Self-similarly expanding regions of phase change yield cavitational instabilities and model deep earthquakes

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ABSTRACT

The dynamical fields that emanate from self-similarly expanding ellipsoidal regions undergoing phase change (change in density, i.e., volume collapse, and change in moduli) under pre-stress, constitute the dynamic generalization of the seminal Eshelby inhomogeneity problem (as an equivalent inclusion problem), and they consist of pressure, shear, and *M* waves emitted by the surface of the expanding ellipsoid and yielding Rayleigh waves in the crack limit. They may constitute the model of Deep Focus Earthquakes (DFEs) occurring under very high pressures and due to phase change. Two fundamental theorems of physics govern the phenomenon: the Cauchy-Kowalewskaya theorem, which, based on dimensional analysis and analytic properties alone, dictates that there is zero particle velocity in the interior, and Noether's theorem that extremizes (minimizes for stability) the energy spent to move the phase boundary so that it does not become a sink (or source) of energy, and which determines the self-similar shape (axes expansion speeds). The expression from Noether's theorem indicates that the expanding region can be *planar*, thus *breaking the symmetry* of the input and the phenomenon manifests itself as a newly discovered one of a "dynamic collapse/ cavitation instability", where very large strain energy condensed in the very thin region can escape out. In the presence of shear, the flattened very thin ellipsoid (or band) will be oriented in space so that the energy due to phase change under pre-stress is able to escape out at minimum loss condensed in the core of dislocations gliding out on the planes where the maximum configurational force (Peach-Koehler) is applied on them. Phase change occurring planarly produces in a flattened expanding ellipdoid a new defect present in the DFEs. The radiation patterns are obtained in terms of the equivalent to the phase change six eigenstrain components, which also contain effects due to planarity through the Dynamic Eshelby Tensor for the flattened ellipsoid. Some models in the literature of DFEs are evaluated and excluded on the basis of not having the energy to move the boundary of phase discontinuity. Noether's theorem is valid in anisotropy and nonlinear elasticity, and the phenomenon is independent of scales, valid from the nano to the very large ones, and applicable in general to other dynamic phenomena of stress induced martensitic transformations, shear banding, and amorphization.

I. Introduction

An analysis is presented for the dynamical elastic fields that emanate from self-similarly expanding ellipsoidal regions where the material undergoes phase change (change in density i.e., volume collapse, and change in moduli under pre-stress). Self-similar expansion of the ellipsoidal surface of discontinuity starts from zero dimension with constant axes expansion speeds, with self-similarity grasping the early response of the system (e.g., Barenblatt, 1996). The problem constitutes the dynamic generalization of the Eshelby inclusion (Eshelby, 1957) problem with the "equivalent eigenstrain" obtained through the constant Dynamic Eshelby Tensor (DET) (Ni and Markenscoff, 2016, a, c). The emanated fields consist of *P*, *S*, and *M* waves, and we express the solution in terms of integrals over their slowness surfaces (rather than the unit sphere) in order to directly obtain the wave-front fields ("radiation patterns") at the *P* and S fronts and also near the surface of the expanding ellipsoid which is the M wave front. Because of the constant stress Eshelby property in the interior domain for self-similar expansion, constant tractions on the crack surfaces can be cancelled by those of an inclusion, and we thus obtain by a limiting process the elliptically expanding crack of Burridge and Willis (1969) where the M waves yield the Rayleigh fields of elastodynamics. The M waves also give the static Eshelby solution for the ellipsoid (Ni and Markenscoff, 2016c) in the particular limit where time goes to infinity and the axes speeds tend to zero, while their product tends to a constant, the axes lengths. In statics, crack solutions have been obtained as limits of Eshelby inclusions in Mura (1982) and more recently by Markenscoff (2018). For the two respective limits, the Burridge and Willis (1969) self-similarly expanding elliptical crack and the static Eshelby inclusion, the M waves may be called "Burridge-Willis-Eshelby Waves". We also refer to the works of Richards (1973, 1976) who solved the elliptically expanding crack by the Cagniard-de Hoop technique.

Markenscoff and Ni (2010) analyzed a spherical inclusion with dilatational transformation strain expanding spherically in general motion $R = R_0 + l(t)$ by applying the expression given by Willis (1965) in terms of the dynamic Green's function integrated over an expanding region. Naturally, when applied to self-similar expansion $R = Vt$ the solution of Markenscoff and Ni (2010) based on integrating the Green's function on an expanding boundary coincides with the one of Ni and Markenscoff (2016, b) based on this self-similar approach, also in this paper. The M waves that arise in the self-similar expansion $R = Vt$ can be shown to be produced from contributions of the wavelets emitted (and traveling at the pressure wave speed) by the expanding surface of the phase boundary at time τ_2 (from the latest position of the expanding boundary that have the time to reach the field point (r,t)) while the wavelets emitted from the earlier position $R = V\tau_1$ on the expanding inclusion that have the time to reach the field point (r,t) are the P waves . The M waves have as degenerate wavefront the expanding surface of the sphere (ellipsoid). The nature of the M waves is seen in the present analysis as a slowness surface, where the governing equation for the expanding inclusion is made homogeneous by the application of the M operator as in Burridge and Willis (1969), and an additional slowness surface (the expanding ellipsoid) is generated (the solution expressed as integrals over the slowness surfaces). All the waves, P, S and M together satisfy zero initial and radiation conditions as required for the self-similar problem.

For self-similarly expanding motion of an ellipsoidal surface due to stretch invariance the anisotropic inclusion with uniform eigenstrain is shown here on the basis of dimensional analysis alone and the ellipticity/ analyticity properties by means of the Cauchy-Kowalewskaya theorem (e. g, John, 1978) to have zero particle velocity in the interior domain. This is confirmed in isotropy by the full solution (Ni and Markenscoff, 2016a, eqtn (3.79)), where, in the corresponding expression of the particle velocity, the contributions of pressure P and shear S waves are shown explicitly to be cancelled in the interior domain by the M waves. As a result, no kinetic energy is radiated to the interior, and there is no focusing at the origin (with high stresses and dissipated energy), as it occurs in non-self-similar motion, such as in inclusions with time dependent eigenstrains (Willis, 1965), and in generally accelerating motion of a spherical non-self-similar accelerating motion in Markenscoff and Ni, 2010.

The self-similarly expanding ellipsoidal region of phase change under pressure is applied here to the modeling of deep focus earthquakes (DFEs), thought to be caused by high-pressure phase transformation. For the concept of transformation strain in geophysics we refer to Rice, 1979. The radiation patterns obtained here for the particle acceleration in terms of the eigenstrains (equivalent to the phase change) can be used to relate seismic signals to information about the source. Nucleation in deep focus earthquakes (DFEs) happens at 400-700 kilometers under the surface of the earth where the mechanism is attributed to "polymorphic phase transitions under high pressures" of over 10GPa (Knopoff and Randall, 1970, Randall and Knopoff, 1970). Analog experiments (Meade and Jeanloz, 1989, 1991) "recorded acoustic emissions and shearing instabilities due to rapid atomic motions across displacive phase transition in Si and Ge at pressures of 70 GPa, well above the brittle-ductile transition in both solids, not by fracturing or cracking of the samples". There is additional literature including works regarding the olivine to spinel transition (Green and Burnley, 1989, Burnley and Green (1989), Green, 2007, Houston and Williams, 1991). Experiments were performed (Schubnel *et al*, 2013) on germanium olivine at 2 to 5 GPa to model in the laboratory the analog of deep focus earthquakes and found that "fault propagation must have been happening instantaneously as phase transition occurs", and there has been a more recent quantitative modeling that is based on static micromechanics (Wang *et al*, 2017). The mechanism " is poorly understood" (Wang et al, 2017) and the phenomenon has remained a deep mystery. Recent publications by Li *et al* (2018) and Romanowicz (2018) attribute the DFEs to anisotropy.

The DFEs are modeled as self-similarly expanding ellipsoidal regions of phase change nucleating under prestress and assumed to grow at constant rate ; the phase change is an equivalent transformation strain obtained by the Dynamic Eshelby Tensor (DET) which depends on the yet undetermined axes speeds that are the variables of the problem. A central result of this paper is the determination of the shape that will be assumed by the self-similarly expanding region of phase change. Following Eshelby (1970) for statics, we postulate here as "equilibrium" shape of the self-similarly expanding phase boundary (it scales as t^2) the one that has invariance of the Hamiltonian (total strain and kinetic energy) under infinitesimal translation of the inhomogeneity position. On the basis of Noether's theorem (1918) it will have to assume the shape that extremizes (minimize*s*, for stability) the energy-rate spent to move the inhomogeneity position, and for the *boundary not to become a sink* (or source) *of energy* it requires that under total loading the dynamic *J* integral (Rice, 1968) vanishes. This will determine the axes speeds. It may be noted that the variation of the Hamiltonian in an infinitesimal variation of the inhomogeneity position is given by the dynamic J integral *if and only if* linear momentum is preserved in the domain (Gupta and Markenscoff, 2012).

The expression from Noether's theorem gives as a possibility that the normal boundary velocity be equal to zero. Thus, a major result is obtained, which is that the expanding region of phase change can propagate ***planarly*** for minimization of the energy spent in the expansion process (for any type of eigenstrain), which constitutes a ***symmetry breaking*** of the input (Markenscoff, 2018). It has been called a ***dynamic collapse instability*** by R. Jeanloz and ***a dynamic cavitational instability*** by J.W. Hutchinson (private communications), *cavitation* for allowing the energy to escape out. If the pre-stress is very high, the strain energy propagating in the very thin region can be finite, and not infinitesimal, with the eigenstrains becoming infinitely large, which explains why the phenomenon happens at the pressures of the deep earthquakes. The Dynamic Eshelby Tensor for the penny-shape flattened ellipsoid obtained here allows the solution of these problems where a 3D phase change is condensed into a 2D flattened ellipsoid ("pancake") so that the assumed shape can be the limit as the ellipsoid becomes flattened to a penny-shape ellipsoid or flattened elliptical cylinder (two-dimensional band, containing not only shear eigenstrain), as in Figure 1(b), (c), respectively. The planarity is consistent with observations in nature and the laboratory in analog experiments for DFEs (Schubnel *et al*, 2013, Wang *et al*, 2017), and in amorphization experiments under shock loading (e.g., (Zhao et al, 2017), and references within.

In summary, it is demonstrated how two fundamental theorems of mathematical physics, the Cauchy-Kowalewskaya and Noether's theorems, taken together result in maximizing the energy rate that is available to radiate out as seismic energy in this model.

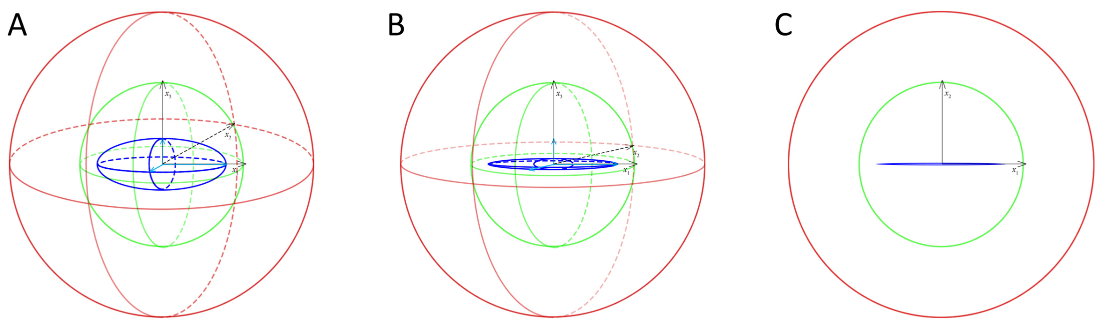

Figure 1: Schematics of the expanding ellipsoidal inclusion, penny-elliptical shape and 2D band, and wave-fronts (P in red, S in green and M in blue)

On the perimeter of the flattened ellipsoid the expansion will be governed by the symmetry-preserving factor of the driving force given by Noether's theorem. The *expression from Noether's theorem gives the physical law (kinetic relation) that drives the expansion relating the applied load to the phase boundary velocity though the dynamic J integral for an interface*.

In the presence of unequal normal stresses, the flattened ellipsoid will orient itself in space on the planes of maximum shear stress where the Peach-Koehler type of configurational force will be maximum and the self-stress due to inertia can be overcome first. This determines the orientation (Figure 2) where the motion starts and the directions (and magnitudes) of the axes speeds along which the dislocations (loops or straight) (produced by infinite large eigenstrains in the infinitesimally thin region) will glide with maximum velocity, and the phase change can escape out gliding condensed in the Burgers vector of a dislocation.

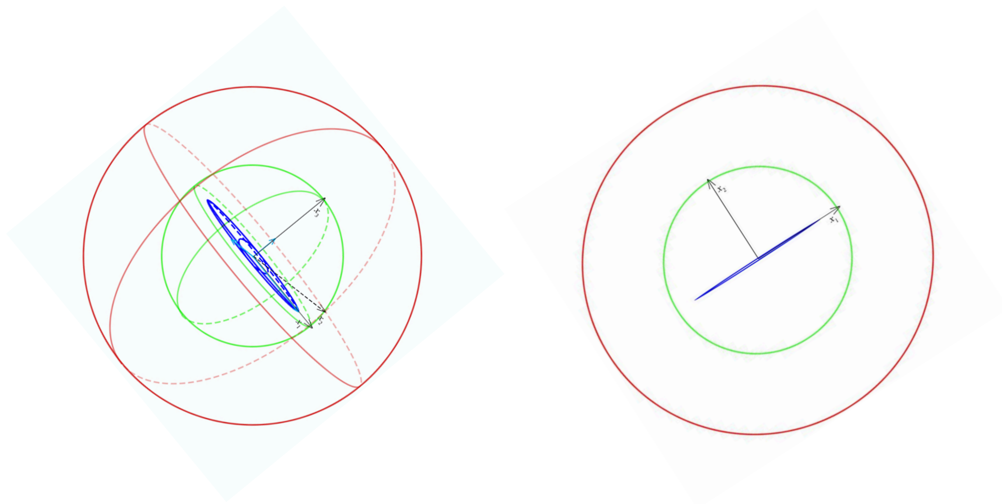

Figure 2: Orientation of the flattened ellipsoid/elliptical cylinder (band) in space with dislocations loops/ straight dislocations emitted on directions of maximum Peach-Koehler force applied on them.

The radiation patterns for the particle acceleration at the wave-front are obtained and they relate it to the equivalent eigenstrain and the expansion speeds. There are six components of the eigenstrain, which are infinitely large in the infinitesimally thin region so that they produce finite displacement discontinuities (actually, rates of displacement discontinuities): the three of them ε_{3i}^{*} produce the components of a Burgers vector of a dislocation loop in the limit of the flattened ellipsoid ("double couple"), while the in-plane components $\varepsilon_{11}^{*},\varepsilon_{22}^{*},\varepsilon_{12}^{*}$ are different in nature. The self-similarly expanding flattened penny-shape inclusion containing in-plane transformation strains may be considered a *new* singular dynamic defect (also square-root singular as the crack) and the analysis has to be produced. In particular, one such defect is volume collapse (change in density) that may break symmetry and propagate planarly with in-plane (infinite large) eigenstrains. This is a defect that is not occurring in shallow earthquakes, and can differentiate them from the deep ones. The radiation patterns have shear contributions unless there are onkly three equal longitudinal eigenstrains, which is not the case if the defect is planar (flattened ellipsoid).

While the phenomenon of expanding regions of phase change under pressure as manifested in DFEs occurs at large scales, it is valid independently of scale, from the nano to the very large ones, and is applicable to other phenomena of phase induced martensitic transformations under pressure (Escobar, et al 2000), amorphization (Zhang, et al, 2017), dynamic shear banding, etc. Noether's theorem is valid in anisotropy and nonlinear elasticity.

II. The waves of phase change from a self-similarly expanding ellipsoid

(a) *P*, *S*, and *M* waves

We consider a self-similarly expanding inclusion with uniform transformation strain (which will be the "equivalent eigenstrain of the phase change" problem) in the interior domain

$$\varepsilon^*_{lm}(\vec{x},t) = \varepsilon^*_{lm} H(t-(s_r^2 x_r^2)^{1/2}) \,, \tag{1}$$

where x_i is the position vector, t the time, $1/s_1, 1/s_2, 1/s_3$, are the axes speeds of the expanding ellipsoid in the argument of the Heaviside step function $H(.)$.

The governing equations expressing conservation of linear momentum in elastodynamics , where C_{ijkl} are the elastic stiffness tensor and ρ is the density are (Ni and Markenscoff, 2016a, c)

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm} \varepsilon^*_{lm} \frac{\partial}{\partial x_k} H(t-(s_r^2 x_r^2)^{1/2}) \tag{2}$$

with initial conditions

$$\vec{u}(\vec{x},0) = 0 \text{ and } \frac{\partial \vec{u}}{\partial t}(\vec{x},0) = 0 \tag{3}$$

and where vanishing radiation conditions at infinity are applied. In terms of the Navier operator of elastodynamics

$$L_{jl}(\frac{\partial}{\partial t}, \nabla) = \rho \frac{\partial^2}{\partial t^2} \delta_{jl} - C_{jklm} \frac{\partial^2}{\partial x_k \partial x_m} \tag{4}$$

with δ_{ij} denoting the Kroenecker delta, the system (2) is written as

$$L_{jl}(\frac{\partial}{\partial t}, \nabla) u_l = -K_j(\nabla) H(t^2 - s_r^2 x_r^2) \quad \text{with} \quad K_j(\nabla) = C_{jklm} \varepsilon^*_{lm} \frac{\partial}{\partial x_k} \tag{5}$$

The system (2) was solved by the Radon transform (e.g., Willis, 1973, Wang and Achenbach, 1994) in Ni and Markenscoff (2016, a). Here we follow the method of Burridge and Willis (1969) for the elliptically expanding crack. We introduce the operator

$$M\{\frac{\partial}{\partial t}, \nabla\} = \frac{\partial^2}{\partial t^2} - \frac{1}{s_r^2} \frac{\partial^2}{\partial x_r^2} \text{ which satisfies the identity } M^2\{\frac{\partial}{\partial t}, \nabla\} H(t^2 - s_r^2 x_r^2) = \frac{8\pi}{s_1 s_2 s_3} \delta(t)\delta(x)$$

so that, with the application of Duhamel's principle the governing system of equations becomes the homogeneous system of equations

$$M^2\{\frac{\partial}{\partial t}, \nabla\} L\{\frac{\partial}{\partial t}, \nabla\} u = 0 \qquad t > 0 \tag{6}$$

with inhomogeneous initial condition

$$\partial^k u_i(0)/\partial t^k = 0 \text{ for } k = 1,2,3,4 \text{ and } \partial^5 u_i(0)/\partial t^5 = -8\pi/(\rho s_1 s_2 s_3) K_i \delta(x) \tag{7}$$

The solution of the system (6) under the initial conditions (7) consists of contributions from the poles of the differential operators L and M, which give waves emanated from the three slowness surfaces, S^P, S^S, S^M (the first two corresponding to the poles of L, and the third one to those of M), which, for an isotropic material with $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ denote the Lame' constants, are:

$$S^P: \ |\xi|^2 = \frac{\rho}{\lambda + 2\mu} = \frac{1}{a^2}, \ S^S: \ |\xi|^2 = \frac{\rho}{\mu} = \frac{1}{b^2}, \ S^M: \ \frac{\xi_1^2}{s_1^2} + \frac{\xi_2^2}{s_3^2} + \frac{\xi_3^2}{s_3^2} = 1 \tag{8}$$

and where a and b denote the pressure and shear wave speeds. The solution of (6) with (7) is obtained following the procedure in Burridge and Willis (1969) in terms of integrals over the slowness surfaces, rather than over the unit sphere as in Ni and Markenscoff, 2016a, c), since in this formulation we can readily obtain the wave front behavior for the radiation patterns for the *P* and *S* waves and also for the M wave front (surface of the ellipsoid). By asymptotically expanding around the *M* wave front we obtain the crack limit and Rayleigh waves as we will show below. The solution for the displacement is

$$\begin{aligned} u_l(\mathbf{x},t) = & \frac{1}{\pi s_1 s_2 s_3} \sum \int_{S^L} \frac{N_{lk}(1,\xi) K_k(\xi)}{\partial D/\partial\gamma(1,\xi) M^2(1,\xi)} (t + \xi\cdot\mathbf{x}) \operatorname{sgn}(t + \xi\cdot\mathbf{x}) \frac{dS}{\left|\nabla\gamma^L(\xi)\right|} \\ & - \frac{1}{4\pi s_1 s_2 s_3} \int_{S^M} \left\{ \rho L_{lk}^{-2}(1,\xi) K_k(\xi) \left[(t + \xi\cdot x)\operatorname{sgn}(t + \xi\cdot x) - (t - \xi\cdot x)\operatorname{sgn}(t - \xi\cdot x) \right] \right. \\ & \left. + L_{lk}^{-1}(1,\xi) K_k(\xi)(\xi\cdot x) H(t - |\xi\cdot x|) \right\} \frac{dS}{\left|\nabla\gamma^M(\xi)\right|} \end{aligned} \tag{9}$$

with, for isotropy,

$$K_k(\vec{\xi}) = C_{jklm}\varepsilon^*_{lm}\xi_j = \lambda\varepsilon^*_{mm}\delta_{jk}\xi_j + \mu(\varepsilon^*_{jk} + \varepsilon^*_{kj})\xi_j$$

and where $N_{ij}(\gamma,\xi_i)$ and $D(\gamma,\xi_i)$ are defined by $L^{-1}{}_{ij}(\gamma,\xi_i) = \dfrac{N_{ij}(\gamma,\xi_i)}{D}$
with $N_{lj}(\gamma,\xi) = \rho[(\gamma^2 - a^2|\xi|^2)\delta_{jl} + (a^2 - b^2)\xi_j\xi_l]$ and $D(\gamma,\xi) = \rho^2(\gamma^2 - b^2|\xi|^2)(\gamma^2 - a^2|\xi|^2)$

$$\gamma^{\mathrm{M}}(\vec{\xi}) = \pm(\xi_k^2/s_k^2)^{1/2}, \ \left|\nabla\gamma(\vec{\xi})\right| = (\xi_k^2/s_k^4)^{1/2} \tag{10}$$

The solution (9) satisfies the Hadamard jump conditions across a moving surface of discontinuity, with normal boundary velocity $\dot{l}$, which are

$$[[\frac{\partial u_l}{\partial t}]] = -\dot{l}[[\frac{\partial u_l}{\partial x_m}]]n_m \quad \text{and} \quad [[\sigma_{ij}n_j]] = -\rho\dot{l}[[\frac{\partial u_i}{\partial t}]] \tag{11}$$

with the brackets denoting the jumps across the moving interface.

For the expanding ellipsoid the normal $\vec{n}$ at a point (x_1, x_2, x_3) on the boundary is $\vec{n} = (s_1^2 x_1, s_2^2 x_2, s_3^2 x_3)/(s_k^4 x_k^2)^{1/2}$, the velocity vector is ($x_1/t, x_2/t, x_3/t$), and it can be written as, $((1/s_1)\cos\theta\sin\phi, (1/s_2)\sin\theta\sin\phi, (1/s_3)\cos\phi)$ --where the angles θ and ϕ are those of spherical coordinates--, and the normal velocity $\dot{l}$ is its projection on the normal vector which is $\dot{l} = 1/(s_k^4 (x_k/t)^2)^{1/2}$.

It may be remarked that the solution obtained for a spherical inclusion in self-similar expansion (Ni and Markenscoff, 2016b) giving the *P*, *S* and *M* waves coincides with the one of Markenscoff and Ni (2010) obtained by the Green's function method for a spherically expanding inclusion with general acceleration when applied to self-similar expansion.

Radiation patterns

We next obtain the radiation patterns for the particle acceleration at the *P* and *S* wave fronts following Lighthill, 1960, and Burridge, 1967. Singularities occur when the plane $t + \vec{\xi} \cdot \vec{x} = 0$ is tangent to the slowness surfaces S^L, S^M at the point $\vec{\xi}_0$. The particle acceleration is

$$\begin{aligned} \frac{\partial^2 u_i(\vec{x},t)}{\partial t^2} &= \frac{1}{\pi s_1 s_2 s_3} \int_{S^L} \frac{N_{lk}(1,\vec{\xi}) C_{jklm} \varepsilon^*_{lm} \xi_j \delta(t + \vec{x}\cdot\vec{\xi})}{\partial D/\partial\gamma(1,\vec{\xi}) M^2(1,\vec{\xi})} \frac{dS}{\left|\vec{\nabla}\gamma^L(\vec{\xi})\right|} \\ &+ \frac{1}{4\pi s_1 s_2 s_3} \int_{S^M} \left\{ -2\rho L^{-2}_{ik}(1,\vec{\xi}) C_{jklm} \varepsilon^*_{lm} \xi_j \delta(t + \vec{x}\cdot\vec{\xi}) \right. \\ &\left. + L^{-1}_{ik}(1,\vec{\xi}) C_{jklm} \varepsilon^*_{lm} \xi_j \left[\delta(t + \vec{\xi}\cdot\vec{x}) + t\delta'(t + \vec{\xi}\cdot\vec{x}) \right] \right\} \frac{dS}{\left|\vec{\nabla}\gamma^M(\vec{\xi})\right|} \end{aligned} \tag{12}$$

Based on the above expression for the particle acceleration, the jump at the wave front $\vec{\xi}_0$ is obtained following Lighthill (1960) and Burridge (1967), as

$$\left[\ddot{u}_l(t,\vec{x})\right] = \frac{-2f\left(\vec{\xi}_0\right)}{s_1 s_2 s_3 K^{1/2}\left(\vec{\xi}_0\right)} \frac{H\left(t + \vec{\xi}_0 \cdot \vec{x}\right)}{|\vec{x}|} \tag{13}$$

where $K(\vec{\xi}_0)$ is the Gaussian curvature of S^L at $\vec{\xi}_0$, and where

$$f\left(\vec{\xi}_0\right) = \frac{N_{lk}\left(1,\vec{\xi}_0\right) K_k\left(\vec{\xi}_0\right)}{\partial D/\partial\gamma\left(1,\vec{\xi}_0\right) M^2\left(1,\vec{\xi}_0\right)} \frac{1}{\left|\vec{\nabla}\gamma^L\left(\vec{\xi}_0\right)\right|} \tag{14}$$

For the P wave-front $K^{1/2} = a$:

$$\left|\vec{\xi}\right|^2 = 1/a^2 \ , \ \gamma^L\left(\vec{\xi}\right) = a\left|\vec{\xi}\right|, \ M^2 = \left\{1 - \frac{1}{a^2}\left(\frac{\hat{x}_1^2}{s_1^2} + \frac{\hat{x}_2^2}{s_2^2} + \frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2 , \ \frac{\partial D}{\partial \gamma} = 2\rho^2 \frac{\lambda + \mu}{\lambda + 2\mu} = \frac{2\rho}{a^2}(\lambda + \mu),$$

$$\frac{N_{lk}(1,\xi)}{\partial D/\partial\gamma(1,\xi) M^2(1,\xi)} = \frac{a^4 \hat{\xi}_l \hat{\xi}_k}{2\rho\left(a^2 - p^2\right)^2} \ , \ p^2 = \frac{\hat{\xi}_k^2}{s_k^2} \ , \quad \left|\vec{\nabla}\gamma^L\left(\vec{\xi}\right)\right| = a \tag{15}$$

Setting $\vec{\xi}_0 = -\frac{\hat{x}}{a}$, where $\hat{x}$ is a unit vector at the P wave-front, the radiation pattern for acceleration emitted by an expanding three-dimensional ellipsoidal inclusion with eigenstrain ε_{ij}^*, at the P front we have the radiation pattern

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{\left(\lambda\delta_{jk}\varepsilon_{mm}^* + \mu\varepsilon_{jk}^* + \mu\varepsilon_{kj}^*\right)\hat{x}_i\hat{x}_k\hat{x}_j}{\pi\rho a^3 s_1 s_2 s_3 \left\{1 - \frac{1}{a^2}\left(\frac{\hat{x}_1^2}{s_1^2} + \frac{\hat{x}_2^2}{s_2^2} + \frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2} \frac{H\left(t - \frac{|\vec{x}|}{a}\right)}{|\vec{x}|} \tag{16}$$

and, analogously, at the S wave-front we have

$$\frac{N_{lk}(1,\xi)}{\partial D/\partial\gamma(1,\xi) M^2(1,\xi)} = \frac{b^4\left[\delta_{lk} - \hat{\xi}_l\hat{\xi}_k\right]}{2\rho\left(b^2 - p^2\right)^2} \tag{17}$$

and we obtain the radiation pattern at the shear wave-front

$$\frac{\partial^2 u_l}{\partial t^2} = \frac{\left[\delta_{lk} - \hat{x}_l\hat{x}_k\right]\left(\lambda\delta_{jk}\varepsilon_{mm}^* + \mu\varepsilon_{jk}^* + \mu\varepsilon_{kj}^*\right)\hat{x}_j}{\pi\rho b^3 s_1 s_2 s_3 \left\{1 - \frac{1}{b^2}\left(\frac{\hat{x}_1^2}{s_1^2} + \frac{\hat{x}_2^2}{s_2^2} + \frac{\hat{x}_3^2}{s_3^2}\right)\right\}^2} \frac{H\left(t - \frac{|\vec{x}|}{b}\right)}{|\vec{x}|} \tag{18}$$

The shear contributions in (18) will only vanish if there are three equal longitudinal eigenstrains, but that is not the case for the flattened ellipsoid, where the third eigenstrain is necessarily unequal (through the dynamic Eshelby Tensor for the penny-shape) so there will always be both P and S wave radiation due to planarity.

There are six components of the eigenstrain, which are infinitely large in the infinitesimally thin region so that they produce finite displacement discontinuities (actually, rates of displacement discontinuities): the three of them ε_{3i}^* are the components of a Burgers vector and produce a dislocation loop in the limit of the flattened ellipsoid. In the limit, as the third axis speed goes to zero, if we set $\lim_{1/s_3 \to 0} 1/s_3 \varepsilon^*_{3s} = b_s$, we retrieve the radiation patterns expression (a double couple, modified with a coefficient for the expanding ellipsoid) in the Burridge and Willis (1969) crack (which has a typo). The in-plane components $\varepsilon^*_{11}, \varepsilon^*_{22}, \varepsilon^*_{12}$ are different in nature (non double couple). Volume collapse propagating planarly in a circular penny-shape will produce equivalent to the volume change eigenstrains $\varepsilon^*_{11}, \varepsilon^*_{22}$, and ε^*_{33} obtained through the Dynamic Eshelby Tensor (for the penny-shape self-similarly expanding ellipsoid) where the planarity effect gives a Poisson's ratio dependence in the eigenstrains which might be misinterpreted as anisotropy in the moment tensor used in seismology.

The above radiation patterns depend on the "equivalent" eigenstrain, which is a function of the input parameters of pre-stress and phase change and the expansion speeds $1/s_i$. The expansion speeds themselves are functions of the phase change and pressure determined in the sequel from Noether's theorem, so that the boundary does not become a sink (or source) of energy.

(b) Self-similarity and dimensional analysis: zero particle velocity in the interior domain

From the field solution for the displacement equation (10) the particle velocity is obtained in the form of integrals over the slowness surfaces as

$$\frac{\partial u_l}{\partial t}(\mathbf{x},t) = \frac{-1}{\pi s_1 s_2 s_3} \sum \int_{S^L} \frac{N_{lk}(1,\xi) C_{jkpq} \varepsilon^*_{pq} \xi_j}{\partial D / \partial \gamma(1,\xi) M^2(1,\xi)} \operatorname{sgn}(t - \xi \cdot \mathbf{x}) \frac{dS}{\left|\nabla \gamma^L(\xi)\right|}$$
$$+ \frac{1}{2\pi s_1 s_2 s_3} \int_{S^M} \left\{ \rho L^{-2}_{lk}(1,\xi) C_{jklm} \varepsilon^*_{lm} \xi_j \operatorname{sgn}(t - \xi \cdot x) \right.$$
$$\left. - L^{-1}_{lk}(1,\xi) C_{jklm} \varepsilon^*_{lm} \xi_j (\xi \cdot x) \delta(t - |\xi \cdot x|) \right\} \frac{dS}{\left|\nabla \gamma^M(\xi)\right|} \tag{19}$$

The critical property of the self-similarly expanding inclusion is that the contributions from the slowness surfaces pertaining to the pressure and shear waves are cancelled by those of the M surface in the interior domain, as the corresponding terms (integrated though over the unit sphere) can be identified in the expression given in Ni and Markenscoff (2016, a, eqtn 3.79). It must be noted here that for self-similar expanding motion these contributions to the particle velocity have constant values in the interior domain (and are not singular at the origin).

The governing system of p.d.e.'s (2) for general anisotropy with initial conditions (2) is stretch-invariant (Ni and Markenscoff, 2016a) so that the number of independent variables can be reduced by one, and it allows for a self-similar solution in terms of the variable $\vec{z} = \vec{x}/t$ and the function $\phi(\vec{z},t) = \frac{\vec{u}(\vec{x},t)}{t}$. In the new variables, the system is easily shown (Ni and Markenscoff, 2016a) to be elliptic in the region $|z| < \upsilon_0^{\min}$ (in the interior of the velocity surfaces for an anisotropic solid corresponding to the operator L), in which region the pertinent determinant does not vanish, while in the exterior the system is hyperbolic.

We then consider the fundamental properties of the system (2) with initial conditions (3). In the interior domain of the expanding anisotropic inclusion the system is an *elliptic* system of partial differential equations with analytic coefficients. The Cauchy-Kowalewskaya theorem (e.g. John, 1978) will provide the solution fully determined in the region of analyticity from the initial conditions (3) (on the displacement and particle velocity). Since these are zero, then the solution will be zero everywhere in the region of analyticity, which is the interior domain of the inclusion. Thus, the vanishing of the particle velocity in the interior domain is dictated by dimensional analysis alone and analytic arguments (answering a question asked by J.R. Rice in 2015 at the Broberg meeting, "what follows from dimensional analysis alone? ") so that the M waves have to be emitted to cancel the pressure and shear ones to fulfill this requirement. Note that for subsonic motion the inclusion lies within the region of ellipticity of the system of p.d.e's and this is the reason that the property is valid only for subsonically expanding inclusions.

(c) The phase change as an equivalent eigenstrain of the expanding ellipsoid with phase change under pressure

Following here the method of Eshelby (1957) for an inhomogeneity under prestress, the full field quantities for the expanding ellipsoidal region with phase change problemunder prestress are the superposition of the uniform pre-stress field σ_{ij}^{0} and the solution of the equations (2) with (3), with eigenstrain equivalent to the change in moduli under prestress as shown in this section.

For isotropic materials the displacement gradient can be explicitly calculated and the Eshelby property of constant constrained strain is proven explicitly (Ni and Markenscoff, 2016a,c). From eqtn (10) we obtain the displacement gradient as integrals over the slowness surfaces,

$$\frac{\partial u_l(x,t)}{\partial x_m} = \frac{1}{\pi s_1 s_2 s_3}\sum\int_{S^L} \frac{\xi_m N_{lk}(1,\xi)K_k(\xi)}{\partial D/\partial r(1,\xi)M^2(1,\xi)}\operatorname{sgn}(t+\xi\cdot x)\frac{dS}{\left|\nabla\gamma^L(\xi)\right|}$$
$$-\frac{1}{4\pi s_1 s_2 s_3}\int_{S^M}\left\{2\rho\xi_M L_{lk}^{-2}(1,\xi)K_k(\xi)H(t-|\xi\cdot x|)\right.$$
$$\left.+\xi_M L_{lk}^{-1}(1,\xi)K_k(\xi)\left[H(t-|\xi\cdot x|-t\delta(t-|\xi\cdot x|)\right]\right\}\frac{dS}{\left|\nabla\gamma^M(\xi)\right|} \tag{20}$$

In the interior domain, in equation (20), by the Schwarz inequality, the step and signum functions are 1, the delta function zero, and thus, the displacement gradient does not depend on the position $\vec{x}$ and we have the Eshelby property extended to dynamics (Ni and Markenscoff, 2016a). From (20) the Dynamic Eshelby Tensor (DET) $S_{ijkl}^{dyn}(1/s_i)$, defined by the relation of the strain to the eigenstrain $\varepsilon_{ij} = S_{ijkl}^{dyn}\varepsilon_{kl}^{*}$, is constant in space and time, depending on the axes speeds $1/s_i$ and was obtained for isotropic materials by Ni and Markenscoff, 2016c, in terms of integrals over the unit sphere. The constant DET allows to solve the problem of a self-similarly expanding ellipsoidal inhomogeneity with "phase change" . Due to change in moduli to C_{ijkl}^{*} as the inhomogeneity expands in a matrix with constants C_{ijkl} under a uniform applied strain ε_{ij}^{0} *at* *infinity (prestress)*, the "phase change" is an equivalent "eigenstrain" ε_{ij}^{*}. If there is a change in density, i.e., a volume collapse, with a corresponding trace of the strain $\varepsilon_{kk}{}^{vc}$, then the volume collapse is equivalent to an eigenstrain ε_{ij}^{*vc} such that

$$\Delta V / V = \varepsilon_{kk}^{vc} = (1/3) S_{mmij}^{dyn} (1/s_r) \varepsilon_{ij}^{*vc} , \tag{21}$$

so that the components of ε_{ij}^{*vc} are dependent on the axes speeds (shape). In the presence of both modulus drop and volume collapse, we will have $\varepsilon_{ij}^{**} = \varepsilon_{ij}^{*} + \varepsilon_{ij}^{*vc}$ as in Eshelby (1957), and equation

$$C_{ijkl}^{*}(\varepsilon_{ij}^{0} + S_{klmn}^{dyn}(1/s_r)\varepsilon_{mn}^{**} - \varepsilon_{kl}^{*vc}) = C_{ijkl}(\varepsilon_{ij}^{0} + S_{klmn}^{dyn}(1/s_r)\varepsilon_{kl}^{**} - \varepsilon_{kl}^{**}) \tag{22}$$

expresses the point-wise Hooke's law (Eshelby, 1957). Eqtn (22) can be separated into a volumetric and a deviatoric part (Eshelby, 1957). The components of equivalent eigenstrain ε_{ij}^{*vc} due to volume collapse in eqtn (21) are determined through the DET and they contain the effects of the geometry, such as expanding penny-shape, for planar propagation.

Equations (22) are also applicable for interior anisotropy C_{ijkl}^{*} (see Eshelby, 1961, for statics) but not exterior anisotropy (when the DET is not known) in a matrix of isotropic material with Lame constants (λ, μ) when they are written as

$$C_{ijkl}^{*}(\varepsilon_{kl} + \varepsilon_{kl}^{0}) = \lambda(\varepsilon_{kk} - \varepsilon_{kk}^{*} + \varepsilon_{kk}^{0})\delta_{ij} + 2\mu(\varepsilon_{ij} - \varepsilon_{ij}^{*} + \varepsilon_{ij}^{0}) \tag{23}$$

Anisotropy has been considered central in DFEs in recent literature (Li *et al* (2018) and Romanowicz (2018)). We may remark that the DET for the sphere Ni and Markenscoff (2016,b) is smaller than the static one for subsonic motion, and, if this is also true for the ellipsoid, the system (22) would be always invertible for ε_{ij}^{*} (see also, Markenscoff, 2016). The equivalent eigenstrain ε_{ij}^{**} depends through (22) on the axes speeds, which will be determined below in terms of the input parameters.

(d) A limit yields the Burridge and Willis (1969) elliptically expanding crack and Rayleigh waves

An important limiting property of the expanding ellipsoidal inclusion with eigenstrain arises from the fact that the Burridge and Willis expanding elliptic crack solution can be retrieved, as anticipated by them, and, hence, the natural name to give to these waves is "Burridge-Willis-Eshelby Waves". The crack solution is obtained in the limit of an ellipsoidal inclusion as the speed of the third axes tends to zero, i.e. $1/s_3 \to 0$, and the eigenstrain ε_{31}^* tends to infinity, in such a way that

$$\lim_{1/s_3 \to 0}\left(1/s_3\, \varepsilon_{3i}^* \right) = \lim_{1/s_3 \to 0}\left(1/s_3 \frac{\partial u_i^*}{\partial x_3} \right) = const = \Delta \dot{u}_i^* \tag{24}$$

tends to a constant, the *rate of displacement discontinuity* $\Delta \dot{u}_i^*$, with crack opening displacement

$$\Delta u_i(x_1, x_2, t) = \Delta \dot{u}_i^* (t^2 - s_1^2 x_1^2 - s_2^2 x_2^2)^{\frac{1}{2}} H(t^2 - s_1^2 x_1^2 - s_2^2 x_2^2) \tag{25}$$

The constant interior stresses (Eshelby property in dynamics) of the self-similarly expanding (flattened) inclusion with eigenstrain ε_{ij}^* can cancel constant applied tractions on the crack faces (since in the Hadamard condition (11) the R.H.S is zero across the crack faces), satisfying the equation

$$\sigma_{3i}^0 + C_{3imn}\left(S_{mnkl}^{dyn} \varepsilon_{kl}^* - \varepsilon_{mn}^* \right) = 0 \ , \tag{26}$$

which determines the corresponding eigenstrain as to cancel the applied tractions σ_{3i}^0 . The idea of obtaining a crack solution from an inclusion in statics is initially due to Eshelby (1957,1961) and treated in Mura, 1982, chapter 5 and is due to the constant stress Eshelby property in the interior domain. Recently a full analysis for the Griffith cracks as limits of Eshelby inclusions was presented in Markenscoff (2019).

The elliptically expanding crack surface is actually the slowness surface S^M that is a degenerate wave-front with the curvature becoming infinite at the elliptical tip of the crack. The near tip field is obtained by the asymptotic expansion near the *M* wave-front according to Burridge and Willis (1969) and we will follow them to obtain the traction on the plane $x_3 = 0$ in front of the expanding flattened elliptical inclusion in the limit as it becomes a crack. The slowness surface S^M is now a cylinder parallel to the ξ_3 axis, $\xi_3 \to \pm\infty$. The near tip field is obtained following Burridge and Willis (1969) for the asymptotic evaluation of the *M* slowness surface contribution in (20). The traction vector $\vec{t}^{(3)}$ on the plane in front of the elliptical edge with the normal boundary velocity $\frac{1}{\dot{l}^2} = s_1^2 \cos^2\theta + s_2^2 \sin^2\theta$, is obtained in the limit as $1/s_3 \to 0$ and $x_3 \to 0$, from

$$t_i^{(3)} = C_{i3lm} \frac{\partial u_l}{\partial x_m} =$$

$$= \lim_{1/s_3 \to 0}\left(\frac{1}{\pi s_1 s_2} \int_{S^L} C_{i3lm} \xi_m N_{lk}(1, \vec{\xi}) C_{kn3q} \xi_n \frac{\varepsilon_{3q}^*}{s_3} \operatorname{sgn}(1 + \vec{\xi} \cdot \frac{\vec{x}}{t}) \frac{dS}{\left| \nabla \gamma^L(\vec{\xi}) \right|} \right.$$

$$-\frac{1}{4\pi s_1 s_2}\int_{S^M}\{2\rho C_{i3lm}\xi_m L_{lk}^{-2}C_{kn3q}\xi_n\frac{\varepsilon_{3q}^*}{s_3}H(t-|\vec{\xi}\cdot\vec{x}|)+\left[C_{i3lm}\xi_m L_{ik}^{-1}(1,\vec{\xi})C_{kn3q}\xi_n\frac{\varepsilon_{3q}^*}{s_3}\right.$$

$$\left.+C_{i3k3}(\xi_1=0,\xi_2=0,\xi_3=1)\frac{\varepsilon_{3k}^*}{s_3}\right]\left[H(t-|\vec{\xi}\cdot\vec{x}|)-t\delta(t-|\vec{\xi}\cdot\vec{x}|)\right]\}\frac{dS}{|\nabla\gamma^L(\vec{\xi})|} \tag{27}$$

In (27) the leading contribution is from the *M* slowness surface, and the leading term of the integral over S^M (the axis parallel to ξ_3) near the inclusion (crack) edge is evaluated at $x_1=\frac{(1+\varepsilon)t}{s_1}$ $,x_2=0,\ x_3=0$, as $\varepsilon\to 0$ from the integral

$$I\approx\lim_{1/s_3\to 0}\frac{1}{4\pi}\frac{1}{(2\varepsilon)^{1/2}}\int_{-\infty}^{\infty}\left[\{C_{i3lm}\xi_m L_{lk}^{-1}(1,\vec{\xi})C_{kn3q}\xi_n+C_{i3q3}\}\frac{\varepsilon_{3q}^*}{s_3}\right]d\xi_3+O(1) \tag{28}$$

with

$$\xi_1=s_1^2x_1/t,\xi_2=s_2^2x_2/t$$

Following Burridge and Willis (1969) the traction vector at a distance of order ε from the axes tips of the expanding inclusion (crack)
for prevailing plane strain, at $x_1=\frac{(1+\varepsilon)t}{s_1}$, $x_2=x_3=0$, along the -x_1 axis, has stress intensity factors

$$K_I=\sqrt{2\varepsilon}\sigma_{33}\sim\Delta\dot{u}_3^*\frac{\mu^2R(s_1)}{2\rho(s_1^2-1/a^2)^{1/2}},\ K_{II}=\sqrt{2\varepsilon}\sigma_{31}\sim\Delta\dot{u}_1^*\frac{\mu^2R(s_1)}{2\rho(s_1^2-1/b^2)^{1/2}} \tag{29}$$

where

$$R(s_1)=4s_1^2\left(s_1^2-1/a^2\right)^{1/2}\left(s_1^2-1/b^2\right)^{1/2}-\left(2s_1^2-1/b^2\right)^2 \tag{30}$$

is the Rayleigh wave function, which vanishes as the speed $1/s_1$ of the -x_1 axis approaches the Rayleigh wave speed c_R. If, at this end, conditions of anti-plane strain prevail due to $\Delta\dot{u}_2^*$, we have asymptotically

$$K_{III}=\sqrt{2\varepsilon}\sigma_{32}\sim\Delta\dot{u}_2^*\frac{(-\mu)(s_1^2-1/b^2)^{1/2}}{2} \tag{31}$$

which vanishes as the boundary propagates at the shear-wave-speed *b*.

III. The shape of the self-similarly expanding ellipsoidal inclusion (determination of axes speeds) and a dynamic cavitational instability

The criterion for the assumed self-similar shape of a region of phase change is based on the extension of the static "equilibrium shape" of a region of martensitic transformation due to Eshelby, 1970, to self-similar dynamics, where the geometrical shape of the ellipsoidal boundary (with constant axes speeds) is *independent of time* (with the surface area scaling as t^2). In non-self-similar expansion wavelets emitted from one position would alter the shape at another point at a later time when they reached that point, and the shape changes as a function of time.

Eshelby, 1970, reasoned on a basis of translational invariance in a thought experiment to reach the conclusion that we must have $\delta E^{tot} = 0$ for *the static* ***equilibrium position of phase boundaries***. Indeed, the "equilibrium shape" is the one that, when assumed,

$$\Pi^H(u_i, u_{i,j}) = \int_\Omega W(x_i, u_i, u_{i,j})dV \tag{32}$$

the *Hamiltonian energy of an elastic body with strain energy density* W *under total loading will be invariant under any family of infinitesimal translations (parallel displacement) of the inhomogeneity position* $x_k^* = x_k + \varepsilon a_k$, $u_i^* = u_i$, (Figure 3).

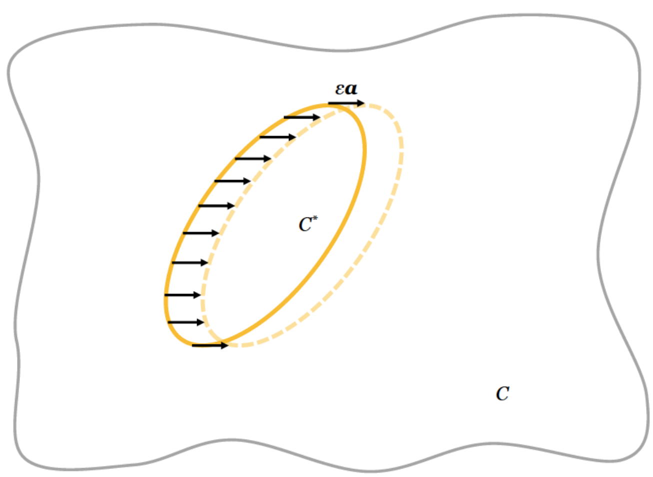

Figure 3: Translation of the inhomogeneity position

Then, according to Noether's theorem (i.e., Gelfand and Fomin p. 177) the shape will be an extremal surface of $u_i(x_i)$ on which the total variation of the total energy of the body under an arbitrary translation $\delta\xi_l = \varepsilon a_l$ of the inhomogeneity position should be equal to zero, i.e.,

$$\delta E^{tot} = \delta\Pi^H = -\int_S \delta\xi_l([[W]]\delta_{lj} - <\sigma_{ij}>[[u_{i,l}]])dS_j = -\delta\xi_l J_l^{dyn} =$$
$$-\int_S \delta\xi_k([[W]] - <\sigma_{ij}>[[u_{i,j}]])dS_k = 0 \tag{33}$$

as in Eshelby,1970. The double brackets [[.]] denote jumps across the phase boundary and <.> denotes the average of the values across the two sides of the interface (surface of discontinuity), with the quantity in parenthesis being the energy-momentum tensor.

To quote Eshelby (1970) : "*Eqtn (33) can be used to find the equilibrium positions of phase and twin boundaries in the presence of stresses produced by the transformation itself, or externally applied or both"*. Eshelby (1970) further argued that, since $\delta E^{tot} = 0$ must be zero for any arbitrary $\delta\xi_k$, the integrand in (33) must vanish, i.e., the "effective normal force" (Eshelby, 1970) , i.e., $[[W]] - <\sigma_{ij}>[[u_{i,j}]] = 0$
must vanish point-wise all along the boundary.

We extend the above static definition to elastodynamics with the Hamiltonian functional containing the strain energy density W and the kinetic energy density T,

$$\Pi(u_i) = \int_{t_1}^{t_2} \int_{\Omega} \left\{ W(x_i, u_{i,j}) + T(\dot{u}_i) \right\} dV\, dt$$

and, for the self-similarly expanding inhomogeneity, ***we postulate that the shape that the boundary will assume, and which is independent of time for self-similar expansion (just scaling in time), must be such that the rate of the Hamiltonian will remain invariant for any family of infinitesimal translations*** $x_k^* = x_k + \varepsilon a_k, u_k^* = u_k$ ***of the inhomogeneity position***. Consequently, ***by Noether's theorem,*** $u_i(x_k, t)$ ***will be an extremal surface both of the rates of the Lagrangean and of the Hamiltonian (since the variation of the Hamiltonian is minus to the variation of the Lagrangean for a purely mechanical system e.g., Stolz, 2003, Markenscoff and Singh, 2015)***, so that

$$\delta \dot{E}^{tot} = \delta\dot{\Pi}^H = 0, \tag{34}$$

which is the generalization of (33) to the elastodynamic Hamiltonian. Thus, Noether's theorem dictates that the system will take a shape that extremizes (minimizes for stability) the total energy spent to move the boundary.

We then give to the inhomogeneity position a translation *per unit time* such that $\varepsilon\dot{a}_k = \varepsilon\upsilon_k$ where υ_k is the defect velocity. With the near-field asymptotic behavior of the dynamic field of the defect satisfying $u_{j,k}\upsilon_k = \dot{u}_j$, we obtain from J^{dyn} (Rice, 1968) the expression for the energy release rate for a moving surface of discontinuity S^d (the surface of the expanding ellipsoid), which is the instantaneous rate of energy flow through S^d, same as for a crack (without the brackets in Freund, 1972) due to self-stresses as

$$\delta \dot{E}^{tot} / \varepsilon = \dot{a}_k J_k^{dyn} = \upsilon_k \int_{S^d} \{[[W+T]]n_k - [[\sigma_{ij} u_{j,k}]]n_i\} dS = \int_{s^d} \{[[W+T]]\upsilon_n + [[\sigma_{ij}\dot{u}_j]]n_i\}\, dS$$
(35)

The integrals in (35) can be easily shown to be path-independent of the shape of the surface of the flux surrounding the boundary S^d as it shrinks onto the surface of the ellipsoid, and, it may be noted, that the above expression of J_k^{dyn} is valid both in linear and nonlinear anisotropic elasticity.

Eqtn (35) with total loading (as eqtn 33) yields

$$\delta \dot{E}^{tot} = \int_{S^d} \{[[(W+T)]]\upsilon_n + n_j[[\sigma_{ij}\dot{u}_i]]\}dS = 0 \tag{36}$$

where υ_n is the normal boundary velocity on a contour surrounding the defect (Freund, 1972) and moving with it, same as $\dot{l}$ in the notation in Appendix A. The interpretation of equation (36), ***and the criterion for growth is that the self-similarly expanding boundary will assume a shape such that the total energy flux through the moving boundary (with all loading included, external plus due to self-stresses) is zero***. Equivalently, we can state ***that the moving boundary will assume a shape for which, under total loading, it will not become a sink or source of energy.***

The expression for the jump in the displacement gradient across the expanding ellipsoid of phase change under pre-stress is obtained in Appendix A by generalizing to dynamics the Hill jump conditions as in Markenscoff (2015). The jump quantities in (35)-(36) for self-similar expansion of an ellipsoidal inclusion are independent of time and depend only on the direction of the normal to the boundary and not time. Consequently, the integral in equation (35) with the self-stresses scales in time as t^2. The scaling of the integral is also valid in the limiting "flat" cases of the ellipsoid as discussed further below.

With the Hadamard jump conditions (11) equation (36) under total loading yields

$$\delta \dot{E}^{tot} = \lim_{S^d \to 0} \int_{S^d} \dot{l}([[W]] - \langle \sigma_{ij} \rangle [[u_{i,j}]])dS = 0 \tag{37}$$

Equation (37) is the expression for the energy release rate with the quantity in parenthesis being the "driving force" . There are two possibilities that the integrand in (37) is zero,

$$\text{either } \dot{l} = 0 \text{ , or } [[W]] - \langle \sigma_{ij} \rangle [[u_{i,j}]] = 0 \tag{38}$$

corresponding to two different "modes" of expansion governed by the boundary velocity $\dot{l}$.

The first possibility of the normal boundary velocity $\dot{l} = 0$ is satisfied on the upper and lower surfaces of the ellipsoid if they become flattened ($1/s_3 \ll 1/s_1, 1/s_2$) and $1/s_3 \to 0$ (circular penny-shape, or elliptical penny-shape, Figure 2B), and in the two-dimensional limit of the ellipsoid becoming an elliptic cylinder , which, in turn, becomes flattened, $1/s_2 \to \infty, s_1/s_3 \to 0$, as a propagating band (Figure 2C). The band may contain any type of eigenstrain, longitudinal and/or shear. Thus, no energy is expended to move the upper and lower surfaces, and this is the mode of minimum energy lost by the system to move the boundary. This **flattened mode** $\dot{l}=0$ is a **symmetry breaking instability**, and may be called a **"dynamic collapse/cavitation instability"** (the names attributed to R. Jeanloz and J.W. Hutchinson respectively). For this to occur the strain energy rate in the very thin ellipsoid in the $\lim 1/s_3 \to 0$ (that scales with a factor $4\pi t^2$) must be finite. i.e.,

$$W/(s_1 s_2 s_3) = \lim_{1/s_3 \to 0} (W/s_3)/s_1 s_2 = const \tag{39}$$

where W denotes the strain energy density in the inclusion, computed by means of the DET for the flattened ellipsoid. This may occur in DFEs where the pre-stress is extremely high. It may be noted here that for the possibility that it becomes a needle (a question raised by R. Jeanloz) we would need to have

$$W/(s_1 s_2 s_3) = \lim_{s_2 \to 0} \lim_{1/s_3 \to 0} (W/s_3 s_2) s_1 = const \tag{40}$$

which means that the strain energy density W would need to tend to infinity faster than in (39).

The second possibility is that the second part in (38), which is the energy-momentum tensor/driving force, is zero on the whole boundary, which is symmetry preserving. Symmetry will require energy to be spent to move the whole spherically expanding boundary, which as a maximum may be unstable for volume collapse, or, which it may not have enough to move the boundary, as discussed next for model B of Knopoff and Randall (1970) of drop of bulk modulus under pressure.

We consider the case that the ellipsoid will flatten, which is the least energy consuming mode to move the boundary and is stable. Equation (37) dictates that **among all the possible orientations in the three dimensional space the ellipsoid will choose to flatten on that plane where the second factor becomes zero, i.e., where on the line part of the boundary the energy release rate becomes zero, so that** $\delta \dot{E}^{tot} = \lim_{S^d \to 0} \int_{S^d} \dot{l}([[W]] - <\sigma_{ij}>[[u_{i,j}]]) dS = 0$ **is satisfied over the whole boundary, and the energy rate to move the boundary will be the minimum among all other orientations within the solid angle**. If the integral in (37) along the line boundary is negative, the boundary will not move. **It will expand on the plane on which the energy-momentum tensor (38b) first becomes zero (under total loading), and this will be the one of maximum Peach-Koehler type of configurational force applied on the moving boundary as to overcome the self-stress to move it**. **This determines both the magnitude and directions of the axes speeds.**

We may remark here that the axes speeds in the Burridge and Willis (1969) elliptically expanding crack may be determined by extremizing the total Gibbs free energy around the elliptical boundary by taking the variations with respect to the two axes speeds equal to zero, analogously to Mura, eqtns (27.25), 1982).

IV. Model of Deep Focus Earthquakes (DFEs)

Based on the self-similarly expanding ellipsoidal inclusion we present a model for deep-focus earthquakes (DFEs). ***Nucleation*** is assumed to occur as the material undergoes phase change (a change in the moduli and density ("volume collapse") under high pressure. We will assume that an instability of phase change starts instantaneously from zero dimension and that the initiation of phase transformation may induce the instability to continue to ***grow*** by the same amount per unit volume and time. Moreover, we assume that the expanding region of "phase change" is ellipsoidal in shape, expanding with constant axes speeds (to be determined) for some time t, until self-similarity is broken (due to interactions) and the scaling ceases. The equivalent phase change is obtained from equations (22) with the axes speeds in the Eshelby Tensor being unknown and determined from equation (37). The field quantities are those obtained from the solution of the self-similarly expanding inclusion and the Dynamic Eshelby Tensor as a function of the eigenstrain and the axes speeds (Ni and Markenscoff, 2016c).

Symmetry preserving spherical expansion and exclusion of DFE models in the literature

We consider the cases where eqtn (38b) is satisfied point-wise on the expanding boundary, as would be the case for spherical symmetry of the input (loading, eigenstrain, geometry) and spherical expansion, where we have for the "driving force" per unit area in the direction of the normal

$$[[W]] - <\sigma_{ij}>[[u_{i,j}]] = 0 \tag{41}$$

We may note that, in the presence of dissipation, the right-hand-side of the equation of (41) will be not be equal to zero, but to a dissipative term $f(\dot{l})$ depending on the boundary velocity, so that point-wise we have

$$[[W]] - <\sigma_{ij}>[[u_{i,j}]] = f(\dot{l}) \tag{42}$$

which is the equation that drives the expansion (kinetic relation, equation of motion), relating the pre-stress and the phase change to the axes expansion speeds involved in the field quantities. The effects of dissipation are to be explored in the future, as this study focuses on inertia only, but they can be large as in dislocations (Clifton and Markenscoff (1981))

For an expanding inclusion with eigenstrain in a homogeneous elastic material Markenscoff and Ni, (2010) have shown that the driving force can be further written as

$$[[W]] - <\sigma_{ij}^{dyn}>[[u_{i,j}]] = -<\sigma_{ij}^{dyn}>[[\varepsilon_{ij}^{*}]] = <\sigma_{ij}^{dyn}>\varepsilon_{ij}^{*} \tag{43}$$

which is the same expression as in statics (Eshelby (1978), valid for eigenstrain loading only and denoting the self-stresses by σ_{ij}^{dyn} . The last expression in (43) corresponds to the one in Truskinovsky (1987). In the presence of pre-stress, due to linearity we superpose $\sigma_{ij} = \sigma_{ij}^{0} + \sigma_{ij}^{dyn}$, and eqtn (41) with (43) reads

$$< \sigma_{ij} > [[\varepsilon_{ij}^{*}]] = 0 \tag{44}$$

so that

$$\sigma_{ij}^{0}\varepsilon_{ij}^{*} + < \sigma_{ij}^{dyn} > \varepsilon_{ij}^{*} = 0 \tag{45}$$

Equation (45) expresses the decoupling of the applied loading and the self-stress for a moving boundary with eigenstrain, as for dislocations Eshelby (1953), and gives physical meaning to equation (37) obtained from Noether's theorem that a Peach-Koehler type of configurational force applied on the phase boundary by the pre-stress must balance the self-force needed to move the boundary, and yields a ***kinetic relation*** in the presence of inertia. For an expanding spherical inclusion the self-stress term in (45) was obtained by Markenscoff and Ni (2010, 2016). In Markenscoff (2010) are given the evolution equations for a plane and spherical phase inclusion with dilatational eigenstrain in general motion.

The energetics to move the boundary of phase discontinuity as dictated by Noether's theorem have not been considered in the seismology literature before, and, in view of that, we are reassessing whether some models proposed in the literature are possible. Equation (45) allows to determine whether the inclusion boundary has enough energy to propagate in self-similar expansion, and at what speed, given the input quantities, so that the deep focus earthquake becomes possible in this mode. Considering the energy expended to move the boundary, we may remark that in dynamic fracture, in the self-similar growth of a plane crack at one-half the shear wave speed, about one third of the input energy is consumed in the fracture process (Freund, 1990, p329). We examine below Model B of Knopoff and Randall (1970).

Model B of Knopoff and Randall (1970) is the case of drop in the bulk modulus under pressure. We have checked the assumption of spherical growth and the calculation outlined below yielded numerically the result that the energy to move the boundary would exceed the available, and, thus, the spherical dynamic growth is shown not to be possible in model B.

The equivalent eigenstrain for change in bulk modulus under pressure is from (22)

$$\varepsilon_{kk}^{*} = 3(K - K^{*})\varepsilon_{kk}^{0} / \{K^{*}S_{mmnn}^{dyn} + (3 - S_{mmnn}^{dyn})K\} \tag{46}$$

with $S_{kkll}^{dyn} = (\frac{1+\nu}{1-\nu})\frac{(1+2V/a)}{(1+V/a)^2}$ (based on Ni and Markenscoff, 2016b),

and equation (45) yields

$$\begin{aligned}&\sigma_{ij}^{0}\varepsilon_{ij}^{*} - 2\mu(3\lambda+2\mu)/(\lambda+2\mu)\varepsilon^{*2}\\&-2\mu(3\lambda+2\mu)/(\lambda+2\mu)(V/a)^2(3-V/a)/[(1+V/a)(1-(V/a)^2]\varepsilon^{*2} = 0\end{aligned} \tag{47}$$

where the first term is of the Peach-Koehler configurational type, and the second term is the static term (same as in Eshelby, 1978) that would need to be overcome before the motion starts,and the third is the “drag-force” due to inertia. It was obtained in Markenscoff and Ni (2016) for an inclusion with eigenstrain in a homogeneous material, which we are applying for the eigenstrain being the equivalent eigenstrain in a homogeneous inclusion. However, this may be approximate, since it may differ for eqtn (42) for an inhomogeneity, where the external stress has to be computed as in Appendix A. Considering that $\sigma_{ij}^{0}\varepsilon_{ij}^{*}=3K\varepsilon_{kk}^{0}\varepsilon^{*}$, and in view of (46) relating the eigenstrain to the applied strain ε_{kk}^{0} , the term ε^{*2} will factor out in (47) , and numerical calculations have shown that the energy to move the boundary would substantially exceed the input energy. In addition, of course, dissipative loss would need to be overcome. Thus, the analysis has to be obtained of whether drop in bulk modulus under pressure has enough energy to propagate planarly as circular penny-shape, on its own.

Essentially, in the Eshelby approach, the equivalent eigenstrain due to drop in moduli under pressure is proportional to the pressure (eqtn (22). If you increase the pressure, the eigenstrain increases, so the pressure increase does not give enough driving force to move the boundary, because the eigenstrain induced by pressure also has increased proportionally. However, if you have change in density (“volume collapse”), this is an equivalent eigenstrain which in magnitude is *independent* of the pressure (eqtn (21)). So more pressure will provide more energy to move the phase boundary in volume collapse that has eigenstrain independent of the pressure (this does not include the deformation due to the pressure itself). So, *volume collapse* under high enough pressure has enough energy to expand spherically and also *simultaneously with bulk modulus drop* (it would provide the difference). However, Noether dictates that in self-similar expansion the inclusion will take the shape that extremizes the energy to move the boundary, and for stability we may assume that the minimum (planarity) will be favored, and volume collapse may expand *planarly*, simultaneously with other phase change. Thus, the model of Randall (1964) with spherical expansion for change in density may not happen corresponding to a maximum of the energy spent to move the boundary that can be unstable.

Symmetry breaking planar expansions of a region of phase change

The symmetry breaking planar expansion is a flattened self-similarly expanding eIlipsoid (penny-shape), where the boundary shape (two axes speeds) are determined from equation (37). Except for special symmetric cases of loading and geometry it is not possible to satisfy point-wise eqtn (38b), and eqtn (37) may be solved as a minimization problem, that is not addressed here further. We will address some particular physical features of planar propagation (flattened ellipsoid).

In the case that the pre-stress is not hydrostatic pressure, but, due to some tension added in some direction, the principal applied stresses become unequal, there will be planes of maximum shear at 45 degrees to the principal directions so that the flattened ellipsoid will orient itself on them (Figure 2). Equation (45) will express that the maximum in- plane shear acting on the flattened band/ellipsoid will produce the maximum Peach-Koehler force acting on a dislocation (line or loop), since **in the very thin expanding ellipsoid the very large eigenstrain ε_{3i}^{*} will give (in the limit with the thickness in the x_3 direction) a finite rate of displacement discontinuity (Burgers vector). Thus, suddenly emitted gliding dislocations will be produced on the**

planes of max shear to glide out at maximum speed, with the strain energy produced by the phase change under pre-stress condensed into the core of the Burgers vector.

The components of the eigenstrain ε_{ij}^{*} in the flattened ellipsoid are six, the three of which will produce a *dislocation loop* (double couple) on a plane on which the ellipsoid is flattened (with the edge components giving a "double couple"), while the longitudinal in plane eigenstrain components in a very thin ellipsoid are a new dynamic defect. Knopoff and Randall (1970) describe a uniaxial longitudinal eigentrain as screw-pairs according to Weertman.

A defect of a flattened ellipsoid will be volume collapse (change in density) that may break the 3D symmetry and propagate the 3D phase change planarly with infinite large in plane eigenstrains. For this circular defect for isotropy, with the two in-plane longitudinal eigenstrains equal, equation (38b) can be satisfied pointwise, and we can determine the axes speeds when we obtain the self-force. This is a defect that is not occurring in shallow earthquakes. It is a "new" singular dynamic defect that can be viewed as the counterpart of the circularly expanding crack (Craggs, 1966), and may manifest itself in this phenomenon predominantly, differentiating the shallow and deep earthquakes.

We may note that, for general time dependent eigenstrains, Willis (1965) found that inclusions do not yield a dislocation in the limit as Eshelby showed in statics; however, it has to be investigated whether this is the case for self-similar motion of the inclusion. It is conjectured that the limit gives a dislocation loop in self-similarly expanding motion based on the fact that the static Eshelby inclusion *is* a limit of the self-similarly expanding one (Ni and Markenscoff, 2016c). The motion of dislocations jumping to constant velocity (but starting from rest) was analyzed in Markenscoff (1980) and Markenscoff and Clifton (1981). The self-force, or "drag-force", or energy-release rate, needed to be provided for a dislocation to jump from rest to a constant velocity was obtained in Clifton and Markenscoff (1981), with the difference here being that they are generated instantaneously and not from rest. Additional dissipation effects for motion of dislocations are discussed in Clifton and Markenscoff (1981).

V. The dynamic Eshelby Tensor for the penny-shape ellipsoid

Both cases of penny-shape ellipse and flattened elliptical cylinder (band) can be computed by means of the Dynamic Eshelby Tensor (DET) for the particular limiting planar cases of the ellipsoid, which are of the same form as the static ones (see Mura, 1982), but with the ratio of axes speeds in place of the ratio of the axes lengths, which has been also confirmed numerically from the full three dimensional expression of the DET in Ni and Markenscoff, 2016 c). This is because the static Eshelby Tensor is obtained from the dynamic one only from the contributions of the *M* waves. For example, Figure 4 plots the component S_{1313}^{dyn} for the ellipsoid given in [Ni and Markenscoff, 2016c, eqtn 3.9] as an integral evaluated numerically, and compares it with the analytical (asymptotic) penny-shape value for it from the expression in Mura , 1982, eqtn, (11.23) for the penny-shape inclusion, but with ratio of the axes speeds in the place of the lengths, i.e.,

$$S_{1313}^{dyn(penny-shape)} = (1/2)\{1+(\nu-2)/(1-\nu)(\pi/4)s_1/s_3\} \tag{48}$$

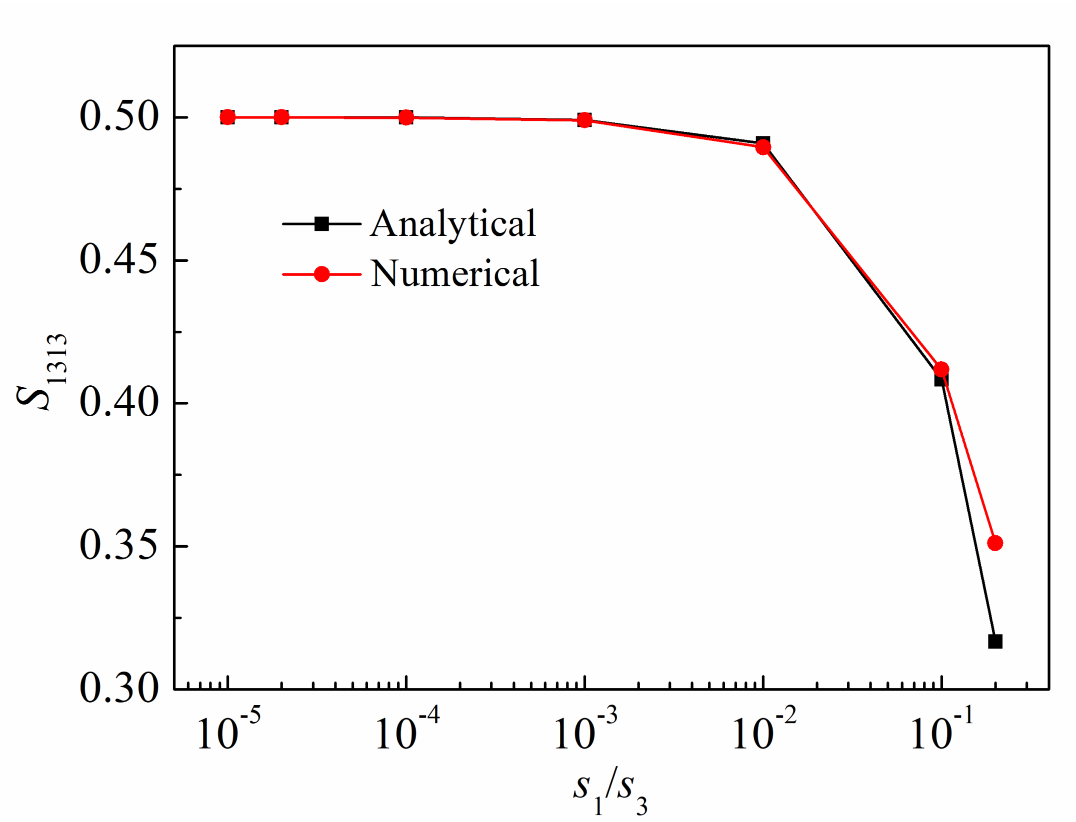

Figure 3. Dynamic Eshelby Tensor component for penny-shape ellipsoid ($\nu = 0.25$)

The agreement is very good for the expansion speed of the third axis (flattened ellipsoid) up to 10^{-2} of the in-plane speed. All computations of the expanding flattened ellipsoids, as in Figures 1b, 1c, can be computed with the expressions for the Dynamic Eshelby Tensor obtained from the corresponding static penny-shape (e.g., Mura, 1982)

VI. Conclusions

In conclusion, we have treated the problem of a self-similarly expanding region of phase change (change in density, i.e., volume collapse, and change in moduli) under pre-stress, with the phase change being an equivalent eigenstrain (obtained through the Dynamic Eshelby Tensor) in the dynamic generalization of the Eshelby ellipsoidal inclusion problem. The axes speeds are the variables of the problem, and, for self-similar expansion, Noether's theorem dictates that they will take the values that extremizes (minimizes for stability) the total energy spent to move the boundary, and that it does not become a sink or source of energy. A particular possibility given by the expression derived from Noether's theorem is that the ellipsoid is "flattened" making the expanding region of phase change *planar*, which constitutes *breaking of the symmetry* of the input and a newly discovered mode of *dynamic cavitational instability or collapse instability*. The planarity is preferred for stability and the ellipsoid takes a flattened shape; in the presence of shear orients itself in space in the directions where the Peach-Koehler force can first generate the defect (dislocations at 45 degrees). The difference with the shallow earthquakes is that the eigenstrains produce cracks with no stresses inside, while in the phase change DFE models we have in-plane components which produce a counterpart new defect. "Volume collapse" (change in density) may break the symmetry and preferably propagate planarly (for energy minimization) so that *the 3D phase change is condensed into the two-dimensional expanding region with phase change (circular in isotropy),* and the planarity will manifest itself in the radiation patterns through the effects of the Dynamic Eshelby Tensor. The flattened penny-shape expanding ellipsoid is a fundamental new problem of a dynamic defect to be solved. The radiation patterns are obtained in terms of the eigenstrains (phase change) and the axes speeds, and the model opens the field for further investigation with regard to seismological data. Both theorems, the Cauchy-Kowalewskaya and Noether's are valid also for anisotropic elasticity. However, for self-similarly expanding anisotropic inclusions we have not yet proven the Eshelby property for the interior domain and we do not have the Dynamic Eshelby Tensor, although an anisotropic interior in an isotropic matrix is obtainable by the isotropic Dynamic Eshelby Tensor as shown here. For full anisotropy in static inclusions the Eshelby property was proven by Willis (1970), and we conjecture that it is true also for self-similarly expanding anisotropic inclusions. The planarity (symmetry breaking) due to Noether's theorem (valid in nonlinear anisotropic elasticity) is independent of scale, valid from the nano to the very large (kilometers) one, and models analogous to the one for DFEs can be developed for other phenomena, such as the dynamic stress inducement of martensitic transformations (Escobar et al, 2000), dynamic shear banding, and amorphization (Zhao, et al, 2017).

Acknowledgment: NSF grant CMMI #1745960 to XM is acknowledged. The author would like to thank Professor Raymond Jeanloz of UC Berkeley for helpful discussions on the deep earthquakes. The contributions of graduate students Xudong Liang of UCSD, and Tan Pengfei and Huang Jiazhao (of the Kun Zhou group) at Nanyang Technical University of Singapore for the numerical evaluations and the figures are acknowledged.

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Appendix A

The jump in the displacement gradient across the self-similarly expanding ellipsoidal inhomogeneity boundary.

We can generalize the Hill (1961) jump conditions to dynamics across the expanding ellipsoidal inclusion, as in Markenscoff (2015) (where a typo needs to be corrected with C^*_{ijkl} in eqtn (28)). We write as in Hill (1961)

$$[[u_{k,l}]] = \lambda_k n_l \tag{A1}$$

where the vector λ_k will be determined point-wise by satisfying the Hadamard jump conditions.

Using the Hadamard equations (11)

$$[[\sigma_{ij}^{0}+\sigma_{ij}]]n_j=\rho\dot{l}[[\partial u_i/\partial t]] \qquad \text{(A2)}$$

and with $\sigma_{ij}^{0}=C_{ijkl}u_{k,l}^{0}$ we obtain

$$\{C_{ijkl}[[u_{k,l}]]+\Delta C_{ijkl}u_{k,l}^{0}+\Delta C_{ijkl}S_{klmn}^{dyn}\varepsilon_{mn}^{**}\}n_j=\rho\dot{l}^{2}[[\partial u_i/\partial x_j]]n_j \qquad \text{(A3)}$$

(having used the other Hadamard condition in (11)), so that from (A3) we have an equation for λ_k

$$K_{ik}^{mb}\lambda_k=-\{\Delta C_{ijkl}\varepsilon_{kl}^{0}+\Delta C_{ijkl}S_{klmn}^{dyn}\varepsilon_{mn}^{**}\}n_j \qquad \text{(A4)}$$

with

$$K_{ik}^{mb}=\{C_{ijkl}-\rho\dot{l}^{2}\delta_{ik}\delta_{jl}\}n_jn_l \quad .$$

We invert as in Mura, 1982,eqtn 6.8, and obtain

$$\lambda_i=-\{\Delta C_{jkmn}\varepsilon_{mn}^{0}+\Delta C_{jkmn}S_{mnrs}^{dyn}\varepsilon_{rs}^{**}\}n_kN_{ij}(\gamma,\vec{n})/D\ (\gamma,\vec{n}) \qquad \text{(A5)}$$

with $\gamma^2=\dot{l}^2\xi_i\xi_i$

The exterior stress σ_{ij}^{+} is then computed and it depends on the applied strain ε_{ij}^{0} , the phase change ΔC_{ijkl} , the ellipsoid axes expansion speeds, the equivalent eigenstrain, the Dynamic Eshelby Tensor and the direction of the normal, but not on time.