“Volume collapse” instabilities in deep-focus earthquakes: a shear source nucleated and driven by pressure

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Symmetry-breaking instabilities in high- pressure phase transformation produce the counterintuitive phenomenon of “volume collapse” producing only shear radiation, with little, or no, volumetric component, even under conditions of full isotropy, and explain the mystery of the long-standing observations in deep-focus earthquakes (400-700 km). Due to instability, at a critical “nucleation pressure”, an arbitrarily small densified region, in the shape of a “pancake-like” flattened ellipsoidal Eshelby inclusion, grows self-similarly as a “lacuna” (zero particle velocity) with the phase transformation occurring under conditions of equilibrium in uniform strain/stress and at constant potential energy (at the vanishing of the M integral, when the radius-expanding driving force $p\varepsilon^{*}_{kk}$ *overcomes the radius-shrinking self-force). The symmetry-breaking flattened shape favors minimization of the energy needed for the boundary to grow large, while for the accommodation of the large collapsing volume in the very thin inclusion deviatoric stresses are developed to avoid openings and overlaps. It is shown that, if an arbitrarily small flattened densified region is generated planarly, and the pressure exceeds the critical nucleation value, then it will necessarily produce a shear seismic source, with little or no, volumetric component, nucleated and driven to propagate by the pressure. The ellipsoid of phase change forms in the direction that minimizes the interaction energy with the pre-stress field and will be close to the direction of max shear pre-stress in the mantle. The obtained stress/deformation fields of a densified 2D flattened elliptical inclusion constitute a new defect that models the “anticrack” in geophysics and densified shear bands. The instability analysis can be extended to the nucleation and growth of the phase transition from water to a solid ice phase under high pressure, with the discovered instabilities providing insight to other phenomena of dynamic phase transformations, such as failure waves, amorphization, planetary impacts, etc.*

I . Introduction

In a series of papers (Ni and Markenscoff, 2016a,b), Markenscoff (2019b), the solution to the dynamic generalization of the seminal Eshelby inclusion problem (Eshelby, 1957), which is the self-similarly expanding ellipsoidal inclusion with uniform transformation strain, was obtained extending the work of Burridge and Willis (1969) and Burridge (1971) on the self-similarly expanding crack. The dynamic problem has the static constant stress “Eshelby property” in the interior domain (Burridge and Willis, 1969, Ni and Markenscoff, 2016a), allowing for the phase change problem to be solved as equivalent eigenstrain (Markenscoff, 2019b). For the self-similarly expanding ellipsoid with uniform transformation strain the governing hyperbolic system of pde’s exhibits a weak lacuna topological property (Atiya *et al*, 1970), (Burridge and Willis, 1969), “a traveling zone of absolute quiet” (Burridge, 1967), so that in the interior of the self-similarly expanding ellipsoid the particle velocity is zero, as shown explicitly in (Ni and Markenscoff, 2016a, Markenscoff, 2021). The constant stress “Eshelby property” has for over half a century deeply affected the understanding of defects and inhomogeneities (Eshelby, 1957, 1961). For an avalanche model of Stage II dislocation plasticity L.M. Brown writes in the *Cavendish Magazine, issue 21, March 2019* : "The Eshelby theorem ensures that the strain/stress is uniform if and only if the inclusion has an ellipsoidal shape" (Markenscoff, 1998). "This ensures that inside the ellipsoidal shape all dislocations move together, as each has the same stress acting on it". The avalanche phenomenon could be connected to the unstable growth of the region of volume collapse treated here. Thus, with the "lacuna" property, the phase transformations in the self-similarly expanding region can take place under equilibrium conditions, and, as shown here, at a critical pressure grow unstably at a constant rate at constant potential energy. Here, we will be focusing on the phase transformation due to change in density ("volume collapse") and the instability created under pressure. The lacuna property and dynamic Eshelby problem for the ellipsoid is also valid for Newtonian fluids (Eshelby 1957, Bilby *et al*, 1975, Markenscoff, 2021) and the dynamic nucleation and growth of phase transition from fluid (water) to a solid phase of ice under high pressures can be analyzed analogously to Markenscoff (2020), or here, providing insight into phase transformations occurring at different depths in Earth's mantle (e.g, Tschauner, *et al* 2018).

The shape of the self-similarly expanding regions of phase change is dictated by Noether's theorem (Noether, 1918), which allows for an instability to occur and the expanding inclusion to assume a flattened "pancake"/band shape (Markenscoff, 2019b), which minimizes the energy needed for it to grow large (Markenscoff, 2020). The deformation field will be analyzed as the asymptotic limit of the flattened Eshelby inclusion (penny-shape) as it accommodates in a vanishingly small thickness a very large change of volume collapsing under pressure, in conditions of full isotropy. We will show, through the vanishing of the M integral (Jackiw, 1972, Budiansky and Rice, 1973) for invariance of the Hamiltonian under scaling, that at a critical value, the pressure acting on the flattened surface of the expanding ellipsoid of phase change provides enough driving force (which equals the pressure times the change in volume ε^*_{kk}) to drive the expansion unstably, overcoming the self-stresses, once an arbitrarily small flattened inclusion has been generated (which is assumed, with the analysis not saying how it was generated). The analysis in this work presents instabilities of regions of phase transformation under pressure that can manifest themselves in deep earthquakes as well as in other phenomena of dynamic phase transformation under shock loading, such as amorphization (Zhao *et al*, 2016a, 2016b) displaying similar features of densified ellipsoidal regions, and failure waves (Kanel *et al*, 1991, Clifton, 1993, Espinosa *et al*, 1997, Said and Glimm, 2018). To demonstrate the main effect of the instability phenomenon we will focus the analysis on phase transformation involving change in density (as equivalent transformation strain of "volume collapse" ε^*_{kk}) in full isotropy; however, all the properties of the dynamic Eshelby problem are extendable to full anisotropy (Willis, 1971). For the application to the specific phase transformations in the deep-focus earthquakes the analysis would need to be reworked along the Willis Fourier transform approach and be evaluated numerically.

Deep-Focus Earthquakes (Frohlich, 2006) (at 400-700kms) are an almost century-old mystery in geophysics, expressed in a recent statement (Wang *et al*, 2017) "how fractures initiate, nucleate, and propagate at these depths remains one of the greatest puzzles in earth science". The deep-focus earthquakes are different from the shallow ones that are due to brittle failure, because, under the very high pressures, the material is above the brittle-ductile transition; consequently, the phase transformation hypothesis (e.g., from metastable olivine to denser wadsleyite or ringwoodite) (Zhan, 2020, Burnley and Green, 1989) has been advocated as the most prevalent one (Zhan, 2020) with supportive evidence of geological material densities and analog experiments (Frohlich, 1989, Meade and Jeanloz, 1989,1991, Schubnel *et al*, 2013). Meade and

Jeanloz (1989) recorded emissions at 70GPA for a phase transition from β -Sn to a simple hexagonal structure in Ge, interpreted as an indication of the deep-focus earthquakes being a "shear instability". The puzzle arises "with the main issue being that a large volume change associated with phase transition would produce an isotropic component in the seismic focal mechanism in disagreement with most observations" (Meade and Jeanloz, 1991), as shear sources with no, or very little, volumetric radiation were observed consistently over a long time (Zhan, 2020, Frohlich, 1989) with the statement that "any successful model of the deep-earthquake source must explain why deep earthquakes are predominantly deviatoric" (Frohlich, 1989). Knopoff and Randall (1970) proposed the model C producing a CLVD (Compensated Linear Vector Dipole) shear radiation, while a volumetric source with change in density was treated in spherical expansion by Randall (1964). Since, despite modeling by classical plasticity and advanced numerical techniques, the generation and propagation of the big earthquakes has remained unanswered e.g., Wang *et al*, 2017, the phase transformation hypothesis seemed to be losing ground and other venues were pursued, such as anisotropy, but the obtained CLVD was qualified as "artifact" of anisotropy in Zhan (2020). While the phenomenon of one fault initiating others as in a "self-organization" model in Green and Burnley (1989) is plausible, the present analysis focuses on one source being nucleated and driven to propagation before such interactions take place, and for which the self-similar expansion solution applies (Barenblatt, 1996). A main characteristic of deep-focus earthquakes is that they occur planarly, e.g., Frohlich, 1989, Schubnel *et al*, 2013. The analysis in this work obtains the stress and deformation field of a densified three-dimensional flattened ellipsoid and of the flattened densified two-dimensional elliptical cylinder (band). This is the correct mechanics of the "densified anticrack" by Burnley and Green (1989), which is the prevalent model for deep earthquakes (Zhan, 2020). The "densified anticracks" are not the opposite of the Griffith crack, as incorrectly assumed in the aforementioned publications, but densification should also affect the in-plane direction; they constitute the new defect analyzed here. It is shown that the densified planar inclusions, in order to accommodate a large volume collapse into a very thin inclusion develop unequal eigenstrains, and, consequently, deviatoric stresses that produce to leading order distortional strain energy density, even under conditions of full isotropy. Most significantly, because of the densification, it is shown here that "densified anticracks" do not "fail in shear mode under deviatoric stress" as stated in Zhan, 2020, Fig 5 (and has been implied in the literature all along), but they fail in shear mode under *pressure* (acting on the volume phase change), which is a central point in the analysis in this paper.

II. The nucleation of a 3D shear source as successive instabilities of "volume collapse" under pressure

We shall assume fully isotropic conditions in material symmetry and a pre-stress of hydrostatic pressure, in order to demonstrate that a shear seismic source is nucleated *even without* deviatoric component in the pre-stress, or other asymmetries. The relaxation of some of the symmetries produces a Double Couple (DC), in addition to the CLVD. The densified region is modeled as a self-similarly expanding Eshelby ellipsoidal inclusion (Markenscoff, 2019b) ("pancake-like") containing uniformly distributed transformation eigenstrain ε_{ij}^{*} equivalent to the change in density. In geophysics the eigenstrain was first called "stress-glut" in Backus and Mulcahy (1976). The governing equation of elastodynamics for the displacement $u_j(x,t)$ of the self-similarly expanding ellipsoidal inclusion, with *constant* axis speeds $1/s_i$, (s_i for slowness), starting from zero dimension with *zero initial and radiation conditions*, is

$$\rho\frac{\partial^2 u_j}{\partial t^2} - C_{jklm}\frac{\partial^2 u_l}{\partial x_k\,\partial x_m} = -C_{jklm}\varepsilon^*_{lm}\frac{\partial}{\partial x_k}H(t-(s_r^2 x_r^2)^{1/2}) \qquad (1)$$

and constitutes the dynamic generalization of the Eshelby inclusion problem (Ni and Markenscoff, 2016a,b). The pre-stress is added to the stresses derived from equation (1). Due to dimensional analysis and analytic properties the system reduces to an elliptic system in the variable $\vec{z} = \vec{x}/t$, and there is no particle velocity in the interior domain (Ni and Markenscoff, 2016a), which is a "lacuna" because the M waves emitted by the expanding phase boundary (as to satisfy the Hadamard jump conditions on the interface) cancel the particle velocity due to the P and S, and lock-in a constant (spatially and temporally) stress field in the interior of the self-similarly expanding ellipsoid (Ni and Markenscoff, 2016a, Markenscoff, 2021). Thus, the phase transformation takes place under conditions of equilibrium (consistent with Incel *et al*, 2019) and under uniform stress; the Eshelby assumption of uniform eigenstrain in the inclusion is justified on the argument that, if the instability starts with a given change in density, it can continue at the same rate under constant potential energy, as shown below. The M waves give the static Eshelby ellipsoidal inclusion as a limiting case (Ni and Markenscoff, 2016b, Markenscoff, 2021).

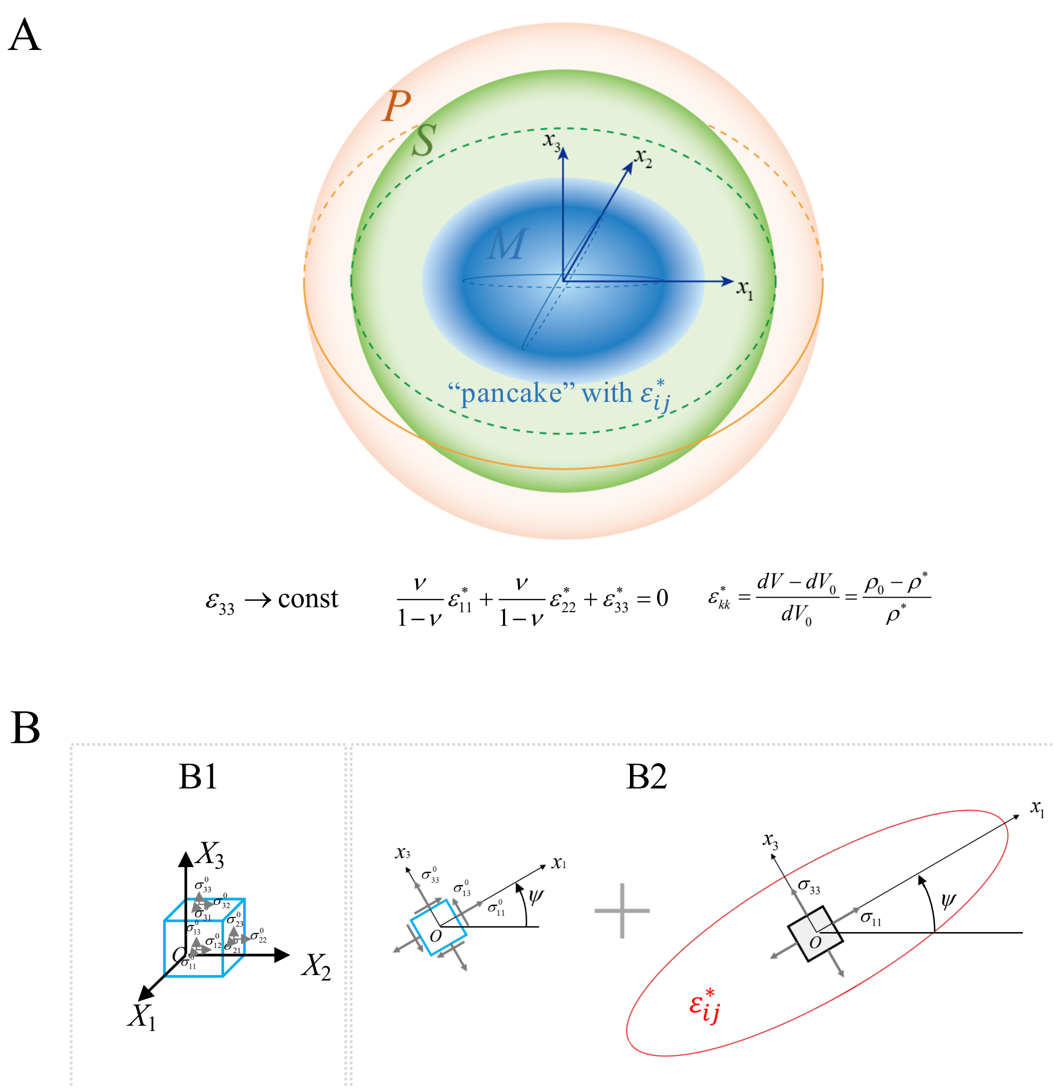


Figure 1: *(A) A self-similarly expanding ellipsoidal inclusion with uniform transformation strain* ε^*_{ij} *(change in density* ε^*_{kk} *) emits pressure (P) waves, shear (S) waves and (M) waves emitted by the phase boundary, which is the degenerate M wave-front (eqtn(1)); the M waves cancel the particle velocity of the P and S in the interior of the inclusion ("lacuna"), and lock-in a constant stress interior field, so that the phase transformation occurs under equilibrium conditions and grows at a constant rate at constant potential energy (Fig 4). The inclusion assumes a flattened "pancake-like" shape (symmetry-breaking) to minimize the energy for the boundary of phase change to grow large and develops deviatoric eigenstrains (and stresses) to maintain material continuity. In (B2) the ellipsoid of phase change* ε^*_{ij} *orients itself in a direction* ψ *that minimizes the interaction energy (eqtn(2)) of the inclusion of phase change ("volume collapse") with the pre-stress field* $\sigma^{(0)}_{ij}$ *(in (B1)).*

The shape and the energetics of the growth of moving defects are governed by Noether's (1918) theorem of the calculus of variations in a variable domain. The surface of the densified inclusion is a surface of discontinuity of the strain energy density and of the displacement gradient, and the mismatch between the two creates "an effective force" (configurational) on the interface, $dF_l = [[P_{lj}]]n_j dS$, where $P_{lj} = W\delta_{lj} - \sigma_{ij}u_{i,l}$ is the "energy-momentum tensor" (e.g. Eshelby, 1970, 1977) , $\delta\dot{E}^{tot} = \int_S \delta\dot{\xi}_l dF_l dS$ is the change in the total energy (Gibbs' free energy (Eshelby, 1961)), due to a displacement of the interface by $\delta\xi_l$. Under total loading, the total configurational force on the interface has to vanish so that *the moving boundary does not become a sink or source of energy* (Markenscoff, 2019b). Ensuring that, the self-similar shape of a surface of discontinuity is obtained by use of Noether's theorem for invariance of the Hamiltonian under translation (Markenscoff, 2019b), and the total energy-release rate on a contour surface S^d surrounding the defect and shrinking onto it (independently of the shape of the shrinking contour) must be zero, i.e.,

$$\delta\dot{E}^{tot} = -\lim_{S^d\to 0}\int_{S^d}\{[[W+T]]\upsilon_n + [[\sigma_{ij}\dot{u}_i]]n_j\}dS = -\lim_{S^d\to 0}\int_{S^d}\dot{l}([[W]] - <\sigma_{ij}>[[u_{i,j}]])dS = 0 \tag{2}$$

where $\upsilon_n \equiv \dot{l}$ denotes the outward normal boundary velocity, the strain energy density in the inclusion with eigenstrain is $W = 1/2\sigma_{ij}(\varepsilon_{ij} - \varepsilon^*_{ij}) = 1/2C_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl})(\varepsilon_{ij} - \varepsilon^*_{ij})$, the double brackets denote jumps, and <.> denotes the average across the surface of discontinuity. Eqtn (2) allows the possibility of a flattened shape with $\dot{l} = 0$ in a *symmetry-breaking instability,* which is consistent with observations on deep earthquakes (e.g., Schubnel, *et al*, 2013). Eqtn (2) also determines the direction in which the ellipsoid forms, minimizing the interaction energy of the ellipsoid of phase change with the pre-stress field if included in the field quantities in (2) (Fig 1 B). As shown in Markenscoff, 2020, the flattened shape minimizes the energy spent to expand the region of phase discontinuity as the radius of the flattened pancake inclusion becomes large (tending to infinity), as compared to the sphere that minimizes it for small radius (tending to zero), --the corresponding *M* integrals vary as a^2 versus a^3 --. We may note that Hurtado and Kim (1999) used the 3D *M* integral for the stability analysis of penny-shaped dislocation instability.

We proceed to analyze the self-similarly expanding penny-shape inclusion with change in density assumed to be produced by unknown eigenstrains with trace equal to the "volume collapse" $\varepsilon_{kk}^{*}=(dV-dV_{0})/dV_{0}=(\rho_{0}-\rho^{*})/\rho^{*}$ given at zero stress, when the inclusion is taken outside the matrix in the Eshelby thought experiment; upon re-insertion in the matrix the strain in the body is $\varepsilon_{ij}=S_{ijkl}^{dyn}\varepsilon_{kl}^{*}$, with S_{ijkl}^{dyn} denoting the Dynamic Eshelby Tensor in the self-similar expansion obtained in (Ni and Markenscoff, 2016b) for isotropy; it is a function of the (constant) axes speeds and the Poisson's ratio of the matrix. The isotropic Eshelby Tensor allows for the isotropic material in the matrix to transform to anisotropic in the interior (Eshelby, 1961), while the constant stress "Eshelby property" is valid in general anisotropy as shown by Willis, 1971, which allows to obtain the solution for an anisotropic material to transform into another anisotropic in self-similar expansion by the use of Fourier transforms. In the *asymptotic analysis* of a *flattened* ellipsoidal inclusion accommodating a very large change in density as the ratio of the axes speeds tends to zero (Fig 1A), in order for *the total strain energy to be finite (and not zero) in the very thin inclusion* ($1/s_{3}\to 0$), the eigenstrains *must tend to infinity* $\varepsilon_{ij}^{*}\to\infty$ as the ratio of the axes speeds (small to large) *tends to zero*, $s_{1}/s_{3}\to 0$, so that their product $\lim_{\varepsilon_{ij}^{*}\to\infty,1/s_{3}\to 0} s_{1}/s_{3}\varepsilon_{ij}^{*}\to const$ is a finite constant. For simplicity, and to focus on the flattened effect of symmetry breaking we will first consider penny-shape axisymmetry in the "pancake", i.e., $\varepsilon_{11}^{*}=\varepsilon_{22}^{*}\neq\varepsilon_{33}^{*}$, and $1/s_{1}=1/s_{2}$. With the dynamic Eshelby Tensor in the *asymptotic limit* as $s_{1}/s_{3}\to 0$ the normal strain is (Markenscoff, 2019b, Mura, 1982, eqtns (11.23)),

$$\varepsilon_{33}=S_{33kl}^{dyn}\varepsilon_{kl}^{*}=2\{\nu/(1-\nu)\}[1-\{(4\nu+1)/8\nu\}\pi(s_{1}/s_{3})]\varepsilon_{11}^{*}+[1-\{(1-2\nu)/4(1-\nu)\}\pi(s_{1}/s_{3})]\varepsilon_{33}^{*} \tag{3}$$

which gives the static limit, in the ratio of the axes lengths $\lim_{t\to\infty,1/s_{i}\to 0} s_{1}t/s_{3}t\to a_{3}/a_{1}$, and consists of the contributions of the *M* waves (Markenscoff, 2019b). In order that the strain $\varepsilon_{33}\to const$ (and not infinite), so as to maintain material continuity (the faces of the pancake not to open or overlap as a very large (infinite in the asymptotic analysis) volume ε_{kk}^{*} is accommodated in a very thin inclusion), we will impose the condition that the infinitely large terms in the eigenstrains cancel each other in equation (3), i.e., the zero-th order term vanishes by satisfying $2\nu/(1-\nu)\varepsilon_{11}^{*}+\varepsilon_{33}^{*}=0$, which we call the "planarity condition". The *unequal principal eigenstrains* have a mean value $\varepsilon_{m}^{*}=\varepsilon_{kk}^{*}/3=-(1-2\nu)/3\nu\varepsilon_{33}^{*}$, which yields the deviatoric components of a CLVD (e.g., Julian *et al*, 1998), as

$$\begin{pmatrix} \varepsilon_{11}^{*}-\varepsilon_{m}^{*} & 0 & 0 \\ 0 & \varepsilon_{11}^{*}-\varepsilon_{m}^{*} & 0 \\ 0 & 0 & \varepsilon_{33}^{*}-\varepsilon_{m}^{*} \end{pmatrix} = -(1+\nu)/\{3(1-2\nu)\}\varepsilon_{kk}^{*}\begin{pmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4a}$$

If we remove the axisymmetry, and consider that the three eigenstrains are unequal, $\varepsilon_{11}^{*}\neq\varepsilon_{22}^{*}\neq\varepsilon_{33}^{*}\neq\varepsilon_{11}^{*}$, and also unequal in-plane axes speeds, $1/s_{1}\#1/s_{2}$, the volume change of the flattened ellipsoid is

$$\Delta V/V=\varepsilon_{kk}=S^{dyn}_{kk11}\varepsilon^*_{11}+S^{dyn}_{kk22}\varepsilon^*_{22}+S^{dyn}_{kk33}\varepsilon^*_{33}=[\nu/(1-\nu)+(1-2\nu)/(1-\nu)\pi/4(s_1/s_3)]\varepsilon^*_{11}+$$
$$[\nu/(1-\nu)+(1-2\nu)/(1-\nu)\pi/4(s_2/s_3)]\varepsilon^*_{22}+[1-(1-2\nu)/(1-\nu)\pi/2(s_1/s_3)]\varepsilon^*_{33} \quad (5)$$

so that, the "planarity condition" obtained from the vanishing of the zero-order terms in ε_{33}, i.e., $\nu/(1-\nu)\varepsilon^*_{11}+\nu/(1-\nu)\varepsilon^*_{22}+\varepsilon^*_{33}=0$, produces *a zero change of volume* $\Delta V/V$ radiation in equation (5), also for the pancake with unequal in-plane eigenstrains and axes speeds.

For unequal in-plane eigenstrains, $\varepsilon^*_{11}\neq\varepsilon^*_{22}$, a double couple (DC) is produced in the last term (in addition to the CLVD's), by decomposing as

$$\begin{pmatrix}\varepsilon^*_{11}&0&0\\0&\varepsilon^*_{22}&0\\0&0&\varepsilon^*_{33}\end{pmatrix}=1/2\begin{pmatrix}\varepsilon^*_{11}&0&0\\0&\varepsilon^*_{11}&0\\0&0&\varepsilon^*_{33}\end{pmatrix}+1/2\begin{pmatrix}\varepsilon^*_{22}&0&0\\0&\varepsilon^*_{22}&0\\0&0&\varepsilon^*_{33}\end{pmatrix}+1/2\begin{pmatrix}\varepsilon^*_{11}-\varepsilon^*_{22}&0&0\\0&\varepsilon^*_{22}-\varepsilon^*_{11}&0\\0&0&0\end{pmatrix}$$

(4b)

A double couple (DC) is also produced (in addition to the CLVD) by a "pancake" shape inclusion with change in shear modulus μ under uniaxial extension, which is Model C in Knopoff and Randall (1970). Since they considered spherical growth of the region of phase change, they correctly predicted only a CLVD radiation for their model C.

For the penny-shape dynamically expanding inclusion the interior stresses $\sigma_{ij}=C_{ijkl}(\varepsilon_{kl}-\varepsilon^*_{kl})$ with inertia are (to leading order)

$$\sigma^{\text{int}}_{11}/2\mu=\sigma^{\text{int}}_{22}/2\mu=-\nu/(1-\nu)(\varepsilon^*_{11}+\varepsilon^*_{22})-\varepsilon^*_{11}=(1+\nu)/2\nu\varepsilon^*_{33}=-(1+\nu)/\{2(1-2\nu)\}\varepsilon^*_{kk}, \quad (6a)$$
$$\sigma_{33}\sim O(s_1/s_3\varepsilon^*_{kk}) \quad (6b)$$
$$\sigma_m=-(1+\nu)2\mu\varepsilon^*_{kk}/3(1-2\nu)\equiv -K\varepsilon^*_{kk} \quad (6c)$$

For infinitely large eigenstrains the in-plane normal stresses due to volume collapse occurring planarly are very large and tensile, while in the normal direction the stress is small, creating large deviatoric stresses; the mean stress is given by (6c).

Having obtained the stress fields we can evaluate eqtn (2) and obtain the energetics of the "driving force" on the boundary of the self-similarly expanding "pancake" shape inclusion. "The driving force" on a surface of discontinuity per unit surface area was evaluated (Markenscoff and Ni, 2010) to be

$$f\equiv[[W]]-<\sigma_{ij}>[[u_{i,j}]]=-<\sigma_{ij}>[[\varepsilon^*_{ij}]]=<\sigma_{ij}>\varepsilon^*_{ij} \quad (7)$$

where the (self) stresses σ_{ij} are those derived from eqtn (1) with zero initial conditions. We will proceed to consider the presence of pre-stress of an applied large hydrostatic pressure $p^{appl}_1=p^{appl}_2=p^{appl}_3=p^{appl}$, so that the total configurational force (per unit area of the phase boundary) is by linear superposition in elasticity $< p_{ij} + \sigma_{ij} > \varepsilon^*_{ij}$, also Markenscoff (2010). We will evaluate eqtn (2) (Fig 2) on a contour surrounding the surface of the self-similarly expanding densified penny-shape inclusion with axes lengths $a_1 = t / s_1 = a_2$, and the ratio $a_3 / a_1 = s_1 / s_3$. With $\dot{l}$ denoting the outward normal boundary velocity (derived in terms of the axes speeds of the ellipsoid in Markenscoff, 2019b), we set $\dot{l} = \dot{\lambda} a_3 = \dot{\lambda} t / s_3$ on the upper and lower faces of the densified "pancake", and $\dot{l} = \dot{\lambda} a_1 = \dot{\lambda} t / s_1$ at the tip perimeter, where $\dot{\lambda}$ is a *dimensionless scaling parameter*. We consider the scaling with $\dot{\lambda}$ and eqtn (2) will reduce to the M_O integral about the origin of the coordinate system. The M integral for scaling symmetry associated with dimensional analysis in theoretical physics (Jackiw, 1972) is a path-independent integral in anisotropic linear elastostatics and is derived from Noether's theorem for invariance of the Hamiltonian Π under scaling (e.g., Markenscoff and Pal Veer Singh (2015)). In self-similar dynamic expansion of the region of phase change, with no kinetic energy in the interior domain, for the M integral, as with the dynamic J integral (energy-release rate under translation (Markenscoff, 2019b, Rice, 1985), we have "contour independence" (rather than path-independence) as the contour shrinks onto the surface of the phase discontinuity and is independent of the shape of the shrinking contour in the limit (e.g. for dislocation, Clifton and Markenscoff, 1981), with the inertia effects to be discussed for the 2D case in the last section. We consider quasi-static expansion and we evaluate the integral in (2) on the surface of the "pancake" ($a_1 = a_2 , a_3 / a_1 << 1$) with the use of (7), and with $\delta \dot{a}_1 = \dot{\lambda} a_1$, and write the (quasi-static) M integral as,

$$\delta \dot{E} = -M_O \dot{\lambda} = \dot{\lambda} \lim_{a_3/a_1 \to 0} a_3 \{(\sigma_{33} + p_3^{appl})\varepsilon^*_{33} + 1/2(\sigma^-_{11} + \sigma^+_{11} + 2p_1^{appl})\varepsilon^*_{11} + 1/2(\sigma^-_{22} + \sigma^+_{22} + 2p_2^{appl})\varepsilon^*_{22}\} 2\pi a_1^2 + \dot{\lambda} a_1 J_{tip}(2\pi a_1)$$

$$= [\lim_{a_3 \to 0, \varepsilon^*_{ij} \to \infty} a_3 \{(\sigma_{33} + p^{cr})\varepsilon^*_{33} + 1/2(\sigma^-_{11} + \sigma^+_{11} + 2p^{cr})\varepsilon^*_{11} + 1/2(\sigma^-_{22} + \sigma^+_{22} + 2p^{cr})\varepsilon^*_{22}\} - J_{tip}] 2\pi a_1 \delta \dot{a}_1 = 0$$

(8)

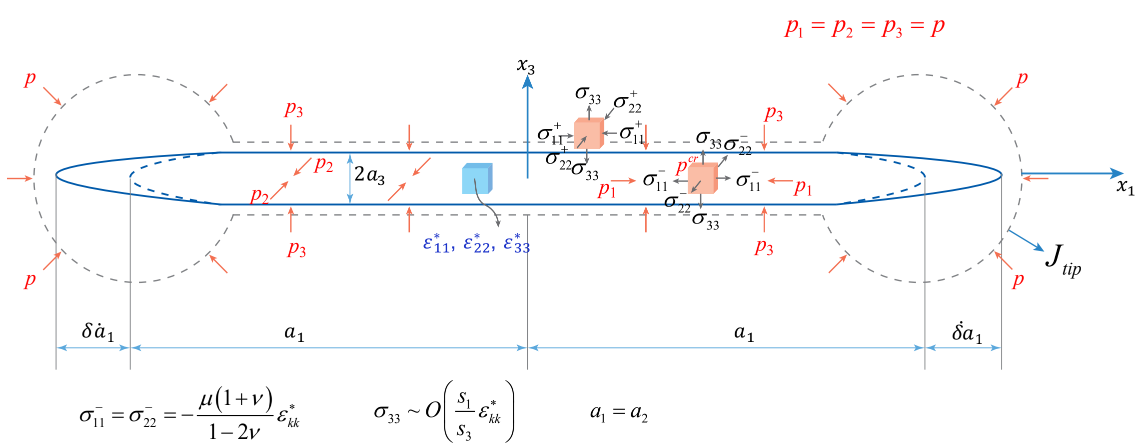


Figure 2: *Nucleation instability is the self-similar growth by* $\delta \dot{a}_i = \dot{\lambda} a_i$ *of an arbitrarily small defect (3D penny-shape inclusion,* $a_1 = a_2$, $a_3 / a_1 << 1$*) at the vanishing of the M integral (i.e., at constant potential energy). The Peach-Koehler "driving force" on a contour surrounding the pancake inclusion is provided by the leading term* $p(\varepsilon_{11}^* + \varepsilon_{22}^* + \varepsilon_{33}^*)$ *(pressure times volume change) in the M integral in eqtn (8), and at the critical pressure it balances the self-stresses* $p_{pancake}^{cr} \varepsilon_{kk}^* + \sigma_{11}^{(\text{int})} \varepsilon_{11}^* = 0$ *(radius-increasing driving force versus radius-shrinking self-force). The* J_{tip} *integral resisting the advance of the tip (in 2D, eqtn (50)) is of lower order in the ratio of the axes lengths* $a_3 / a_1 << 1$ *than the driving force* $p\varepsilon_{kk}^*$.

In eqtn (8) the J_{tip} integral is equal to the energy-release-rate G to advance the tip of the penny-shape incrementally by δa_1 (Rice, 1985) and is obtained in section III, eqtn (50) for the 2D quasi-static case**,** while in eqtn (54) are included inertia considerations. In eqtn (8) the minus sign superscript denotes the limit of the quantities to the inner side of the surface of discontinuity and the plus sign the limit to the outer side. We note that the traction is continuous across the top/lower surfaces due to the vanishing of the boundary velocity for a "pancake", that the external stress components $\sigma_{11}^+, \sigma_{22}^+$ parallel to the interface are of the order of the ratio of the axes speeds (and, thus, lower than the interior in-plane stresses), and that the interior stresses $\sigma_{11}^-, \sigma_{22}^-, \sigma_{33}^-$ are the stresses with inertia obtained in eqtns (6). Eqtn (8) has the same expression for equal or unequal eigenstrains in the pancake plane (which produce an additional DC), so that the pressure acting on the change in volume is driving both the CLVD and DC propagation.

The vanishing of the *M* integral in eqtn (8) produces a ***nucleation instability*** at the vanishing of the quantity in the brackets, which gives the critical value of the hydrostatic pressure, at which *at any arbitrarily small densified pancake-shape inclusion can grow incrementally by* $\delta \dot{a}_1 = \dot{\lambda} a_1$ (scaling with $\dot{\lambda}$) at the same rate ε_{kk}^* (uniform eigenstrain for Eshelby inclusions) ***at constant potential energy of the system.*** In the words of an anonymous referee "the radius-expanding driving force on the surface of discontinuity overcomes the radius-shrinking self-force" in eqtn (8), where the Peach-Koehler force $p^{cr} \varepsilon_{kk}^*$ balances the self-forces, to the leading order, for $p_{pancake}^{cr} \varepsilon_{kk}^* + \sigma_{11}^{(\text{int})} \varepsilon_{11}^* = 0$, yielding *the critical pressure for the nucleation of a pancake-like densified 3D inclusion* as*,*

$$p_{pancake}^{cr} = (1+\nu)(1-\nu) / \{2(1-2\nu)^2\} \mu \varepsilon_{kk}^* \tag{9a}$$

The critical pressure to *nucleate a densified 2D band* is obtained in section III, eqtn (53), to leading order as

$$p_{band}^{cr} = (1-\nu) / (1-2\nu)^2 \mu \varepsilon_{kk}^* \tag{9b}$$

Equations (9) show that, to leading order, the critical nucleation pressure depends only on the "volume collapse", Poisson's ratio, and shear modulus, and are plotted in Fig 3. The pressure to nucleate a spherical inclusion was obtained in Markenscoff (2020) as $p^{cr(sph)} = (2/9)\mu(1+\nu)/(1-\nu)\varepsilon_{kk}^*$, (Fig 3, in blue), and it is much smaller than the pressure needed to nucleate a flattened shape densified defect in eqtns (9), which additionally contains a

CLVD shear deformation. This suggests that spherical inclusions may have been nucleated but not be able to grow larger, as they would require too much energy to grow to a large radius (Markenscoff, 2020), consistent with observations of planar zones and non-volumetric radiation in deep-focus earthquakes. It should be noted that the self-similarly expanding Eshelby solution does not allow for a change in shape and self-similarity must be maintained, and the arbitrarily small generated densified inclusion must be considered of penny-shape (pancake-like) shape. The theory is not considering how the arbitrarily small flattened densified inclusion has been "generated". "Nucleation" has been defined in Markenscoff (2020) as growth at constant potential energy from an (already "generated") arbitrarily small size, and is used here in the same sense. In eqtn (8) the term of the Peach-Koehler driving force $p^{cr}\varepsilon^*_{kk}$ is larger than the resisting J_{tip} , (eqtn (50) in 2D), so that the growth will be unstably fast --unhindered, as a collapsing "house of cards"-- in the absence of other dissipation, once nucleated.

The total volumetric stress in the inclusion is the sum of the applied pressure $p^{A(incl)} = K\varepsilon^{(0)}_{kk}$ (pre-stress) plus the mean stress p^I in the inclusion (Eshelby, 1961),

$$p^{inc} = p^{A(inc)} + p^I \equiv K\varepsilon^{(0)}_{kk} + K(\varepsilon_{kk} - \varepsilon^*_{kk}) \tag{10}$$

and, by comparison of eqtn (9) to eqtn (6c) (Fig 3), *the critical pressure to nucleate* the densified flattened inclusion is always *larger* in magnitude than the mean stress $p^I = \sigma_{mean} = K(\varepsilon_{kk} - \varepsilon^*_{kk})$ (or p^I), if the inclusion is a 2D band, while for the penny-shape it is larger for Poisson's ratio >0.2 , which is indeed the case for olivine (0.29) and geological materials at these depths (Dziewonski and Anderson, 1981)· With a densification eigenstrain ε^*_{kk} of the order of 10% , the nucleation pressure in eqtn (9) is in general agreement with values of the pressure at the depths of the deep-focus earthquakes (e.g. Dziewonski and Anderson,1981). Also, to be remarked, in serpentine acoustic emissions were recorded at pressures corresponding to depths below the seismic barrier at 700kms (Meade and Jeanloz, 1991), thus implying that some other phenomenon causes the seismic barrier (R. Jeanloz, private communication).

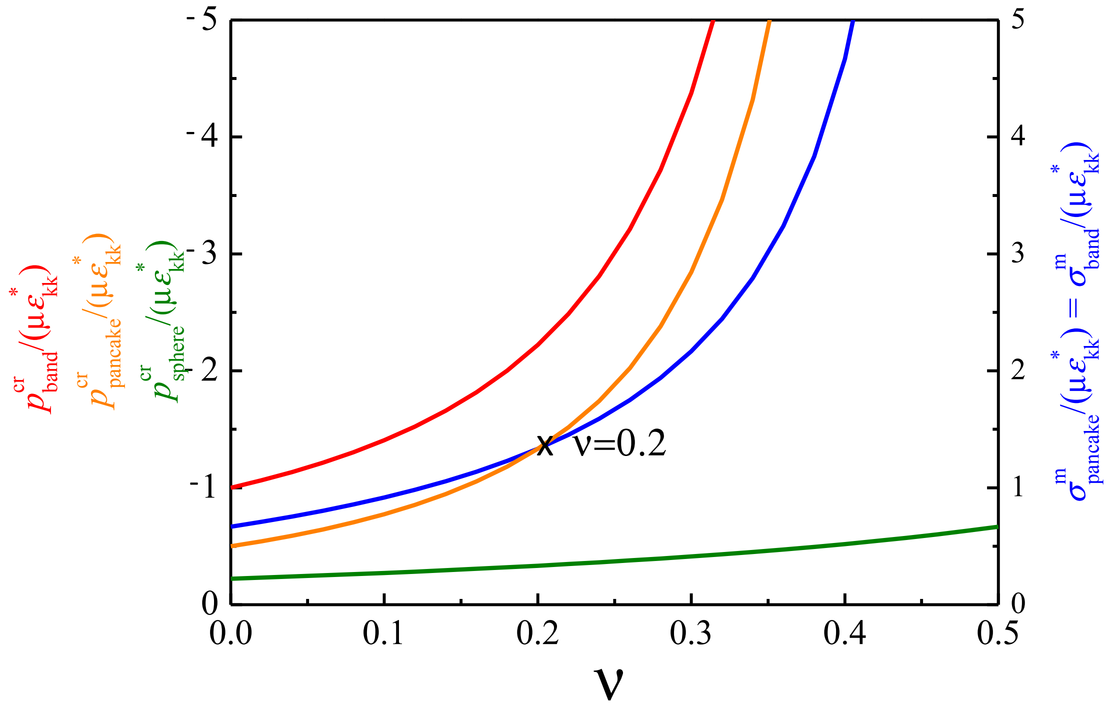


Figure 3: Shear seismic source: *The magnitude of the critical pressure (* $p^{cr}/\mu\varepsilon^*_{kk}$ *) at nucleation of a 3D "pancake" (orange) and a 2D band (red) as function of Poisson's ratio (Poisson's ratio is > 0.2 for the geological materials -- for olivine is 0.29-- at these depths (Dziewonski and Anderson (1981)); the large tensile mean stress (blue) in the inclusion due to "volume collapse"* ε^*_{kk} *is cancelled by the large compressive pressure of the pre-stress, while, for small deviatoric pre-stress, the deviatoric stresses (due to deviatoric eigenstrains in the flattened shape,- "planarity"-) remain large and predominant, producing to leading order distortional strain energy density, and a shear- seismic source. The pressure needed to nucleate a flattened shape is higher than the one to nucleate a sphere (green), but the spherical inclusions require more energy to grow to a large radius.*

Thus, for a pre-strain $\varepsilon^{(0)}_{kk}$ of the order of the "volume collapse" ε^*_{kk}, the compressive pre-stress cancels the tensile stress $-K\varepsilon^*_{kk}$, and the remaining volumetric term $K\varepsilon_{kk}$ is of lower order according to eqtn (5) with the zero-th order terms vanishing (planarity condition). However, the deviatoric stresses in the inclusion

$$\mu(\grave{\varepsilon}^A_{ij} + \grave{\varepsilon}_{ij} - \grave{\varepsilon}^*_{ij}) \qquad (ij=11,22,33) \tag{11}$$

are large due to the large deviatoric eigenstrains $\grave{\varepsilon}^*_{ij}$, since the deviatoric pre-stress $\mu\grave{\varepsilon}^A_{ij}$ at the depths of deep-focus earthquakes is of much smaller order (Frohlich, 2006) than the pressure, and it cannot cancel the ones due to the deviatoric eigenstrains (caused by the pancake geometry/

"planarity condition"). Thus, the leading order contribution to the strain energy density in the flattened inclusion with large volume collapse under high pressure will be the deviatoric component, and the densified inclusion will constitute a shear seismic source, with zero, or negligible, volumetric component. Although the Eshelby property is not valid in nonlinear elasticity, the *asymptotic* analysis for the flattened inclusion differentiates the order of the eigenstrains (large/asymptotically infinite) from the order of the strains (small/finite), with the order of the pre-stress (large for pressure, small for deviatoric pre-stress) governing the behavior of the system, and yielding the predominantly shear-source behavior.

We have shown that, under full isotropy in material, geometry and loading, a CLVD can be nucleated at a critical pressure. If we relax the symmetries, so that the eigenstrains $\varepsilon_{11}^* \neq \varepsilon_{22}^*$ are unequal, or assume the presence of a deviatoric pre-stress with change in shear modulus, this will produce a double couple (DC) (Markenscoff, 2019b) added to the present analysis. The additional effects of change in bulk modulus can be included as equivalent eigenstrains, as in Markenscoff, 2020.

For change in shear modulus (from μ to μ^*) under the applied strain $\varepsilon_{31}^{(0)}$, we have the eigenstrain

$$\varepsilon_{31}^* = (\mu - \mu^*)\varepsilon_{31}^{(0)} / \{2S_{3131}^{dyn}\mu^* + (1-2S_{3131}^{dyn})\mu\} \tag{12a}$$

with $S_{3131}^{penny-shape} = 1/2\{1+(\nu-2)/(1-\nu)\pi/4s_1/s_3\}$ to the leading order for the penny-shape "pancake". This produces in the "pancake" a shear strain, $\varepsilon_{31}' = S_{31ij}^{dyn}\varepsilon_{ij}^*$, and a shear stress $\sigma_{31}' = 2\mu(\varepsilon_{31}' - \varepsilon_{31}^*) = \mu(\nu-2)/[2(1-\nu)]\pi s_1/s_3\varepsilon_{31}^*$ to be added to the pre-stress $\sigma_{31}^{(0)}$ so that $\sigma_{31}^{tot} = \sigma_{31}^{(0)} + \sigma_{31}'$.

For change both in bulk and shear moduli in an inhomogeneous inclusion under applied loading we write in the notation of Eshelby (Eshelby, 1961, Bilby *et al*, 1975)

$$K^*(\varepsilon_{kk}^A + \varepsilon_{kk}^C - \varepsilon_{kk}^{*cd}) = K(\varepsilon_{kk}^A + \varepsilon_{kk}^C - \varepsilon_{kk}^{**}) \tag{12b}$$

$$\mu^*(`\varepsilon_{ij}^A + `\varepsilon_{ij}^C - `\varepsilon_{ij}^{*cd}) = \mu(`\varepsilon_{ij}^A + `\varepsilon_{ij}^C - `\varepsilon_{ij}^{**}) \tag{12c}$$

where $\varepsilon_{ij}^{**} = \varepsilon_{ij}^{*cd} + \varepsilon_{ij}^{*inh}$ is the total equivalent eigenstrain due to change in density and change in moduli (inhomogeneity), and the constrained strain with superscript C in Eshelby is the strain denoted without superscript in this text; see also, Mura, 1982, Markenscoff, 2020.

The ellipsoid forms in the direction ψ of minimization of the interaction energy (which is eqtn (2)), of the stress field due to the phase change in the pancake with the pre-stress (Fig 1 (B2)). For the 2D densified band the plane of the pancake will form with Ox_1 along the direction of the maximum shear pre-stress (bisecting the directions of max and min normal stresses) in the mantle, while for 3D pancake in the 3D pre-stress field it can be more complex (Markenscoff, in preparation). Within the pancake, the CLVD radiation direction will be according to eqtn (4a).

The above instability analysis is extendable to a Newtonian fluid, for which the Eshelby ellipsoidal inclusion property is also valid (Eshelby, 1957); it was applied by Bilby, *et al*, 1975, for the change of shape of a viscous ellipsoidal region embedded in a slowly deforming matrix having a different viscosity and was extended to self-similar expansion by Markenscoff (2021). For the aqueous fluid, the displacement in eqtn (1) is the particle velocity, and the lacuna property also holds, so that the expanding region of phase transformation has zero particle velocity in the interior where the *M* waves cancel the *P* and *S* (Markenscoff, 2021). The phenomenon of the dynamic phase transition of a Newtonian fluid (water) to a phase of ice can be analyzed by this approach, and it has important implications as to where it may occur in Earth's mantle (e.g., Bina and Navrotsky, 2000). The Eshelby method with the isotropic Eshelby Tensor (Ni and Markenscoff, 2016b) is applicable to a phase transformation where the interior transforms into anisotropic, while the matrix is isotropic (Eshelby, 1961). For solids, the critical instability pressure for spherical nucleation and growth of an inclusion undergoing change in density and bulk modulus at constant potential energy was obtained by Markenscoff (2020) as a function of Poisson's ratio; however, for the water to ice transition, the relation between density and modulus has to be obtained from the equation of state as related to the pressure, which requires solving a nonlinear problem. The pancake-like expansion favors expansion to a larger radius than the sphere (Markenscoff, 2020) minimizing the energy spent to move the phase boundary. In the fluid, the pancake-like geometry will induce large deviatoric eigenstrains and deviatoric stresses according to the analysis presented here; however, if the applied loading can cancel these large deviatoric stresses, which may happen under *uniaxial* applied loading (producing a CLVD as in eqtn (4a)), then, to leading order, only volumetric strain energy will remain in the ice inclusion, inducing the emission of only pressure waves to the outside aqueous fluid. The phenomenon of water to ice transition (with volume increase) under uniaxial compression is the counterpart to the deep-focus earthquakes (with volume decrease) under pressure in solids described above, --volumetric versus the distortional strain energies, respectively--. The presence of ice VII inclusions in diamond discovered in Tschauner, *et al,* 2018, indicates the presence of an adjacent aqueous fluid during the formation of the diamond in the Lower Mantle/Transition Zone, which may also imply a more complex interaction between the two dynamic phenomena, the ice formation and the phase transformation into diamond, or the deep-focus earthquakes. The strains in the self- similarly expanding pancake-like inclusion with ice, which are rates of deformation in the fluid, are of lower order (finite) than the rates of the eigenstrain (which tend to infinity in the asymptotic analysis), and thus, the interior of the pancake inclusion with ice deforms as a solid, yielding the fluid to solid transition.

III. Volume collapse in a two-dimensional self-similarly expanding flattened Eshelby elliptical densified inclusion: "anticrack" in geophysics, or densified shear band

(a) Volume collapse in a flattened elliptical cylinder

We consider the two-dimensional problem of an isotropic material undergoing change in density (densification) under high pressure with the finite "volume collapse" occurring planarly in a 2D band as the *asymptotic* limit of an elliptical cylindrical Eshelby inclusion. A densified flattened elliptical inclusion under pressure has been called "anticrack" (Green and Burnley, 1989, Burnley and Green, 1989) in the geophysics literature and was incorrectly treated as a Griffith crack by omitting the effect of densification in the longitudinal direction inside the anticrack; it also differs from the rigid line inclusion which were called "anticrack" by Dundurs and Markenscoff (1989). More importantly, the analysis shows that it is the pressure acting on the volume phase change that drives the anticrack, correcting the literature where the deviatoric stresses are considered to be the driving force, e.g. Zhan, Fig 5a, 2020. The analysis presented here is also applicable to amorphized bands such as in Zhao *et al*, 2016a, 2016b, creating densified shear bands.

The following analysis for the two-dimensional problem follows the three-dimensional one above. We consider that the planar change in density induces equivalent transformation strains with components (to be determined) $\varepsilon_{11}^* \neq 0$, $\varepsilon_{22}^* \neq 0$,$\varepsilon_{33}^* = 0$ and the shear eigenstrains to be zero, in an Eshelby elliptical cylinder inclusion with cylinder axis in the x_3 direction (Fig 4(a)). The change in density is treated as an imaginary equivalent eigenstrain that creates a change in volume equal to $\varepsilon_{11}^* + \varepsilon_{22}^* = \varepsilon_{kk}^* = (dV - dV_0) / dV_0 = (\rho_0 - \rho^*) / \rho^*$
given at zero stress, when the inclusion is taken outside the matrix, in the Eshelby thought experiment .

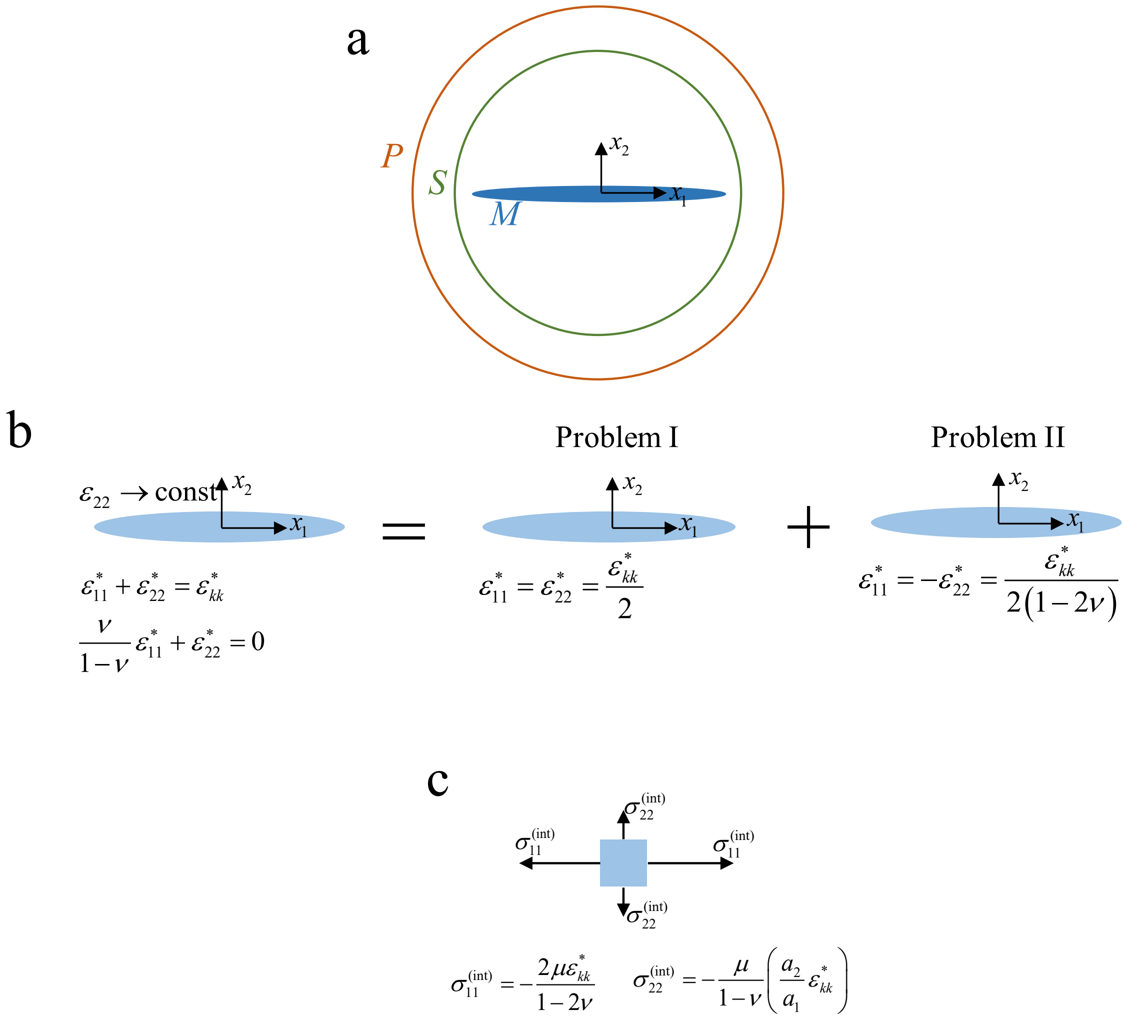


Figure 4: **a** *A self-similarly expanding Eshelby inclusion (a flattened ellipse* $a_2 / a_1 \rightarrow 0$ *with uniform eigenstrain) emits pressure (P) waves, shear (S) waves and (M) waves, emitted by the phase boundary, of which it is the degenerate wave-front; they cancel each other in the interior of the inclusion (a "lacuna" of zero particle velocity) locking-in a constant strain interior field*; **b** *A densified flattened inclusion with change in density* ε_{kk}^* *that is large in the asymptotic analysis*

develops deviatoric eigenstrains (symmetry breaking) so that the faces do not open or overlap $\lim_{a_2/a_1\to 0,\varepsilon^*_{kk}\to\infty} a_2/a_1\varepsilon^*_{kk} \to const$ *, and the fields decompose into a symmetric part I (center of dilatation) and an antisymmetric part II (* $\varepsilon^*_{11} = -\varepsilon^*_{22}$ *, which is a symmetric center of shear);* ***c*** *the interior longitudinal stress* σ_{11} *is of the order of the eigenstrain (large) and tensile, while* σ_{22} *is small, so that a very large deviatoric strain energy is produced (the seismic energy). The pre-stress is superposed to the above self-stresses.*

The change in volume (densification/"volume collapse") is negative, so that the eigenstrain ε^*_{kk} is negative. The components of the eigenstrain, considered unknown now, will satisfy a relation among them, to be derived below, such as to maintain material continuity when the material densifies a finite volume collapsing into a very thin inclusion.

The strains in the 2D flattened elliptical cylinder (band) of "volume collapse" are determined through the Eshelby tensor $\varepsilon_{ij} = S_{ijkl}\varepsilon^*_{kl}$, with the Eshelby tensor S_{ijkl} pertaining to the elliptic cylinder (Mura, 1982, eqtn (11.22)) to be taken asymptotically as the ratio of the axes $a_2/a_1 \to 0$ (the long axis being $a_3 \to \infty$), and yielding the interior strains

$$\varepsilon_{11} = (1/2(1-\nu))[\{2a_2/a_1 + (1-2\nu)a_2/a_1\}\varepsilon^*_{11} - (1-2\nu)a_2/a_1\varepsilon^*_{22}] \quad (13a)$$

$$\varepsilon_{22} = \nu/(1-\nu)\varepsilon^*_{11} + \varepsilon^*_{22} - (1-2\nu)/2(1-\nu)a_2/a_1(\varepsilon^*_{11} + \varepsilon^*_{22}) \quad (13b)$$

where ν is the Poisson's ratio. The density/volume change in the inclusion constrained by the matrix is

$$(\rho_0 - \rho')/\rho' = \varepsilon_{kk} = S_{kkij}\varepsilon^*_{ij} = (1+\nu)/3(1-\nu)\varepsilon^*_{kk} \quad (14)$$

and the stresses are

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) \quad (15)$$

As in the 3D case, in the *asymptotic analysis* of a *flattened* ellipsoidal inclusion accommodating a very large change in density as the ratio of the axes speeds tends to zero (Fig 4), in order for the total strain energy to be finite (and not zero) in the very thin inclusion ($a_2/a_1 \to 0$), the eigenstrains *must tend to infinity* $\varepsilon^*_{ij} \to \infty$ as the ratio of the axes speeds (small to large) *tends to zero*, $a_2/a_1 \to 0$, so that their product

$$\lim_{\varepsilon^*_{ij}\to\infty, a_2\to 0} (a_2/a_1)\varepsilon^*_{ij} \to const \quad (16)$$

For a volume collapse ε^*_{kk} to occur planarly we shall assume that, for material continuity with no opening or overlapping of the band faces, to the leading order we must have in (13b) the strain $\varepsilon_{22} \to const$, so that we have the vanishing of the zero-th order terms in (13b), which gives, what we call "the planarity condition" among the eigenstrains

$$\nu/(1-\nu)\varepsilon_{11}^{*}+\varepsilon_{22}^{*}=0 \tag{17}$$

so that a deviatoric part of the eigenstrains is created in the anticrack. From (17) we have

$$\varepsilon_{11}^{*}=(1-\nu)/(1-2\nu)\varepsilon_{kk}^{*} \qquad \varepsilon_{22}^{*}=-\nu/(1-2\nu)\varepsilon_{kk}^{*} \tag{18}$$

with ε_{11}^{*} being negative for increase in density and decrease in volume, while ε_{22}^{*} is positive. The strains in (13a) and (13b) will be finite under the assumption of eqtn (17). Thus, with finite strains in (13), but with infinite eigenstrains, with stresses infinitely large due to (16), the strain energy density will be infinite, and the total strain energy in the very thin (zero volume) "anticrack" will be finite, and not zero. We will determine the stresses, interior and exterior to the flattened elliptical cylinder under the above conditions. We decompose the problem into a symmetric (volumetric) (I) and an anti-symmetric (deviatoric) (II) part (Fig 4, b), which, in view of (17) and (18), are

$$\begin{pmatrix} \varepsilon_{11}^{*} & 0 \\ 0 & \varepsilon_{22}^{*} \end{pmatrix}=\begin{pmatrix} \varepsilon_{11}^{*} & 0 \\ 0 & -\nu/(1-\nu)\varepsilon_{11}^{*} \end{pmatrix}=\begin{pmatrix} \dfrac{(1-\nu)}{(1-2\nu)}\varepsilon_{kk}^{*} & 0 \\ 0 & \dfrac{-\nu}{(1-2\nu)}\varepsilon_{kk}^{*} \end{pmatrix}=$$

$$\begin{pmatrix} \varepsilon_{kk}^{*}/2 & 0 \\ 0 & \varepsilon_{kk}^{*}/2 \end{pmatrix}+\begin{pmatrix} \dfrac{\varepsilon_{kk}^{*}}{2(1-2\nu)} & 0 \\ 0 & -\dfrac{\varepsilon_{kk}^{*}}{2(1-2\nu)} \end{pmatrix} \tag{19}$$

(b) The interior and exterior stress and displacement fields of a densified band*: *("anticrack").

According to eqtn (19) the problem of volume collapse into a two-dimensional densified band decomposes into a symmetric one and an antisymmetric one, which is a symmetric center of shear; the antisymmetric center of shear has nonzero eigenstrains $\varepsilon_{12}^{*}=\varepsilon_{21}^{*}$. In the sequel we will use (*x,y*) rather than (*1,2*) notation for the coordinate axes. We will first solve the symmetric Problem I, which is a center of dilatation with eigenstrains

$$\varepsilon_{xx}^{*}=\varepsilon_{yy}^{*}=\varepsilon_{kk}^{*}/2 \tag{20}$$

negative for reduction in volume. The interior stresses in the flattened cylinder are directly obtained through the Eshelby tensor for the cylinder (e.g., Mura, 1982) by the asymptotic expansion in the flat limit expansion in the small variable a_2/a_1 and are obtained as

$$\sigma_{xx}^{I(int)}=-2\mu\varepsilon_{xx}^{*}/(1-\nu)=-\mu\varepsilon_{kk}^{*}/(1-\nu) \tag{21}$$

$$\sigma_{yy}^{I(\mathrm{int})}=-2\mu/(1-\nu)(a_2/a_1)\varepsilon_{yy}^{*}=-\mu/(1-\nu)(a_2/a_1)\varepsilon_{kk}^{*} \tag{22}$$

$\sigma_{xy}^{I(\mathrm{int})} = 0$

In the flattened cylinder the stress in eqtn (21) is infinitely large tension. The exterior fields are found by distributing in the flattened cylinder centers of dilatational eigenstrain with density

$$p(x) = (p_0 / a_1)(a_1^2 - x^2)^{1/2}\,,\ p_0 = \lim_{a_2 \to 0, \varepsilon_{xx}^* \to \infty} 2a_2(\varepsilon_{kk}^* / 2)\,,\ \lim_{\varepsilon_{kk}^* \to \infty, V \to 0} (1/2\varepsilon_{kk}^*)V = p = const \qquad (23)$$

with p being defined as the strength of the inclusion, where V is the volume of the inclusion per unit thickness along the cylinder axis.

The stresses exterior to the flattened ellipsoid are found by integration of the stresses due to distributed centers of dilatational eigenstrain of strength $p(\xi)$ at the position ξ, each producing the stresses

$$\sigma_{xx}^{I(ext)}(x,y) = -\sigma_{yy}^{I(ext)}(x,y) = -\mu p(\xi) / \{\pi(1-\nu)\}\{-1/r^2 + 2x^2/r^4\} \qquad (24)$$

$$\sigma_{xy}(x,y) = 0 \qquad (25)$$

with $r^2 = (x-\xi)^2 + y^2$, so that the external stresses of the inclusion at a field point in the plane is the integral of their contributions

$$\sigma_{xx}^{I(ext)}(x,y) = -\sigma_{yy}^{I(ext)}(x,y) = -\mu / \{\pi(1-\nu)\} \int_{-a_1}^{a_1} p(\xi)\{-1/r^2 + 2x^2/r^4\} d\xi \qquad (26)$$

The integral is evaluated as a principal value as in Kaya and Ergogan, 1987, also in Markenscoff, 2019a, so that along the *Ox* axis we obtain the square-root singular field exterior field

$$\sigma_{xx}^{I(ext)}(x,0) = -\sigma_{yy}^{I(ext)}(x,0) = \mu / (1-\nu) p_0 / a_1 \begin{cases} 1 & |x| < a_1 \\ 1 - |x| / (x^2 - a_1^2)^{1/2} & a_1 < |x| \end{cases} \qquad (27)$$

The displacements are

$$u_x(x,0) = p_0 / \{2(1-\nu)a_1\} \begin{cases} x & |x| < a_1 \\ x - (x^2 - a_1^2)^{1/2} \operatorname{sgn} x & a_1 < |x| < \infty \end{cases} \qquad (28)$$

and

$$u_y(x,0^+) - u_y(x,0^-) = p_0 / \{a_1(1-\nu)\}(a_1^2 - x^2)^{1/2} \qquad \text{for } |x| < a_1 \qquad (29)$$

where p_0 is given by eqtn (23). The solution obtained in eqtn (27) provides the stress intensity factor at the square-root singular tips, and also confirms the continuity of the traction on the faces of the inclusion by comparison of eqtns (22) and (27). The above external field is one of zero dilatation (pure shear), which is consistent for an inclusion *of any shape with dilatational eigenstrain* according to Eshelby, 1961 (and, previously, Crum as referenced by Eshelby, 1961; the additional antisymmetric part II is a modification removing the singularity from the strain affecting the application of the divergence theorem in Crum, or Bitter-Crum).

In the antisymmetric Problem II, the second matrix on the RHS of (19) can be called a symmetric center of shear. The eigenstrains are

$$\varepsilon^*_{xx} = -\varepsilon^*_{yy} = 1/\{2(1-2\nu)\}\varepsilon^*_{kk} = e^{*II} \tag{30}$$

with e^{*II} negative for reduction in volume and the interior stresses are obtained through the Eshelby Tensor for the elliptical cylinder expanded asymptotically for $a_2/a_1 \to 0$ in the flattened limit to yield

$$\sigma^{II(\text{int})}_{xx} = -2\mu\nu/(1-\nu)\varepsilon^*_{xx} = -\mu\nu/\{(1-\nu)(1-2\nu)\}\varepsilon^*_{kk} \tag{31}$$

$$\sigma^{II(\text{int})}_{yy} = 4\mu/(1-\nu)(a_2/a_1)^2\varepsilon^*_{xx} = 2\mu/\{(1-\nu)(1-2\nu)\}(a_2/a_1)^2\varepsilon^*_{kk} \tag{32}$$

$$\sigma^{II(int)}_{xy} = 0$$

From eqtn (31), it is seen that that internal in-plane stress is infinitely large uniaxial tension.

To obtain the external fields we consider the distributed centers of shear with eigenstrains given by (30), with the strength *q* defined analogously to *p*, in (23), with the eigenstrain e^{*II}.

The stress field (in the plane *Oxy*) of a symmetric center of shear with the eigenstrains in (30) in a circular cylinder is obtained following the stress function determination by matching the boundary conditions at the inclusion boundary satisfied by the stress function $U = \mu q/\{2(1-\nu)\}\cos 2\theta$, --the internal stresses agree with the Eshelby inclusion--, so that

$$\sigma^{II}_{xx}(x,y) = 2\mu q/\{\pi(1-\nu)\}\{3x^2/r^4 - 4x^4/r^6\} \tag{33}$$

$$\sigma^{II}_{xy}(x,y) = 2\mu q/\{\pi(1-\nu)\}\{2xy/r^4 - 4x^3y/r^6\} \tag{34}$$

$$\sigma^{II}_{yy}(x,y) = 2\mu q/\{\pi(1-\nu)\}\{1/r^2 - 5x^2/r^4 + 4x^4/r^6\} \tag{35}$$

and, at *y*=0:

$$\sigma^{II}_{xx}(x,0) = -2\mu q/\{\pi(1-\nu)\}(1/x^2) \tag{36}$$

$$\sigma^{II}_{yy}(x,0) = \sigma^{II}_{xy}(x,0) = 0 \tag{37}$$

We note that in polar coordinates the solution takes the form:

$$\sigma^{II}_{rr} = -2\mu q/\{\pi(1-\nu)\}\cos 2\theta/r^2 \tag{38}$$

$$\sigma^{II(ext)}_{\theta\theta} = 0 \tag{39}$$

Integrating the stresses in (31) due to centers of eigenstrain distributed over the flattened inclusion, with the principal value taken according to Kaya and Ergogan, 1987, also in Markenscoff, 2019a, we obtain the square-root singular field

$$\sigma_{xx}^{II(ext)}(x,0) = -2\mu q_0 / a_1\{\pi(1-\nu)\}\int_{-a_1}^{a_1}(a_1^2-\xi^2)^{1/2}/(\xi-x)^2 d\xi = 2\mu q_0/\{a_1(1-\nu)\}\begin{cases} 1 & |x|<a_1 \\ 1-|x|/(x^2-a_1^2)^{1/2} & a_1<|x| \end{cases}$$

$$\sigma_{yy}^{II(ext)} = 0 \qquad (40)$$

where $q_0 = \lim_{a_2\to 0,\varepsilon_{xx}^*\to\infty} 2a_2\varepsilon_{xx}^* = \lim_{a_2\to 0,\varepsilon_{kk}^*\to\infty} a_2\varepsilon_{kk}^*/(1-2\nu)$ (41)

The displacements are:

$$u_x(x,0) = q_0/a_1\begin{cases} x & |x|<a_1 \\ x-(x^2-a_1^2)^{1/2}\operatorname{sgn}x & a_1<|x|<\infty \end{cases} \qquad (42)$$

$$u_y(x,0^+)-u_y(x,0^-) = -q_0/a_1\{(1-2\nu)/(1-\nu)\}(a_1^2-x^2)^{1/2} \quad \text{for } |x|<a_1 \qquad (43)$$

which is positive for $q_0<0$.

We note that eqtn (32) is consistent with eqtn (37) regarding the continuity of the normal tractions on the band faces to the leading order, and, of course, that the hoop stress σ_{xx}^{ext} is finite (eigenstrain times small axis length) on the top side of the " densified anticrack" faces in (40). It is smaller in order than the internal stress, as it experiences a jump across the boundary by the infinite amount of the (infinitely large) eigenstrain. At the tips the stress in the direction of the crack is square root infinite (tensile for volume reduction). The total stress is the sum of the fields I and II, which for the interior stresses are:

$$\sigma_{xx}^{(\text{int})} = \sigma_{xx}^{I(\text{int})} + \sigma_{xx}^{II(\text{int})} = -\mu\varepsilon_{kk}^*/(1-2\nu) \qquad (44)$$

$$\sigma_{yy}^{(\text{int})} = \sigma_{yy}^{I(\text{int})} + \sigma_{yy}^{II(\text{int})} \sim (-)\mu/(1-\nu)(a_2/a_1\varepsilon_{kk}^*) \qquad (45)$$

$$\sigma_{xy}^{(int)} = 0 \qquad (46)$$

Moreover, in plane strain, the out of plane stress is $\sigma_{zz} = \nu(\sigma_{xx}^{I+II} + \sigma_{yy}^{I+II})$ and the mean stress in the " densified anticrack" is to the leading order

$$\sigma_m = -(1+\nu)\mu\varepsilon_{kk}^*/3(1-2\nu) \qquad (47)$$

which is tensile for reduction in volume, and coincides with eqtn (6c) . Eqtn (44) shows that there is infinitely large tensile stress in the " densified anticrack" inclusion in the direction of the band, and that the directions of the maximum shear are at 45^0 angles to the inclusion as seen in Zhao *et al*, 2016b. The distribution of the stresses is shown in Figure 5.

The displacements for the sums of the displacements of problems I and II are:

$$u_y^{I+II}(x,0^+) - u_y^{I+II}(x,0^-) = 0 \tag{48}$$

$$u_x^{I+II}(x,0) = [p_0 / 2(1-\nu) + q_0] / a_1 \begin{cases} x & |x|<a_1 \\ x-(x^2-a_1^2)^{1/2}\,\mathrm{sgn}\,x & a_1<|x|<\infty \end{cases} \tag{49}$$

where we note in (48) that the there is indeed no difference in the vertical displacements of the faces of the "densified anticrack" , consistent with material continuity and the "planarity condition". The stress distributions for Problems I and II are graphed in Figure 5.

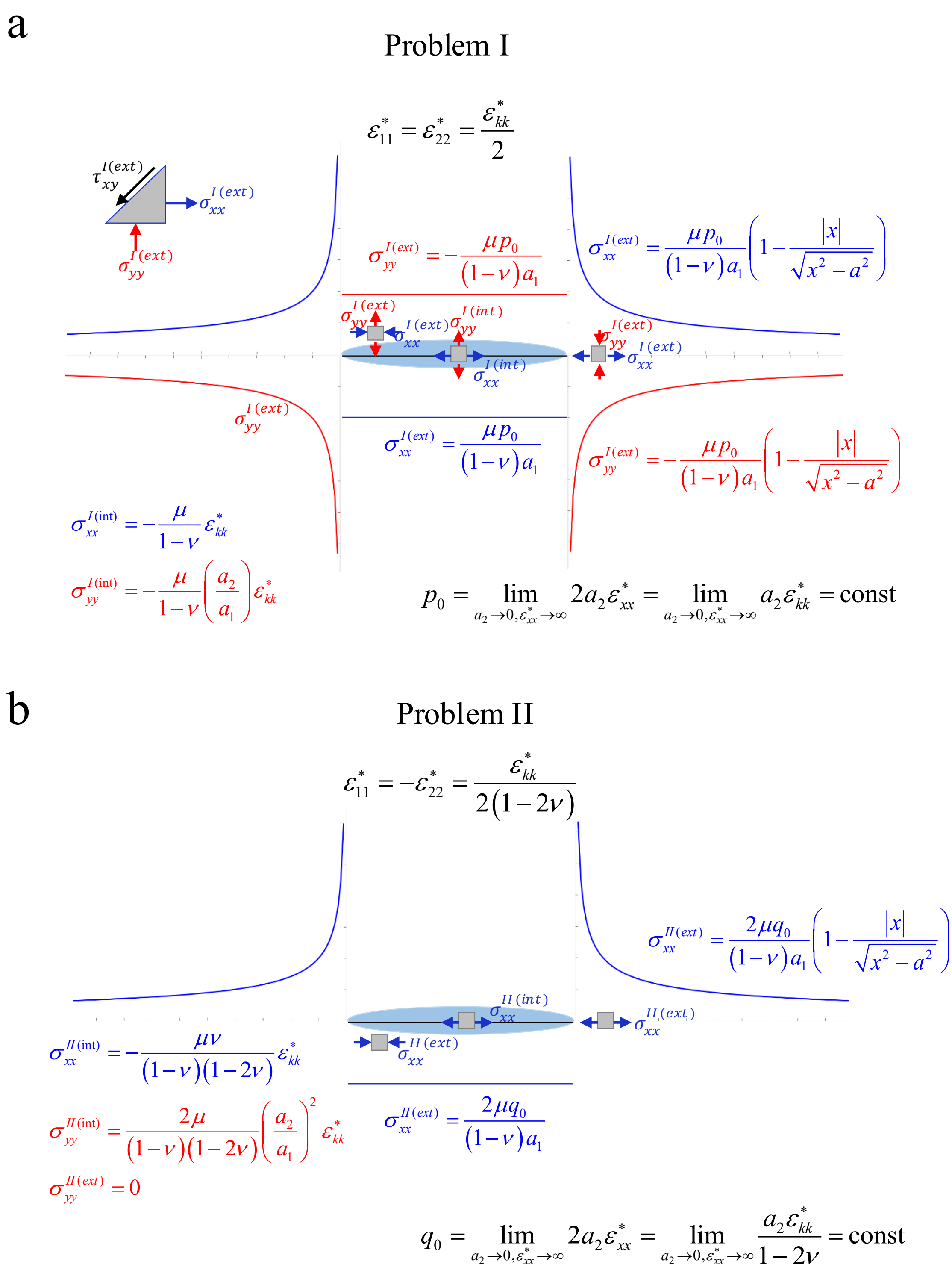


Figure 5: The internal and external stresses due to a densified 2D band ("anticrack" in geophysics,) are obtained as superposition of the distribution of symmetric dilatational centers of eigenstrains (symmetric Problem I) and of symmetric centers of shear (antisymmetric Problem II). The plot shows the stress distribution at y=0.

(c) The energy-release rate to incrementally advance the densified band

The J_{tip}^{band} integral which is equal to the energy-release rate needed to advance incrementally the tip of the "anticrack" (densified band with densification ε_{kk}^{*}) is obtained in statics by the incremental work (e.g., Rice, 1985) done by the singular longitudinal stress at the tip acting on the corresponding longitudinal strain during an infinitesimal advancement of the "densified band" (which is the only work done at the tip), as

$$J^{band}\Delta a = G\Delta a = 1/2\int_{a_1}^{a_1+\Delta a}\sigma_{xx}^{I+II}(x,0)[u_x^{I+II}(x^+,0)-u_x^{I+II}(x^-,0)]dx =$$

$$-(1/2)\mu(p_0+2q_0)/(1-\nu)[p_0/2(1-\nu)+q_0]/a_1\int_{a_1}^{a_1+\Delta a}(x-a_1)^{1/2}/(x-a_1)^{1/2}dx \qquad (50)$$

$$\equiv -\mu f(\nu)\lim_{a_2\to 0,\varepsilon_{kk}^{*}\to\infty}(a_2/a_1\varepsilon_{kk}^{*})^2 a_1\Delta a$$

with $f(\nu)=(3-2\nu)(3-4\nu)/\{4(1-2\nu)^2(1-\nu)^2\}$ (51)

The derivation in the next section provides the driving force to overcome the value of the *J* integral for a given densification and advance the "anticrack", with other dissipative effects being neglected.

(d) The energetics of the growth in scaling and propagation of a densified shear band under pressure, quasi-statically and with inertia

We will consider the presence of pre-stress of an applied large hydrostatic pressure $p_1^{appl}=p_2^{appl}=p_3^{appl}=p^{appl}$ so that the stresses in the inclusion are the superposition of those due to the eigenstrains plus the hydrostatic pressure. We are evaluating eqtn (2) on a contour surrounding the surface of the self-similarly quasi-statically growing densified 2D band inclusion of length $a_1=t/s_1$, with the ratio $a_2/a_1=s_1/s_2$ remaining constant for self-similarity, and in the presence of applied pressure (Fig 6). With $\dot{l}$ denoting the outward normal boundary velocity (derived in terms of the axes speeds in Markenscoff, 2019b), we set $\dot{l}=\dot{\lambda}a_2=\dot{\lambda}t/s_2$ on the upper and lower faces of the densified band, and $\dot{l}=\dot{\lambda}a_1=\dot{\lambda}t/s_1$ at the tips, where $\dot{\lambda}$ is a *dimensionless scaling parameter* as is considered in the M_O integral about the origin of the coordinate system (Budiansky and Rice, 1973). Thus, from eqtn (2) we can produce the *M* integral for a self-similarly quasi-statically growing band inclusion where all field quantities have been obtained above, and we write the counterpart of eqtn (8) in two-dimensions, as

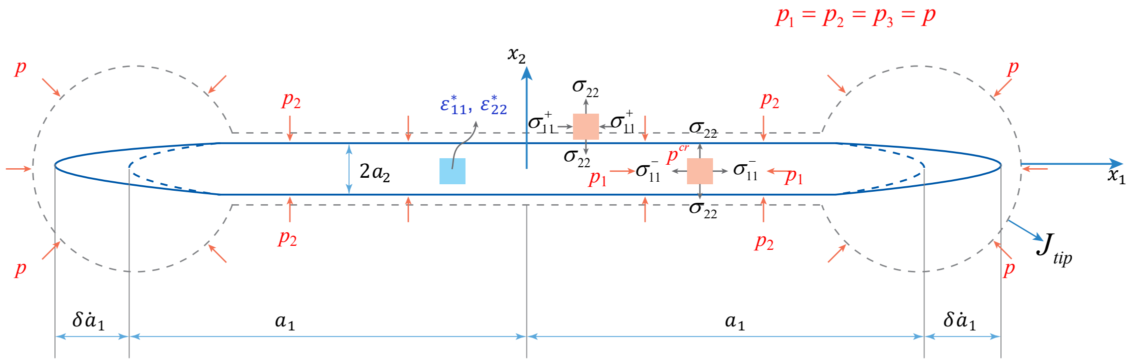
$$\delta\dot{E} = -M_O\dot{\lambda} = \dot{\lambda}[a_2\{(\sigma_{22}+p_2^{appl})\varepsilon_{22}^{*}+1/2(\sigma_{11}^{-}+\sigma_{11}^{+}+2p_1^{appl})\varepsilon_{11}^{*}\}2a_1]+\dot{\lambda}a_1J_{tip}^{band} =$$

$$[2(a_2/a_1)\{p_2^{cr}(\varepsilon_{11}^{*}+\varepsilon_{22}^{*})+1/2(\sigma_{11}^{-})\varepsilon_{11}^{*}+(\sigma_{22}\varepsilon_{22}^{*}+\sigma_{11}^{+}\varepsilon_{11}^{*})\}-\mu f(\nu)\lim_{a_2\to 0,\varepsilon_{kk}^{*}\to\infty}(a_2/a_1\varepsilon_{kk}^{*})^2]a_1\delta\dot{a}_1=0$$

(52)

with $J_{tip}^{band} = -\mu f(\nu) \lim_{a_2 \to 0, \varepsilon_{kk}^* \to \infty} (a_2 / a_1 \varepsilon_{kk}^*)^2 (2a_1)$, $f(\nu) = (3-2\nu)(3-4\nu) / \{2(1-2\nu)^2(1-\nu)^2\}$

obtained in eqtn (50)/(51) to advance the tip of the densified band incrementally by δa_1 .



$$\sigma_{11}^- = \sigma_{11}^{(\text{int})} = -\frac{2\mu}{1-2\nu}\varepsilon_{kk}^* \qquad \sigma_{11}^+ = \sigma_{11}^{(\text{ext})} = \frac{\mu(3-2\nu)}{\pi(1-\nu)(1-2\nu)}\left(\frac{a_2}{a_1}\varepsilon_{kk}^*\right) \qquad e_{33}^* = 0$$

$$\sigma_{22} \sim -\mu / (1-\nu)(a_2 / a_1 \varepsilon_{kk}^*)$$

Figure 6: *Nucleation instability is the growth at constant potential energy of an arbitrarily small defect (densified 2D band): the driving force on a contour surrounding the inclusion is provided by the Peach-Koehler term* $p(\varepsilon_{11}^* + \varepsilon_{22}^*)$ *(pressure times change in volume) in the M integral and balances the self-stress resistance to growth in scaling* $\dot{\lambda}$ *. It yields the critical pressure at which the band grows incrementally by* $\delta \dot{a}_1 = \dot{\lambda} a_1$ *at constant potential energy. The J integral resistance to advance the tip is of lower order in the ratio of the axes lengths* $a_2 / a_1 << 1$ *than the Peach-Koehler driving force, so that the band will grow unstably at the critical nucleation pressure overcoming the effect of the internal self-stress, at* $p_{band}^{cr}\varepsilon_{kk}^* + 1/2\sigma_{11}^{(\text{int})}\varepsilon_{11}^* = 0$

The vanishing of the *M* integral, which is growth at constant potential energy, on the LHS of eqtn (52), produces a ***nucleation instability*** at the vanishing of the quantity in the brackets, which gives the critical value of the hydrostatic pressure, at which *any arbitrarily small densified band can have an incremental extension in scaling* $\delta \dot{a}_1 = \dot{\lambda} a_1$ ***at constant potential energy of the system***. The work ($p_2\varepsilon_{22}^* + p_1\varepsilon_{11}^* = p\varepsilon_{kk}^*$) produced by the pressure acting on the change in volume ε_{kk}^* (Peach-Koehler force, radius-increasing) balances the self-forces (radius-shrinking) on the surface of discontinuity due to the internal self-stresses including the *J* integral at the tips (obtained in (50)), and, eqtn (52) to leading order in the ratio of the axes speeds, and yields *the critical pressure for nucleation of a 2D densified band* as

$$p_{band}^{cr}\varepsilon_{kk}^* + 1/2\sigma_{11}^{(\text{int})}\varepsilon_{11}^* = 0 \qquad or \qquad p_{band}^{cr} = (1-\nu)/(1-2\nu)^2 \mu\varepsilon_{kk}^* + O(s_1 / s_2\varepsilon_{kk}^*\mu) \tag{53}$$

The critical pressure in a 2D band always exceeds the mean (tensile) stress in eqtn (47), (Fig 3), and cancels it, so that, to leading order, there will remain a large distortional strain energy.

We will examine now the ***effects of inertia*** on the nucleation and propagation of a self-similarly expanding 2D inclusion through the *M* integral, evaluated in self-similar dynamic expansion on a contour shrinking onto the surface of discontinuity; it will be independent of the shape of the contour ("contour independence" rather than "path-independence") in the limit as it shrinks onto the defect, as with the dynamic *J* integral in dynamic fracture mechanics and also in dislocations jumping to constant velocity shown in Clifton and Markenscoff,1981. For the dynamically self-similarly expanding flattened inclusion the values of the interior stresses are as those with the corresponding axes speeds ratio, locked-in by the *M* waves (Markenscoff, 2019). From the Hadamard jump conditions there is continuity of the traction σ_{22} due to the vanishing of the boundary velocity on the top/lower surfaces, and the external dynamic stress σ_{11}^{+} is of the order of the ratio of the axes speeds (and, consequently, of lower order than the interior stress σ_{11}^{-}). As we have not explicitly obtained the dynamic outside fields and the J^{dyn} integral with inertia for the above problem, we will take as an approximation for its value the dynamic energy-release rate obtained for a screw dislocation jumping from rest to a constant velocity given in Clifton and Markenscoff, 1981, eqtn (40). A CLVD is a screw pair (attributed to Weertman in Knopoff and Randall, 1970), which in 3D is a screw cone pair at 45^0 angles e.g., Julian *et al*, 1998); as a rough approximation, we shall obtain the order of magnitude of the pressure needed in (52) for overcoming the "drag force" due to inertia to dynamically emit a screw dislocation. We write eqtn (52) relating the driving force of the pressure to the energy-release rate, as in Clifton and Markenscoff, 1981, eqtn (40) for the emission of a screw dislocation with Burgers vector *b,* (which relates to the eigenstrain as $b \sim \lim_{a_2 \to 0, \varepsilon_{kk}^* \to \infty} 2a_2\varepsilon_{kk}^*$), with v_d denoting the dislocation velocity and c_2 the shear wave speed, as

$$\delta\dot{E} = -M_O\dot{\lambda} = \dot{\lambda} \lim_{a_2 \to 0, \varepsilon_{ij}^* \to \infty} a_2\{(\sigma_{22} + p_2^{appl})\varepsilon_{22}^* + 1/2(\sigma_{11}^{-} + \sigma_{11}^{+} + 2p_1^{appl})\varepsilon_{11}^*\}2a_1 + \dot{\lambda}a_1 J_{tip}^{dyn} \sim$$
$$\lim_{a_2 \to 0, \varepsilon_{kk}^* \to \infty} (2a_2\varepsilon_{kk}^*)p(\dot{a}_1) - C\mu b^2/(2\pi t)[1 - \{1-(v_d/c_2)^2\}^{1/2}]/(1-(v_d/c_2)^2)^{1/2} = 0$$
(54)

with the constant *C* containing a Poisson's ratio and angular dependence effect to relate the screw dislocation to the CLVD . Ignoring higher order terms, and assuming small v_d/c_2 , we obtain from (54) the pressure needed to overcome the inertia "drag force" on a screw dislocation jumping from rest to a constant velocity v_d as,

$$p^{appl} \sim C\mu b(v_d/c_2)/\{4\pi t c_2\} \tag{55}$$

which decays with time as 1/*t.* The nucleation time *t* will not be considered zero, but such that $tc_2 \sim 5b$, (for dislocations, the time is taken to be large relative to a period of lattice vibrations, and the expression (55) being in agreement for a screw dislocation with Eshelby, 1953). We can, thus, conclude that the "nucleation pressure" given by eqtn (53) is large enough to overcome the "drag force" on a screw dislocation jumping to a constant velocity in (54), and that the nucleation and propagation instability by the *M* integral can be fully dynamic for small expansion speeds in this approximation. Obtaining the relation between the pressure and the expansion speeds, as obtained for a dislocation in Clifton and Markenscoff, 1981, requires further analysis for the calculation of the dynamic energy-release rate for the self-similarly expanding 3D penny-shape densified inclusion from the external tip fields based on the solution of Ni and Markenscoff, 2016a; then, the dynamic *M* integral, as in eqtn (54) will provide such relation for the purely mechanical inertia effects. This remains open for future investigation.

IV. Conclusions

In summary, the nucleation and propagation of deep-focus earthquakes are a manifestation of a new physical phenomenon in solid mechanics exhibiting successive instabilities in high pressure dynamic phase transformations with "volume collapse". If, at a point, a densified flattened ellipsoidal inclusion of an *arbitrarily* small size ("pancake-like" in 3D, or densified band in 2D) is generated (-- the model does not say how--), then, due to an instability, at a *critical* "nucleation" pressure (at the vanishing of the *M* integral), the densified flattened inclusion can grow with the same volume change ε_{kk}^{*} *at constant potential energy.* The dynamic Eshelby problem of the self-similarly expanding ellipsoidal inclusion with uniform transformation strain has the remarkable "lacuna" physical property (zero particle velocity in the interior domain/no kinetic energy), thus making the phase transformation to take place under conditions of equilibrium and under uniform interior stress (Eshelby property). The Eshelby assumption of uniform eigenstrain inside the inclusion is justified on the argument that, if the instability starts with a given change in density, it can continue at the same rate under constant potential energy. The flattened densified inclusion is treated as the asymptotic limit of the ellipsoid (penny-shape), and it was shown that (for Poisson's ratio >0.2) the critical pressure (compression) is higher than the mean tensile stress in the densified inclusion, so that it cancels it, and, thus, to leading order, the strain energy density is distortional, producing a predominantly *shear* seismic source. The *M* integral shows that the growth of the densified inclusion (which is a shear seismic source) is driven by the pressure (acting on the change in volume of the phase transformation), even under conditions of full isotropy, and even in the absence of deviatoric pre-stress. The flattened shape of the ellipsoid is the manifestation of a symmetry-breaking instability that favors the minimization of the energy (*M* integral) needed for the inclusion to grow large (while the sphere minimizes it for small radius). The solution and methodology are completely extendable to full anisotropy (Willis, 1971) and are also extendable to a Newtonian fluid (Markenscoff, 2021) in self-similar dynamic expansion of an inclusion of a fluid of different viscosity, extending the approach of Bilby *et al*, (1975). The high- pressure dynamic nucleation and growth of an arbitrarily small inclusion in the water to a solid ice phase transition can be studied in spherical expansion by extending Markenscoff, 2020, or in pancake-like expansion as here, with significant implications on Earth's structure, water budget and seismicity. Although the Eshelby property is not valid in nonlinear elasticity the *asymptotic* analysis for the flattened inclusion differentiates the order of the eigenstrains (large/infinite) from the order of the strains (small/finite), with the order of the pre-stress (large for pressure, small for deviatoric pre-stress) governing the behavior of the system. While in the deep-focus earthquakes the pre-stress is a static field constant in time, the phenomenon and analysis are also generally applicable to other dynamic phenomena, such as failure waves, phase transformation of metastable materials occurring in the short time interval and under shock-loading conditions, as the amorphization transformations, etc. Thus, the discovered phenomenon of shear instabilities in dynamic phase transformations under pressure unravels the mystery of the cause of the deep-focus earthquakes and provides insight into the concepts and analytical tools for a host of other dynamic phase transformation phenomena in materials under extreme conditions.

Acknowledgements. The author acknowledges many helpful discussions with Prof. Raymond Jeanloz, UC Berkeley. The support of NSF grant CMMI No. 1745960 is gratefully acknowledged. The preparation of the figures is by Mr. Zhaoqiang Song, graduate student, UCSD.

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