4.7}$$

After integration by parts, we obtain

$$\hat{u}_{3,1}=\widehat{(u_{3,1})}=u_{3,1}\star g_a, \tag{4.8}$$

$$\hat{u}_{3,2}=u_{3,2}\star g_a,\quad \hat{u}_{3,t}=u_{3,t}\star g_a, \tag{4.9}$$

$$\hat{u}_{3,jj}=u_{3,jj}\star g_a, \tag{4.10}$$

for $j=1,2,t$, and $\hat{u}_3(x,y,t)$ satisfies the Navier equation of elastodynamics for $y\neq 0$

$$\frac{\partial^2\hat{u}_3(x,y,t)}{\partial x^2}+\frac{\partial^2\hat{u}_3(x,y,t)}{\partial y^2}=\frac{1}{c_2^2}\frac{\partial^2\hat{u}_3(x,y,t)}{\partial t^2}. \tag{4.11}$$

4.2. The self-force based on smearing

As discussed above, the smeared field $\hat{u}_3$ is infinitely differentiable for $y>0$ and $y<0$, and continuous and bounded when $y\to 0^+$ or $y\to 0^-$. However, due to the discontinuity condition (4.6), $\hat{u}_3$ has a finite jump discontinuity across the slip plane $y=0$. The stress $\hat{u}_{3,1}$ and the field velocity $\hat{u}_{3,t}$ also have finite jumps across the slip plane $y=0$,

$$\begin{aligned}\hat{u}_{3,1}(x,0^+,t) &= -\hat{u}_{3,1}(x,0^-,t)=-\frac{b}{2}g_a(x-l(t))\\ &= -\frac{b}{2\pi}\frac{a}{(x-l(t))^2+a^2}\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}\hat{u}_{3,t}(x,0^+,t) &= -\hat{u}_{3,t}(x,0^-,t)=\frac{bv(t)}{2}g_a(x-l(t))\\ &= \frac{b}{2\pi}\frac{av(t)}{(x-l(t))^2+a^2}.\end{aligned} \tag{4.13}$$

As an even function in y, $\hat{u}_{3,2}$ is continuous across $y=0$.

Therefore, there is a *surface singularity* on the slip plane $y=0$, on which, the displacement field $\hat{u}_3$, the stress $\mu\hat{u}_{3,1}$, and the field velocity $\hat{u}_{3,t}$ have finite jumps, respectively. In order to keep the smeared dislocation moving non-uniformly on the slip plane, a force opposite and equal to the self-force on the slip plane must be applied on the dislocation. To define the force on the slip plane, using a similar approach treating the point singularity, we exclude an infinitesimal neighborhood of the surface singularity, such as the infinite strip V_A with height of A both in the positive and negative y-directions for

$A \ll 1$, and the surfaces $\partial V_A = S_A \cup S_{-A}$ with S_A and S_{-A} parallel and symmetric with distances of A and $-A$ to the slip plane, respectively. Here and below, all the volumes and surfaces are assumed to be of unit length in the z-direction. Then we consider the infinite strip V_A as an inhomogeneity. In an analogous way treating the point defect, the self-force F_1 in the x-direction on the surface singularity is given by the dynamic J-integral

$$\begin{aligned} F_1 &= \lim_{A\to 0}\int_{V\setminus V_A}\frac{\partial}{\partial t}[\rho\hat{u}_3\hat{u}_{3,1}]\,\mathrm{d}V + \int_{S_+\cup S_-}[(\hat{W}-\hat{T})\delta_{j1} - \hat{\sigma}_{3j}\hat{u}_{3,1}]\,\mathrm{d}S_j \\ &= \int_V \frac{\partial}{\partial t}[\rho\hat{u}_3\hat{u}_{3,1}]\,\mathrm{d}V + \int_{S_+\cup S_-}[(\hat{W}-\hat{T})\delta_{j1} - \hat{\sigma}_{3j}\hat{u}_{3,1}]\,\mathrm{d}S_j, \end{aligned} \tag{4.14}$$

where V is a volume containing the slip plane, and $\partial V = S_+ \cup S_-$ with $S_\pm$ as surfaces in the half-space $y>0$ and $y<0$, respectively. Without loss of generality, we may assume V is also infinite strip with the finite heights both in the positive and negative y-direction and $S_\pm$ parallel to the slip plane.

In view of the fact that the smeared fields are sufficiently smooth for $y>0$ and $y<0$, respectively, and the field variables $\hat{u}_{3,j}$ for $j = 1, 2, t$ are well-behaved near infinity, it is easy to show that in the infinite strips for $y>0$ and $y<0$, respectively, the Gauss divergence theorem and the conservation laws (2.9) are valid. Hence, the dynamic J-integral in (4.14) is independent of the choice of the volume V. And by using the conservation laws (2.9) in (4.14), we have

$$\begin{aligned} F_1 &= \lim_{A\to 0}\left\{\int_{V\setminus V_A}\frac{\partial}{\partial t}[\rho\hat{u}_3\hat{u}_{3,1}]\,\mathrm{d}V + \int_{S_+\cup S_-}[(\hat{W}-\hat{T})\delta_{j1} - \hat{\sigma}_{3j}\hat{u}_{3,1}]\,\mathrm{d}S_j\right\} \\ &= \lim_{A\to 0}\int_{S_A+\cup S_{-A}}[(\hat{W}-\hat{T})\delta_{j1} - \hat{\sigma}_{3j}\hat{u}_{3,1}]\,\mathrm{d}S_j. \end{aligned} \tag{4.15}$$

The last equation is further reduced to

$$F_1 = \lim_{A\to 0}\int_{-\infty}^{\infty}\left[\hat{\sigma}_{32}(x,-A,t)\hat{u}_{3,1}(x,-A,t) - \hat{\sigma}_{32}(x,A,t)\hat{u}_{3,1}(x,A,t)\right]\mathrm{d}x. \tag{4.16}$$

It is clear that the force F_1 is in general not zero since there is a jump discontinuity on the slip plane $y = 0$ (for continuous fields it would naturally be zero).

Further considering that $\hat{\sigma}_{32} = \mu\hat{u}_{3,2}$ and $\hat{u}_{3,1}$ are even and odd functions of y, respectively, from (4.16), we have

$$F_1 = \lim_{A\to 0^+} I_{SA} \equiv \lim_{A\to 0^+}\left[-2\mu\int_{-\infty}^{\infty}\hat{u}_{3,2}(x,A,t)\hat{u}_{3,1}(x,A,t)\,\mathrm{d}x\right]. \tag{4.17}$$

We will prove that

(1) $\hat{u}_{3,1}$ is essentially a delta sequence as $y \to 0$ and expressed as

$$\hat{u}_{3,1}(x,y,t) = -\frac{b}{2\pi}\frac{(\gamma y + a)}{(x-l(t))^2 + (\gamma y + a)^2} + G(x,y,t), \tag{4.18}$$

where again $\gamma = \sqrt{1 - v^2/c_2^2}$, a is the smearing parameter, and $G(x,y,t) \to 0$ as $y \to 0$ uniformly for $|x| < M$, for any positive M.

(2) Noting that $\hat{u}_{3,2}$ is a convolution of $u_{3,2}$ with a delta sequence, so that I_{SA} as an integration of $\hat{u}_{3,2}\hat{u}_{3,1}$ is approximately a convolution of $u_{3,2}$ with a delta sequence (of a different parameter). Namely,

$$I_{SA} = -2\mu\int_{-\infty}^{\infty}\hat{u}_{3,2}\hat{u}_{3,1}\,\mathrm{d}x \sim -\mu b[u_{3,2}\star g_{(2a+\gamma A)}](l(t)), \tag{4.19}$$

where

$$g_{(2a+\gamma A)} = \frac{2a+\gamma A}{\pi(x^2 + (2a+\gamma A)^2)}.$$

(3) Because, as A approaches to 0, the delta sequence concentrates on a neighborhood of the core of the dislocation $x_1 = l(t)$, so that in the convolution $u_{3,2} \star g_{(2a+\gamma A)}$, only the near field of $u_{3,2}$ plays a role. As $A \to 0$, the convolution of the near-field expansion of $u_{3,2}$ with a delta sequence yields

$$I_{SA} \sim \mu b f_{32} \ln(2a) + \frac{\mu b}{2}[g_{32}(0,t) + g_{32}(\pi,t)], \tag{4.20}$$

which represents the contribution from the logarithmic term and the $O(1)$ term in the near-field expansion of $u_{3,2}$, where again f_{32} and g_{32} are the near-field coefficients. The leading term of the near-field expansion of $u_{3,2}$ gives no contribution due to the symmetry, and the contribution from remaining terms of the near-field expansion is in higher order of a where a is the smearing parameter.

(4) Therefore, we have the expression for the self-force

$$F_1 = -\frac{\mu b \dot{v}}{4\pi c_2^2 \gamma^3} \ln(2a) + \mu b g_{32}(0,t) + O(a \ln a), \tag{4.21}$$

where $g_{32}(0,t)$ has been evaluated in Eq. (3.77).

Now we proceed to prove (1)–(4).

Proof of (1). From Eqs. (3.19) and (3.21), $u_{3,1}$ is expressed by

$$u_{3,1} = -\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} + g(x,y,t), \tag{4.22}$$

with $g(x,y,t)$ an odd continuous function in y and satisfying

$$\lim_{y \to 0} g(x,y,t) = 0, \tag{4.23}$$

for every x.

$\hat{u}_{3,1}$ is then written as

$$\begin{aligned}\hat{u}_{3,1} &= u_{3,1} \star g_a \\ &= \left[-\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} + g(x,y,t)\right] \star \frac{a}{\pi(x^2 + a^2)}.\end{aligned} \tag{4.24}$$

Calculate the first convolution, for $y > 0$,

$$\begin{aligned}\frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} \star \frac{a}{x^2 + a^2} &= \int_{-\infty}^{\infty} \frac{\gamma y}{(\xi - l(t))^2 + \gamma^2 y^2} \left[\frac{a}{(x - \xi)^2 + a^2}\right] \mathrm{d}\xi \\ &= \int_{-\infty}^{\infty} \frac{\gamma y}{\eta^2 + \gamma^2 y^2} \left[\frac{a}{(x - l(t) - \eta)^2 + a^2}\right] \mathrm{d}\eta \\ &= \frac{\pi(\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} = \pi^2 g_{(a+\gamma y)}(x - l(t)),\end{aligned} \tag{4.25}$$

where the following integral is used, for $p > 0$,

$$\int_{-\infty}^{\infty} \frac{p}{x^2 + p^2} \left[\frac{a}{(z - x)^2 + a^2}\right] \mathrm{d}x = \frac{\pi(p + a)}{z^2 + (p + a)^2}. \tag{4.26}$$

Then we obtain

$$\hat{u}_{3,1}(x,y,t) = -\frac{b}{2\pi} \frac{(\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} + G(x,y,t), \tag{4.27}$$

where $G(x,y,t)$ is defined by the integral

$$G(x,y,t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi,y,t)}{(x - \xi)^2 + a^2} \mathrm{d}\xi. \tag{4.28}$$

From (4.12),

$$\lim_{y\to 0}\hat{u}_{3,1}(x,y,t) = -\frac{b}{2\pi}\frac{a}{(x-l(t))^2+a^2}, \tag{4.29}$$

and for $|x|\leqslant M$ the convergence is uniform, where M may be any positive number. Hence, from Eqs. (4.27) and (4.29), we have

$$\lim_{y\to 0} G(x,y,t) = \lim_{y\to 0}\left[\hat{u}_{3,1}(x,y,t) + \frac{b}{2\pi}\frac{(\gamma y+a)}{(x-l(t))^2+(\gamma y+a)^2}\right] = 0. \tag{4.30}$$

The convergence is again uniform for $|x|\leqslant M$. □

Proof of (2). Substituting (4.27) into the integral expression of I_{SA}, we have

$$\begin{aligned} I_{SA} = & -2\int_{-\infty}^{\infty}\mu\hat{u}_{3,2}\hat{u}_{3,1}\,\mathrm{d}x \\ &= \frac{\mu b}{\pi}\int_{-\infty}^{\infty}\hat{u}_{3,2}(x,A,t)\frac{(\gamma A+a)}{(x-l(t))^2+(\gamma A+a)^2}\,\mathrm{d}x \\ &\quad -2\mu\int_{-\infty}^{\infty}\mu\hat{u}_{3,2}(x,A,t)G(x,A,t)\,\mathrm{d}x \\ &\equiv I_{S1}+I_{S2}, \end{aligned} \tag{4.31}$$

where I_{S1} and I_{S2} are defined correspondingly.

For the integral I_{S2}, in view of the well behavior of the integrand at infinity, the improper integral is uniformly convergent with respect to A. Because $\hat{u}_{3,2}$ is bounded, and as shown in Eq. (4.30) $G(x,A,t)\to 0$ as $A\to 0$ uniformly for $|x|\leqslant M$ where M is a positive number, then the limit of $A\to 0$ may be taken under the integral, and we have

$$\lim_{A\to 0} I_{S2} = 0. \tag{4.32}$$

As for the integral I_{S1},

$$\begin{aligned} I_{S1} &= \frac{\mu b}{\pi}\int_{-\infty}^{\infty}\hat{u}_{3,2}(x,A,t)\frac{(\gamma A+a)}{(x-l(t))^2+(\gamma A+a)^2}\,\mathrm{d}x \\ &= \mu b[\hat{u}_{3,2}\star g_{(a+\gamma A)}](l(t)). \end{aligned} \tag{4.33}$$

Noting that $\hat{u}_{3,2} = u_{3,2}\star g_a$ and applying the associativity property of the convolution, we have

$$\begin{aligned} I_{S1} &= \mu b[\hat{u}_{3,2}\star g_{(a+A)}](l(t)) = \mu b[(u_{3,2}\star g_a)\star g_{(a+\gamma A)}](l(t)) \\ &= \mu b[u_{3,2}\star(g_a\star g_{(a+\gamma A)})](l(t)) \\ &= \mu b[u_{3,2}\star g_{(2a+\gamma A)}](l(t)), \end{aligned} \tag{4.34}$$

where from Eq. (4.26),

$$\begin{aligned} g_a\star g_{(a+\gamma A)} &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{a}{\eta^2+a^2}\left[\frac{a+\gamma A}{(x-\eta)^2+(a+\gamma A)^2}\right]\mathrm{d}\eta \\ &= \frac{1}{\pi}\frac{(2a+\gamma A)}{(x)^2+(2a+\gamma A)^2} = g_{(2a+\gamma A)}. \end{aligned} \tag{4.35}$$

Hence, as $A \to 0$,

$$I_{SA} \sim I_{S1} = \mu b[u_{3,2} \star g_e](l(t)) = \frac{\mu b}{\pi}\frac{1}{\pi}\int_{-\infty}^{\infty} u_{3,2}(\xi, A, t)\frac{e}{(\xi - l(t))^2 + e^2}\,d\xi = \frac{\mu b}{\pi}\frac{1}{\pi}\int_{-\infty}^{\infty} u_{3,2}(\eta + l(t), A, t)\frac{e}{\eta^2 + e^2}\,d\eta, \tag{4.36}$$

where $e \equiv 2a + \gamma A$. □

Proof of (3). From the meaning of the asymptotic expansion, there is a sufficiently small number $\zeta > 0$, such that when $\varepsilon^2 = \eta^2 + A^2 \leqslant \zeta^2$, for $A \geqslant 0$, $u_{3,2}(\eta + l(t), A, t) = u_{3,2}(\varepsilon, \theta, t)$ has the asymptotic expansion

$$u_{3,2}(\varepsilon, \theta, t) = \frac{b}{2\pi}\frac{\gamma\eta}{\eta^2 + \gamma^2 A^2} + f_{32}\ln\varepsilon + g_{32}(\theta, t) + p_4(\varepsilon, \theta, t), \tag{4.37}$$

where $0 \leqslant \theta = \tan^{-1}(A/\eta) \leqslant \pi$, f_{32} and g_{32} are the near-field coefficients, and

$$|p_4| \leqslant M(\theta)\varepsilon, \tag{4.38}$$

and $M(\theta)$ is a bounded function.

We decompose the integral domain in I_{S1} into two parts: (i) the near-field: $\eta^2 + A^2 \leqslant \zeta^2$; (ii) the far field: $\eta^2 + A^2 > \zeta^2$ and write the integral I_{S1} as

$$I_{S1} = I_N + I_F \equiv \frac{\mu b}{\pi}\int_{-q}^{q}\left[\frac{b}{2\pi}\frac{\gamma\eta}{\eta^2 + \gamma^2 A^2} + f_{32}\ln\varepsilon + g_{32}(\theta, t) + p_4(\varepsilon, \theta, t)\right]\frac{e}{\eta^2 + e^2}\,d\eta + \frac{\mu b}{\pi}\int_{\eta > q} u_{3,2}(\eta + l(t), A, t)\frac{e}{\eta^2 + e^2}\,d\eta, \tag{4.39}$$

where $q = \sqrt{\zeta^2 - A^2}$, I_N and I_F are the integrals over the near field and far field, respectively.

We shall show that for $a \ll 1$,

(i) $$\lim_{A \to 0} I_F = O(a). \tag{4.40}$$

(ii) $$\lim_{A \to 0} I_N = \mu b f_{32}\ln(2a) + \mu b g_{32}(0, t) + O(a\ln a) \tag{4.41}$$

To show Eq. (4.40), we note that for $\eta > q = \sqrt{\zeta^2 - A^2}$, or equivalently, $\eta^2 + A^2 > \zeta^2$, $u_{3,2}(\eta + l(t), A, t)$ is bounded, i.e., for $\eta > q$,

$$\frac{\mu b}{\pi}|u_{3,2}(\eta + l(t), A, t)| \leqslant M_1, \tag{4.42}$$

for some bound $M_1 > 0$. From the definition of I_F and Eq. (4.42), it follows that

$$|I_F| \leqslant M_1\int_{\eta > q}\frac{e}{\eta^2 + e^2}\,d\eta = 2M_1\int_{q}^{\infty}\frac{e}{\eta^2 + e^2}\,d\eta = 2M_1\int_{q/e}^{\infty}\frac{d\zeta}{\zeta^2 + 1} = 2M_1 e/q + \text{h.o.t.}(e), \tag{4.43}$$

where again $e = 2a + \gamma A$ and $q = \sqrt{\zeta^2 - A^2}$. Hence, as $A \to 0$, I_F is of the order of $O(a)$ for $a \ll 1$.

For I_N, we write

$$I_N = \frac{\mu b}{\pi}\int_{-q}^{q}\left[\frac{b}{2\pi}\frac{\gamma\eta}{\eta^2+\gamma^2 A^2} + f_{32}\ln\varepsilon + g_{32}(\theta,t) + p_4(\varepsilon,\theta,t)\right]\frac{e}{\eta^2+e^2}\,d\eta. \tag{4.44}$$

By symmetry, the integral of the first term is zero, i.e.,

$$\int_{-q}^{q}\left[\frac{b}{2\pi}\frac{\gamma\eta}{\eta^2+\gamma^2 A^2}\frac{e}{\eta^2+e^2}\right]d\eta = 0. \tag{4.45}$$

The integral of the second term is

$$\frac{\mu b}{\pi}\int_{-q}^{q} f_{32}\ln(\eta^2+A^2)\frac{e}{\eta^2+e^2}\,d\eta = \frac{\mu b}{\pi}\int_{-\infty}^{\infty} f_{32}\ln(\eta^2+A^2)\frac{e}{\eta^2+e^2}\,d\eta - \frac{\mu b}{\pi}\int_{q}^{\infty} f_{32}\ln(\eta^2+A^2)\frac{e}{\eta^2+e^2}\,d\eta, \tag{4.46}$$

where again f_{32} is independent of η.

For the first integral in Eq. (4.46),

$$\frac{\mu b}{\pi}\int_{-\infty}^{\infty} f_{32}\ln(\eta^2+A^2)\frac{e}{\eta^2+e^2}\,d\eta = \mu b f_{32}\ln(A+e) = \mu b f_{32}\ln(2a+2A). \tag{4.47}$$

In the second integral in (4.46), by changing variables of integration, we obtain

$$\left|\int_{q}^{\infty}\ln(A^2+\eta^2)\frac{e}{\eta^2+e^2}\,d\eta\right| = \left|\int_0^{e/q}\frac{\ln(A^2+e^2/\xi^2)}{1+\xi^2}\,d\xi\right| \leqslant \int_0^{e/q}\left|\frac{2\ln(\xi)}{1+\xi^2}\right|d\xi + \int_0^{e/q}\left|\frac{\ln(A^2\xi^2+e^2)}{1+\xi^2}\right|d\xi. \tag{4.48}$$

The last two integrals are of the order of $O(a\ln a)$ as $A\to 0$, since

$$\int_0^{e/q}\left|\frac{2\ln(\xi)}{1+\xi^2}\right|d\xi \leqslant \left|\int_0^{e/q} 2\ln(\xi)\,d\xi\right| = |e/q(1-\ln(e/q)| = O(e/q\ln(e/q)) \tag{4.49}$$

and

$$\int_0^{e/q}\left|\frac{\ln(A^2\xi^2+e^2)}{1+\xi^2}\right|d\xi \leqslant \int_0^{e/q}|\ln(A^2(e/q)^2+e^2)|\,d\xi = O(e/q\ln(e)). \tag{4.50}$$

The integral of the third term in Eq. (4.44) is rewritten as

$$\frac{\mu b}{\pi}\int_{-q}^{q} g_{32}(\theta,t)\frac{e}{\eta^2+e^2}\,d\eta = \frac{\mu b}{\pi}\int_{-\delta}^{\delta} g_{32}(\theta,t)\frac{e}{\eta^2+e^2}\,d\eta + \frac{\mu b}{\pi}\left[\int_{-q}^{-\delta} + \int_{-q}^{-\delta}\right] g_{32}(\theta,t)\frac{e}{\eta^2+e^2}\,d\eta, \tag{4.51}$$

where $\delta = o(a)$ is a small number. Then

$$\left|\int_{-\delta}^{\delta} g_{32}(\theta,t)\frac{e}{\eta^2+e^2}\,d\eta\right| \leqslant 2M_0\int_0^{\delta}\frac{e}{\eta^2+e^2}\,d\eta = 2M_0\tan^{-1}(\delta/e) = O(a), \tag{4.52}$$

since g_{32} is bounded in the near field for $t>0$.

For the second integral on the right-hand side of Eq. (4.51), the limit of $A \to 0$ may be taken under the integral, so we have

$$\begin{aligned}\lim_{A\to 0}\frac{\mu b}{\pi}\int_{-q}^{q} g_{32}(\theta,t)\frac{e}{\eta^2+e^2}\,\mathrm{d}\eta &= \frac{\mu b}{\pi}[g_{32}(0,t)+g_{32}(\pi,t)]\int_{\delta}^{\zeta}\frac{e}{\eta^2+e^2}\,\mathrm{d}\eta + o(a)\\ &= \frac{\mu b}{2}[g_{32}(0,t)+g_{32}(\pi,t)]\tan^{-1}(\zeta/a)+O(a)\\ &= [g_{32}(0,t)+g_{32}(\pi,t)]\left(\frac{\pi}{2}-e/q\right)+O(a)\\ &= \frac{\pi}{2}[g_{32}(0,t)+g_{32}(\pi,t)]+O(a).\end{aligned} \tag{4.53}$$

Furthermore, as $A \to 0$,

$$\left|\int_{-q}^{q} p_1\frac{e}{\eta^2+e^2}\,\mathrm{d}\eta\right| \leqslant M_2\int_{-q}^{q}\frac{e\,\eta}{\eta^2+e^2}\,\mathrm{d}\eta = M_2 e\ln(q^2+e^2) = O(a). \tag{4.54}$$

Hence, combining the above results, we obtain, for $a \ll 1$,

$$\lim_{A\to 0} I_{SA} = \mu b f_{32}\ln(2a) + \frac{\mu b}{2}[g_{32}(0,t)+g_{32}(\pi,t)] + O(a\ln a). \tag{4.55}$$

The evaluation (4.40) and (4.41) are then proved. □

Proof of (4). Consequently, the self-force is thus given by

$$\begin{aligned}F_1 &= \lim_{A\to 0} I_{SA} = \lim_{A\to 0}[I_{S1}+I_{S2}] = \lim_{A\to 0}[I_{\mathrm{F}}+I_{\mathrm{N}}]\\ &= \frac{\mu b f_{32}}{2}\ln(2a) + \frac{\mu b}{2}[g_{32}(0,t)+g_{32}(\pi,t)] + O(a\ln a).\end{aligned} \tag{4.56}$$

Note that

$$g_{32}(\pi,t) = \int_0^{\pi} g'_{32}(\theta,t)\,\mathrm{d}\theta + g_{32}(0,t), \tag{4.57}$$

where as in Eq. (3.56), $g'_{32}(\theta,t)$ is given by

$$g'_{32} = -\frac{b\dot{v}\cos\theta\sin\theta}{4\pi c_2^2\gamma}\left[\frac{(7\gamma^2-5)\cos^2\theta-\gamma^2(\gamma^2-3)\sin^2\theta}{(\cos^2\theta+\gamma^2\sin^2\theta)^3}\right]. \tag{4.58}$$

It is then easy to see that

$$\int_0^{\pi} g'_{32}(\theta,t)\,\mathrm{d}\theta = 0, \tag{4.59}$$

it follows that $g_{32}(\pi,t) = g_{32}(0,t)$. Therefore,

$$F_1 = \frac{\mu b f_{32}}{2}\ln(2a) + \mu b g_{32}(0,t) + O(a\ln a), \tag{4.60}$$

which completes the proof of (4). □

Therefore, from Eq. (4.60) and the expression (3.77) for $g_{32}(0,t)$, we have the self-force on an accelerating screw dislocation based on a smearing (ramp-core) method,

$$F_1 = F_1^{\mathrm{in}} + F_1^{\mathrm{non}} + \text{h.o.t.}, \tag{4.61}$$

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where F_1^{in} and F_1^{non} are the inertial part and the non-inertial part of the self-force, are given, for $t>0$, respectively as follows:

$$F_1^{\text{in}} = \frac{\mu b^2 \dot{v}(t)}{2\pi} \left\{ \frac{-\ln(2a)}{2c^2\gamma^3} + \frac{[1+\ln(\beta/(2\gamma^2)]}{2c^2\gamma^3} + \frac{[4(1-2\beta)+\beta^2(2-\beta)]}{2c^2(1-\beta)^4} \ln\left(\frac{1+\gamma}{\beta}\right) + \frac{(1+\beta^2)}{2c^2\beta(1-\beta)^2\gamma} \right\}, \tag{4.62}$$

$$\begin{aligned} F_1^{\text{non}} = \frac{\mu b^2}{2\pi} \Bigg\{ & \int_0^{\eta(\omega)} \frac{c(t-\tau) l(\tau)\,\mathrm{d}\tau}{(l(t)-l(\tau))^2 [c^2(t-\tau)^2 - (l(t)-l(\tau))^2]^{1/2}} + \frac{1}{l(t)} \\ & + \int_{\eta(\omega)}^t \frac{\ln(l(t)-l(\tau)) K(t,\tau,v(\tau))\,\mathrm{d}\tau}{[c^2(t-\tau)^2 - (l(t)-l(\tau))^2]^{5/2}} \\ & - \frac{c(t-\eta(\omega))}{(l(t)-\omega)^2 [c^2(t-\eta(\omega))^2 - (l(t)-\omega)^2]^{1/2}} \\ & + \frac{c \ln(l(t)-\omega)[(t-\eta(\omega))v(\eta(\omega)) - (l(t)-\omega)](l(t)-\omega)}{v(\eta(\omega))[c^2(t-\eta(\omega))^2 - (l(t)-\omega)^2]^{3/2}} \Bigg\}, \end{aligned} \tag{4.63}$$

where $a>0$ is the dimensionless smearing parameter defined in Eq. (4.4), $c = c_2$, $v(\tau) = \dot{l}(\tau)$, and $K(t,\tau,v(\tau))$ is defined by

$$\begin{aligned} K(t,\tau,v(\tau)) = & \frac{c}{v^2(\tau)[c^2(t-\tau)^2 - (l(t)-l(\tau))^2]^{5/2}} \\ & \times \{ v^3(\tau)(t-\tau)[c^2(t-\tau)^2 + 2(l(t)-l(\tau))^2] \\ & - 2v^2(\tau)(l(t)-l(\tau))[2c^2(t-\tau)^2 + (l(t)-l(\tau))^2] \\ & - 3c^2 v(\tau)(t-\tau)(l(t)-l(\tau))^2 \\ & - \dot{v}(\tau)(l(t)-l(\tau))^2[c^2(t-\tau)^2 - (l(t)-l(\tau))^2]\}. \end{aligned} \tag{4.64}$$

In the expression (4.63), the integrals represent the terms dependent on the history of the motion.

In the above expressions, it has been verified by direct calculation that the length-scale effects of L_0 arising from the $\ln \varepsilon$ and $O(1)$ terms in Theorem 3 cancel each other.

4.3. The effective mass

From the self-force, we derive the effective mass for an accelerating screw dislocation. As discussed in Section 2, the effective mass is defined analogously to Newton's law,

$$F^{\text{in}} = \frac{\mathrm{d}}{\mathrm{d}t}(m_{\text{e}} v), \tag{4.65}$$

and given by

$$m_{\text{e}} = \frac{1}{v(t)} \int_0^t F^{\text{in}}\,\mathrm{d}t, \tag{4.66}$$

for $t>0$, and where $v(t) = \dot{l}(t)$, F^{in} is the inertial part of the self-force F.

Noting that expression (4.62) is valid for $t>0$, e.g., it is valid on the closed interval $[t_\varepsilon, t]$, for a positive t_ε near 0, and that at $t=0$, $m_{\text{e}} v$ is zero, we may decompose the integral in (4.66) into two parts

$$\int_0^t F^{\text{in}}\,\mathrm{d}t = \int_{t_\varepsilon}^t F^{\text{in}}\,\mathrm{d}t + \int_0^{t_\varepsilon} F^{\text{in}}\,\mathrm{d}t = \int_{t_\varepsilon}^t F^{\text{in}}\,\mathrm{d}t + \int_0^{t_\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}(m_{\text{e}} v)\,\mathrm{d}t. \tag{4.67}$$

For the first term of the right-hand side of the last equation, the near-field expansion (4.62) is applicable, the integrated terms at the lower and upper limits for the first and second integrals, respectively, cancel out, and the integrated term at $t=0$ for the second term of the right-hand side of the last equation vanishes. Then, the net contribution to the integral in Eq. (4.66) is

$$\int_0^t F^{\text{in}}\,\mathrm{d}t = \Phi(t), \tag{4.68}$$

where Φ is the time dependent part of the antiderivative of the near-field expression (4.62). From Eqs. (4.62), (4.66) and (4.68), we hence obtain the expression for the effective mass

$$\begin{aligned} m_{\mathrm{e}} = \frac{\mu b^2}{2\pi}\Bigg\{ & \frac{-\ln(2a)}{2c^2\gamma} + \frac{1}{2c^2\gamma}\left[\ln\left(\frac{\beta}{2\gamma}\right) - 1\right] \\ & + \frac{1}{2c^2\beta}\tan^{-1}\left(\frac{\beta}{\gamma}\right) + \frac{(2-\beta)\gamma}{3c^2\beta(1-\beta)^2} + \frac{1}{2c^2\beta}\ln\left(\frac{\beta}{1+\gamma}\right) \\ & + \frac{1}{2c^2\beta}\int_0^\beta \frac{4-8r+2r^2-r^3}{(1-r)^4}\ln\left(\frac{1+\sqrt{1-r^2}}{r}\right)\mathrm{d}r\Bigg\}, \end{aligned} \tag{4.69}$$

for $t>0$, and $c=c_2$, $\beta = v/c$, $\gamma = \sqrt{1-\beta^2}$.

5. The self-force and effective mass based on the theory of distributions

5.1. Regularization of divergent integrals

When an integral does not exist as the usual improper integral, then the divergent integral may possibly be regularized based on the theory of distributions.

Distributions are linear continuous functionals on the space of the fundamental functions (or, test functions). One example of the space of fundamental functions is $\mathscr{K} = C_0^\infty(R^n)$, i.e., the space of infinitely differentiable functions with compact supports in R^n. A locally integral function f on R^n corresponds to a linear continuous functional on $\mathscr{K}$, which is defined as

$$(f,\phi) = \int_{R^n} f(x)\phi(x)\,\mathrm{d}x, \tag{5.1}$$

where $\phi \in \mathscr{K}$, and x denotes an n-dimensional vector.

If f is not locally integrable, then the integral (5.1) does not exist for all $\phi \in \mathscr{K}$. Suppose that f only has isolated singularities, then the regularization $\mathscr{R}eg.f$ of f satisfies the following conditions (Gel'fand and Shilov, vol. I, 1964):

(i) $\mathscr{R}eg.f$ is a linear continuous functional on $\mathscr{K}$.
(ii) For all $\phi \in \mathscr{K}$ which vanishes in a neighborhood of a singularity x_0 of f,

$$(\mathscr{R}eg.f,\phi) = \int_{R^n} f(x)\phi(x)\,\mathrm{d}x. \tag{5.2}$$

There are several ways to construct a regularization, e.g.:

(1) *The CPV.*

$$(\mathscr{R}eg.f,\phi) = ⨍ f(x)\phi(x)\,\mathrm{d}x, \tag{5.3}$$

if it exists. The CPV has been used in fracture mechanics, and used to calculate the energy of elastic defects (Dascalu and Maugin, 1994).

(2) *The Hadamard finite part.*

$$(\mathscr{R}eg.f,\phi) = ⨎ f(x)\phi(x)\,\mathrm{d}x, \tag{5.4}$$

if the right-hand side exists, which denotes the Hadamard finite part of a divergent integral. The Hadamard finite part of a divergent integral is defined by removing the divergent part of the integral and keep the finite part. (see, Hadamard, 1952; Kanwal, 1998; Estrada and Kanwal, 1994).

The finite part integral arises naturally in problems of bridged cracks in fracture mechanics, see e.g., Nemat-Nasser and Hori (1987), Willis and Nemat-Nasser (1990), Hori and Nemat-Nasser (1990), Nemat-Nasser and Hori (1999), and Ni and Nemat-Nasser (2000).

(3) If $f(x)$ has an algebraic singularity at x_0, i.e., $f(x)|x-x_0|^m$ is locally integrable for an integer $m>0$, then

$$(\mathscr{R}eg.f,\phi)=\int_{U\setminus B_a} f(x)\phi(x)\,\mathrm{d}x+\int_{B_a} f(r,\theta_j)\Bigg[\phi(r,\theta_j) - \sum_{|k|<m-n+1}\frac{1}{k!}\partial_r^k\phi(x_0,\theta_j)r^k\Bigg]r^{n-1}\,\mathrm{d}r\,\mathrm{d}\Omega, \tag{5.5}$$

where B_a is a closed ball of the dimensionless radius a at x_0, a is an arbitrary positive number, r and θ_j, for $1\leqslant j\leqslant n-1$ are the spherical coordinate variables at x_0, and $\mathrm{d}\Omega$ is the area element of the unit ball at x_0 (Gel'fand and Shilov, 1964; Kanwal, 1998).

For the regularization of a singular distribution defined by those three ways, it is easy to verify that the conditions (i) and (ii) of a regularization are satisfied.

For a divergent integral, we may decompose the integrand into two factors with one in the function space $\mathscr{K}$ and for the other its regularization is available. Then we view the divergent integral as a linear continuous functional, corresponding to a singular distribution, evaluated on an element in the infinitely differentiable test function space, which gives the regularization of the divergent integral.

5.2. The self-force based on regularization

As shown in Section 3, in the surface-independent dynamic J-integral for the self-force on an accelerating (screw) dislocation, the volume integral does not exist even in the sense of CPV. Specifically, here we seek to use (5.5) to regularize the divergent volume integral in Eq. (2.16),

$$\int_V g\,\mathrm{d}v\equiv\int_V\frac{\partial}{\partial t}[\rho\dot{u}_3u_{3,l}]\,\mathrm{d}v. \tag{5.6}$$

As discussed before, we choose V to be a cylindrical volume around the dislocation line. The problem may hence be considered to be two-dimensional, all volumes and surfaces have an unite length in the z-direction. The regularization is performed in the two-dimensional framework. Namely, we may choose a sufficiently small positive number q, and define the regularization of the integral through the regularization of a singular distribution, specifically by using (Gel'fand and Shilov, 1964; Kanwal, 1998):

$$(\mathscr{R}eg.g,\phi)=\int_{V\setminus B_q} g(x)\phi(x)\,\mathrm{d}V + \int_{B_q} g(r,\theta)\Bigg[\phi(r,\theta)-\sum_{|k|<m}\frac{\partial^k}{\partial r^k}\phi(x_0,\theta)r^k/k!\Bigg]r\,\mathrm{d}r\,\mathrm{d}\theta, \tag{5.7}$$

where B_q is a closed circle of dimensionless radius q at the core of dislocation x_0, r and θ are the circular coordinate variables at x_0. The infinitely differentiable function $\phi(x)\in\mathscr{K}=C_0^\infty$ is chosen to be identical to 1 on V with a support in a small neighborhood of V. Then, the regularization of Eq. (5.6) is written as

$$\mathscr{R}eg. \int_V g(x)\,\mathrm{d}v = \int_{V\backslash B_q} g(x)\,\mathrm{d}V. \tag{5.8}$$

The self-force on a generally accelerating screw dislocation in the x-direction is then expressed as

$$F_1 = \int_{V\backslash B_q} g(x)\,\mathrm{d}V + \int_S [(W-T)\delta_{1j} - \sigma_{3j}u_{3,j}]\,\mathrm{d}S_j, \tag{5.9}$$

where $S = \partial V$. By using the conservation laws (2.9) over the homogeneous region $V\backslash B_q$, (5.9) is further reduced to

$$F_1 = \int_{S_q} [(W-T)\delta_{1j} - \sigma_{3j}u_{3,j}]\,\mathrm{d}S_j, \tag{5.10}$$

where $S_q \equiv \partial B_q$. In other words, it follows that

$$F_1 = I_q, \tag{5.11}$$

where $I_q = I_\varepsilon$ ia evaluated by (3.80) for $\varepsilon = q$. Therefore, from Eq. (3.80), we have

$$F_1 = -\frac{\mu b^2 \dot{v}}{4\pi c_2^2 \gamma^3}\ln q + \frac{\mu b^2 \dot{v}}{4\pi c_2^2\gamma^3}\left[\ln(\gamma(1+\gamma)/2) - \frac{4+\beta^4-\beta^2(7+2\gamma)}{(1+\gamma)^2}\right] + \mu b g_{32}(0,t) + \text{h.o.t.} \tag{5.12}$$

Now $0 < q \ll 1$ is a fixed number, and the self-force F_1 is well-defined.

From Eq. (5.12) and the expression (3.77) for $g_{32}(0,t)$, the explicit expression of F_1 based on the theory of distributions is obtained as follows:

$$F_1 = F_1^{\text{in}} + F_1^{\text{non}} + \text{h.o.t.}, \tag{5.13}$$

where F_1^{in} and F_1^{non} are the inertial part and the remain part of the self-force, are given, for $t > 0$, respectively, as follows, where again $c = c_2$,

$$\begin{aligned} F_1^{\text{in}} = {} & \frac{\mu b^2 \dot{v}(t)}{2\pi}\left\{\frac{-\ln(q)}{2c^2\gamma^3} + \frac{[1+\ln(\beta/(2\gamma^2)]}{2c^2\gamma^3}\right. \\ & \left. + \frac{[4(1-2\beta)+\beta^2(2-\beta)]}{2c^2(1-\beta)^4}\ln\left(\frac{1+\gamma}{\beta}\right) + \frac{(1+\beta^2)}{2c^2\beta(1-\beta)^2\gamma}\right\} \\ & + \frac{\mu b^2 \dot{v}}{4\pi c^2\gamma^3}\left[\ln(\gamma(1+\gamma)/2) - \frac{4+\beta^4-\beta^2(7+2\gamma)}{(1+\gamma)^2}\right], \end{aligned} \tag{5.14}$$

$$\begin{aligned} F_1^{\text{non}} = {} & \frac{\mu b^2}{2\pi}\left\{\int_0^{\eta(\omega)} \frac{c(t-\tau)l(\tau)\,\mathrm{d}\tau}{(l(t)-l(\tau))^2[c^2(t-\tau)^2 - (l(t)-l(\tau))^2]^{1/2}} + \frac{1}{l(t)}\right. \\ & + \int_{\eta(\omega)}^t \frac{\ln(l(t)-l(\tau))K(t,\tau,v(\tau))\,\mathrm{d}\tau}{[c^2(t-\tau)^2-(l(t)-l(\tau))^2]^{5/2}} \\ & - \frac{c(t-\eta(\omega))}{(l(t)-\omega)^2[c^2(t-\eta(\omega))^2-(l(t)-\omega)^2]^{1/2}} \\ & \left. + \frac{c\ln(l(t)-\omega)[(t-\eta(\omega))v(\eta(\omega)) - (l(t)-\omega)](l(t)-\omega)}{v(\eta(\omega))[c^2(t-\eta(\omega))^2-(l(t)-\omega)^2]^{3/2}}\right\}, \end{aligned} \tag{5.15}$$

where q is the regularization parameter, $c=c_2$, $v(\tau)=\dot{l}(\tau)$, and $K(t,\tau,v(\tau))$ is defined by

$$\begin{aligned}K(t,\tau,v(\tau)) = & \frac{c}{v^2(\tau)[c^2(t-\tau)^2-(l(t)-l(\tau))^2]^{5/2}}\\ & \times\{v^3(\tau)(t-\tau)[c^2(t-\tau)^2+2(l(t)-l(\tau))^2]\\ & -2v^2(\tau)(l(t)-l(\tau))[2c^2(t-\tau)^2+(l(t)-l(\tau))^2]\\ & -3c^2v(\tau)(t-\tau)(l(t)-l(\tau))^2\\ & -\dot{v}(\tau)(l(t)-l(\tau))^2[c^2(t-\tau)^2-(l(t)-l(\tau))^2]\}.\end{aligned} \tag{5.16}$$

In the expressions (5.15), the integrals represent the terms dependent on the history of the motion. From the comparison between the self-force, Eqs. (4.62) and (4.63), evaluated by the smearing (ramp-core) approach and the one, Eqs. (5.14) and (5.15), obtained by the distribution approach, it is seen that the evaluations of the self-force on a generally accelerating screw dislocation by both methods agree up to the leading terms.

5.3. The effective mass

As discussed before, the effective mass is defined analogously to Newton's law

$$F^{\mathrm{in}}=\frac{\mathrm{d}}{\mathrm{d}t}(m_{\mathrm{e}}v), \tag{5.17}$$

and given by

$$m_{\mathrm{e}}=\frac{1}{v(t)}\int_0^t F^{\mathrm{in}}\,\mathrm{d}t, \tag{5.18}$$

for $t>0$, and where F^{in} is the inertial part of the self-force.

In a similar manner as discussed in previous section, by using the expression (5.14) of F^{in}, we obtain the effective mass

$$\begin{aligned}m_{\mathrm{e}}=\frac{\mu b^2}{2\pi}\Bigg\{ & \frac{-\ln(4q)}{2c^2\gamma^3}+\frac{8}{3c^2\gamma}+\frac{\ln(1+\gamma)}{2c^2\gamma}-\frac{\gamma}{3c^2(1-\beta)^2}\\ & +\frac{\tan^{-1}(\beta/\gamma)}{2c^2\beta}-\frac{3\sin^{-1}\beta}{2c^2\beta}-\frac{\ln((1+\beta)/(1-\beta))}{4c^2\beta}\\ & +\frac{\ln\beta}{2c^2\gamma}+\frac{2\gamma}{3c^2\beta(1-\beta)^2}+\frac{\ln(\beta/(1+\gamma))}{2c^2\beta}+\frac{1}{c^2\beta^2}\\ & +\frac{4}{3c^2\beta^2\gamma(1+\gamma)}-\frac{5}{3c^2\beta^2\gamma}\\ & +\frac{1}{2c^2\beta}\int_0^\beta\frac{4-8r+2r^2-r^3}{(1-r)^4}\ln\left(\frac{1+\sqrt{(1-r^2)}}{r}\right)\mathrm{d}r\Bigg\},\end{aligned} \tag{5.19}$$

for $t>0$, and again $c=c_2$, $\beta=v/c$, $\gamma=\sqrt{(1-\beta^2)}$.

6. Concluding remarks

By using the dynamic J-integral, explicit and complete expressions for the self-force and effective mass of a generally accelerating screw dislocation are obtained. An essential part for this calculation is the complete evaluation for the near-field expansions (Section 3). Due to the divergence of the volume integral in the dynamic J-integral for the case of a generally accelerating motion, two methods are used to treat it: smearing the singularity of the core (ramp-core), and regularization based on the theory of distributions. In the smearing (ramp-core) method, since there is surface-independence of the dynamic J-integral, the infinite strip is chosen for the evaluation as mathematically simpler and physically meaningful. The self-force is then given

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in terms of an integral over the infinite length of the strip containing the smeared fields, which is then reduced to a convolution of the Volterra fields with a delta sequence. In the convolution, only the near-field (of the Volterra) plays a role. Finally, after taking the limit of the width of the strip $A \to 0$, the self-force is obtained by Eq. (4.60) in terms of the smearing parameter a and the near-field coefficients derived by a singular perturbation analysis in Section 3. In the method based on the theory of distributions (Gel'fand and Shilov, 1964, vol. I; and Kanwal, 1998), the divergent volume integral in the dynamic J-integral is regularized, and the evaluated self-force depends on a length parameter q. The small parameter used in both methods, which corresponds to the cut-off core radius—and is undetermined here—will have to be determined by matching the self-force of this continuum dislocation model to the self-force of a lattice scale one, such as the model of Kresse and Truskinovsky (2003).

The methods developed here for the calculation of the self-force and effective mass of a generally accelerating screw dislocation can be also applied to those of a generally accelerating edge dislocation, which will be presented in a separate paper (part II).

In principle, the approaches applied in the present analysis can be extended to calculate the self-force on a moving dislocation loop, the radiated near fields of which have been investigated by using singular perturbation analysis in Markenscoff and Ni (1990).

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References

Al'shitz, V.I., Indenbom, V.L., Shtol'berg, A.A., 1971. The Peierls dynamic force. Soviet Phys. JEPT 33 (6), 1240–1245.

Atkinson, C., Eshelby, J.D., 1968. The flow of energy into the tip of a moving crack. Int. J. Fracture Mechanics 4, 3–8.

Atluri, S.N., 1986. Engineering approaches and path-independent integrals in fracture mechanics. In: Atluri, S.N. (Ed.), Computational Methods in Fracture Mechanics. Springer, Berlin (Chapter 5).

Atluri, S.N., Nishioka, T., 1983. Path-independent integrals, energy release rates and general solutions of near-tip fields in mixed mode dynamic fracture mechanics. Eng. Fract. Mech. 18, 1–22.

Bui, H.D., 1977. Stress and crack-displacement intensity factors in elastodynamics. In: Fracture vol. 3, ICF4, Waterloo, Canada, June 19–24, 1977, pp. 91–95.

Callias, C., Markenscoff, X., 1988. Singular asymptotics of integrals and the near-field radiated from nonuniformly moving dislocations. Arch. Ration. Mech. Anal. 102 (3), 273–285.

Clifton, R.J., Markenscoff, X., 1981. Radiation from non-uniformly moving dislocation and elastic precursor decay. J. Mech. Phys. Solids 29 (3), 227–251.

Dascalu, C., Maugin, G.A., 1993. Forces matérielles et taux de restitution de l'énergie dans les corps élastiques homogénes avec défauts. C. R. Acad. Sci. Paris, t. 317, Série II, 1135–1140.

Dascalu, C., Maugin, G.A., 1994. The energy of elastic defects: a distributional approach. Proc. R. Soc. Lond. A. 445, 23–27.

EDM, 1993. Encyclopedic Dictionary of Mathematics, second ed., 1993. The Mathematical Society of Japan, the MIT Press, Cambridge, MA, London, England.

Eshelby, J.D., 1951. The force on an elastic singularity. Philos. Trans. R. Soc. Lond. A244, 87–112.

Eshelby, J.D., 1953. The equation of motion of a dislocation. Phys. Rev. 90 (2), 248–255.

Eshelby, J.D., 1970. Energy relation of the energy-momentum tensor in continuum mechanics. In: Kannien, M.F., Adler, W.F., Rosenfeld, A.R., Taffee, R.I. (Eds.), Inelastic Behavior of Solids.

Eshelby, J.D., 1975. The elastic energy-momentum tensor. J. Elasticity 5, 321–335.

Eshelby, J.D., 1977. Lecture on the elastic energy-momentum tensor. In: Lecture Notes (handwritten), Collected Works Eshelby, J.D., Markenscoff, X., Gupta, A. (Eds.), Springer, 2006.

Estrada, R., Kanwal, R.P., 1994. Asymptotic Analysis: A Distributional Approach. Birkhäuser, Boston.

Fletcher, D.C., 1976. Conservation laws in linear elastodynamics. Arch. Ration Mech. Anal. 60, 329–353.

Frank, F.C., 1949. On the equation of motion of crystal dislocations. Proc. Phys. Soc. A 62, 131–134.

Freund, L.B., 1972. Energy flux into the tip of an extending crack in an elastic solid. J. Elasticity 2, 341–349.

Freund, L.B., 1990. Dynamic Fracture Mechanics. Cambridge University Press, Cambridge.

Friedrichs, K.O., 1953. On the differentiability of solutions of linear elliptic differential equations. Commun. Pure. Appl. Math. 6, 299–326.

Gavazza, S.D., Barnett, D.M., 1976. The self-force on a planar dislocation loop in an anisotropic linear elastic medium. J. Mech. Phys. Solids 24, 171–185.

Gel'fand, I.M., Fomin, S.V., 1963. Calculus of Variations. Prentice-Hall Inc., Englewood Cliffs, NJ.

Gel'fand, I.M., Shilov, G.E., 1964. Generalized Functions, vol. 1. Academic press, New York.
Günther, W., 1962. Uber einge Randintegrale der Elastodymanik. Abh. Braunschw. Wiss. Ges. 14, 54–72.
Gurtin, M.E., 2000. Configurational Forces as Basic Concepts of Continuum Physics. Springer, New York.
Hadamard, J., 1952. Lectures on Cauchy's Problem in Linear Differential Equations. Dover, New York.
Hirth, J.P., Lothe, J., 1968. Theory of Dislocations. McGraw-Hill, New York.
Hirth, J.P., Zbib, H., Lothe, J., 1998. Force on high velocity dislocations. Modeling Simul. Mater. Sci. Eng. 6, 165–169.
Hori, M., Nemat-Nasser, S., 1990. Asymptotic solution of a class of strongly singular integral equations. SIAM J. Appl. Math. 50, 716–725.
Kanwal, R.D., 1998. Generalized Functions: Theory and Technique, second ed. Birkhäuser, Boston.
Kresse, O., Truskinovsky, L., 2003. Mobility of lattice defects discrete and continuum approaches. J. Mech. Phys. Solids 51, 1305–1332.
Kienzler, R., Herrmann, G., 2000. Mechanics in Material Space. Springer, Berlin, Heideberg.
Kienzler, R., Maugin, G.A., 2001. Configurational Mechanics of Materials, Kienzler, R., Maugin, G.A., (Eds.), Springer, Wien, New York.
Lubarda, V., Markenscoff, X., 2006. A variable core model and the Peierls stress for the mixed (screw-edge) dislocation. Appl. Phys. Lett. 89, no. 151923.
Lubarda, V., Markenscoff, X., 2007. Configurational force on a lattice dislocation and the Peierls stress. Arch. Appl. Mech. 77, 147–154.
Markenscoff, X., 1980. The transient motion of a nonuniformly moving dislocation. J. Elasticity 10, 193–201.
Markenscoff, X., Ni, L., 1990. The singular nature of the stress field near an arbitrarily moving dislocation loop. J. Mech. Phys. Solids 38 (4), 481–490.
Markenscoff, X., Ni, L., 1993. The near-field of an accelerating dislocation in an anisotropic solid. J. Elasticity 30, 211–217.
Markenscoff, X., Ni, L., 2001a. The transient motion of a dislocation with a ramp-like core. J. Mech. Phys. Solids 49, 1603–1619.
Markenscoff, X., Ni, L., 2001b. The transient motion of a ramp-core dislocation jumping from rest to supersonic speed. J. Appl. Mech. 68, 656–659.
Maugin, G.A., 1993. Material Inhomogeneity in Elasticity. Chapman & Hall, London.
Maugin, G.A., 1994. On the *J*-integral and energy-release rates in dynamic fracture. Acta Mech. 105, 33–47.
Maugin, G.A., Trimarco, C., 1992. Pseudomomentum and material forces in nonlinear elasticity: variational formulations and application to brittle fracture. Acta Mech. 94, 1–28.
Nemat-Nasser, S., Hori, M., 1987. Toughening by partial or full bridging of cracks in ceramic and fiber reinforced composites. Mech. Mater. 6, 245–269.
Nemat-Nasser, S., Hori, M., 1999. Micromechanics: Overall Properties of Heterogeneous Materials, second revised ed. Elsevier Science, North-Holland, Amsterdam.
Ni, L., Markenscoff, X., 2003. The logarithmic singularity of an accelerating edge dislocation. Philos. Mag. 83 (31–34), 3723–3734.
Ni, L., Nemat-Nasser, S., 2000. Singular perturbation solutions of a class of systems of singular integral equations. SIAM J. Appl. Math. 61 (4), 1219–1236.
Noether, E., 1918. Invariant variations problem. Klg-Ges. Wiss. Nach. Góttingen Math. Physik. Kl. 2, 235.
Rice, J.R., 1968. A path-independent integral and the approximate analysis of strain concentrations by notes and cracks. J. Appl. Mech. 35, 379–388.
Rogula, D., 1977. Forces in material space. Arch. Mech. 29, 705–715.
Sakamoto, M., 1991. High-velocity dislocations: effective mass effective line tension and multiplication. Philos. Mag. A. 63, 1241–1248.
Schechter, M., 1977. Modern Methods in Partial Differential Equations-An Introduction. McGraw-Hill, New York.
Weertman, J., Weertman, J.R., 1980. Moving dislocations. In: Nabarro, F.R.N. (Ed.), Dislocations in Solids. North-Holland, Amsterdam.
Willis, J.R., Nemat-Nasser, S., 1990. Singular perturbation solution of a class of singular integral equations. Q. Appl. Math. 48, 741–753.
Yosida, K., 1980. Functional Analysis, sixth ed. Springer, Berlin, Heidelberg.

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