The formation of 'optimal' vortex rings, and the efficiency of propulsion devices

By P. F. LINDEN[1] AND J. S. TURNER[2]

[1]Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411, USA

[2]Research School of Earth Sciences, Australian National University, Canberra, ACT 0200, Australia

The formation of an axisymmetric vortex ring by forcing fluid impulsively through a pipe is examined. An idealized model of the circulation, impulse and energy provided by the injected plug is developed, and these quantities are equated to the corresponding properties of the class of rings with finite cores described by Norbury (1973). It is shown that, as the length-to-diameter aspect ratio L/D of the plug increases, the size of the core increases in comparison with all the fluid carried along with the ring, until the limiting case of Hill's spherical vortex is reached. For aspect ratios larger than a certain value it is not possible to produce a single ring while conserving circulation, impulse, volume and energy. This implies that the limiting vortex is 'optimal' in the sense that it has maximum impulse, circulation and volume for a given energy input. While this matching calculation makes the physical mechanism clear, the L/D ratio that can be achieved in practice is more appropriately taken from the direct experimental measurements of Gharib *et al.* (1998) who concluded that the limiting value is $L/D = 4$. This is close to the value found in our calculation.

1. Introduction

For at least a hundred years there has been a great deal of interest in calculations of the properties of vortex rings, such as their propagation speed and stability, and the relation of these to the vorticity distribution. Many eminent mathematicians and physicists contributed to the inviscid theory at the end of the 19th century, and identified simple limiting cases such as the thin-core vortex and Hill's spherical vortex. Some of them also conducted experiments in air or water, and the ease with which smoke rings can be produced by forcing fluid through a sharp-edged opening or through a nozzle was (and still is) common knowledge.

Particularly during the past twenty years there have been numerous, increasingly sophisticated analytical, experimental and numerical studies of the generation and evolution of vortex rings. Shariff & Leonard (1992) give a helpful and accessible review. In the laboratory, Baird, Wairegi & Loo (1977) attempted to relate the properties of rings to the formation parameters, and Maxworthy (1972, 1977), Didden (1979), Glezer (1988) and Glezer & Coles (1990) have reported extensive experiments on ring formation and development. Applications of various kinds have been suggested; Turner (1960) and Fohl (1967) proposed that vortex rings might be used to project smoke or other effluents to great heights in the atmosphere, thus reducing the need for tall chimneys. The properties of buoyant vortex rings (Turner 1957)

are such that, once these can be formed with a tight buoyant core, there is a great height advantage to be gained compared with forcing material continuously from a chimney. Preliminary attempts to apply this idea in practice have shown, however, that in addition to instabilities of the rings themselves, there is significant shedding near the generator, for reasons that were not clearly understood.

Another intriguing question concerns the use of vortex rings by fish as a means of propulsion. Vortex rings are shed from the tail of a swimming fish and it is observed that fish over a wide range of sizes produce vortices at a Strouhal number, based on the speed of the fish, of about 0.3. This observation raises the question of whether this choice is optimal and if so, why. A related question concerns the size of the vortex ring and, therefore, the amplitude of a single flap, used to accelerate the fish for a short distance or to change direction.

Most of the more recent experiments cited above used the ejection of fluid from a cylinder with a piston, but concentrated on small ratios L/D of the stroke, or length L of the ejected cylinder of fluid, to the diameter D of the opening. This deficiency has recently been addressed by Gharib, Rambod & Shariff (1998) whose experiments in water concentrated on the large L/D case. They posed the question: what is the largest circulation that a vortex ring can achieve by increasing L/D, keeping the average piston velocity fixed? The vorticity flux provided by the separated shear layer at the edge of the orifice is fed into the vortex ring, and the circulation should be proportional to L; but is there an upper limit as L/D becomes large? We concentrate for simplicity on the experiments using a single nozzle diameter in which the piston was brought impulsively up to a constant speed and held there during the fluid ejection, though they also used other acceleration and nozzle conditions. Briefly, their technique and results are as follows. They visualized the flows by marking the input fluid with fluorescent dye, and used digital particle image velocimetry to map the velocity and vorticity distributions.

The measured flow fields generated by small stroke ratio ($L/D = 2$) showed only a single vortex ring. Almost all the discharged fluid was entrained into the ring, and the vorticity was contained in a region approximately the same size as the dyed core. For a much larger $L/D = 14.5$ there was a leading vortex ring followed by a trailing jet of dyed fluid with further vortex-like disturbances on it. The front vortex ring was clearly separated from the fluid behind it, showing that the formation had been completed and no more vorticity was being entrained into the ring. In this case the region of dyed fluid carried along by the front ring was wider, and larger than the vortex core. The transition between these two distinct states was observed to occur at a mean stroke ratio of approximately 4, with a range of 3.6 to 4.5 in other experiments. Gharib *et al.* (1998) refer to this as the 'formation number', which can also be interpreted as non-dimensional time, scaled with the piston velocity and the nozzle diameter. They conclude that at this value of L/D the maximum circulation a vortex ring can attain during its formation is reached. We note here that a similar vortex-like behaviour has been recorded in experiments by Crow & Champagne (1971) on the instability of a steady jet forced through a circular orifice. This phenomenon and its implications will be discussed further in §4.

Gharib *et al.* (1998) interpreted their results using an analytical model based on a variational principle proposed by Kelvin and later by Benjamin (1976). This states that a steadily translating vortex ring must have maximum energy with respect to perturbations that preserve the impulse and vorticity. They suggest that at a certain stage in the formation process, near $L/D = 4$, the piston apparatus is no longer able to supply energy at a rate compatible with this energy requirement. This model will

not be discussed in detail: it is the purpose of the present paper to give a different, but complementary, description of the formation process. We believe this approach will shed further light on the physical constraints in a form that will be useful in certain potential applications to vortex ring generators.

2. Matching the properties of the input fluid and the vortex ring

The basic idea behind our theoretical model is that the properties of a vortex ring can be predicted by equating the values of the circulation, impulse, volume and kinetic energy of the injected plug of length L, diameter D to the corresponding values for a class of rings with finite cores. The calculation is in the spirit of G. I. Taylor's (1953) paper, in which he considered the relation between the impulsive motion of a circular disc moved normal to its plane in a fluid, and the thin-core vortex ring resulting if the disc were then 'dissolved away'. He showed that the properties of such a ring could be uniquely determined in this way. In a similar manner, Fohl (1968) calculated the parameters of the thin-core vortex ring that is created by abruptly accelerating a sphere of fluid to a uniform velocity throughout its volume. The properties of the flow around the sphere, regarded as a solid body moving in an inviscid fluid, can be determined, and equated to those of the resulting vortex ring. Saffman (1975) used related ideas to calculate the properties of the vortex ring produced by the roll-up of a cylindrical vortex sheet formed by forcing fluid through a circular orifice. He evaluated explicitly the energy and impulse of the sheet, and again matched these and the circulation to the corresponding properties of a thin-core vortex ring, but he did not explore a range of length-to-diameter ratios.

The following results are based on the particular family of finite-core vortex rings computed by Norbury (1973), of which the classical thin-core vortices and Hill's spherical vortex are end members. It is convenient for the present purpose to have a definite class of vortices on which to base the calculations, but now that results such as those of Gharib *et al.* (1998) are available, they could readily be repeated for more realistic vorticity distributions, as measured in a viscous fluid. The purpose of the present paper is to establish the validity and value of the matching idea in principle, not to make detailed numerical predictions. Thus we have concentrated on the simplest case of a plug input with a constant velocity profile across it, and mention only in passing the changes to be expected by varying other parameters (such as the velocity profile) that we have obtained in further calculations not reported in detail here.

After the present manuscript had been completed, the paper by Mohseni & Gharib (1998) was brought to our attention. These authors have used a similar matching procedure, but there are significant differences of emphasis between their treatment and ours.

2.1. *The properties of the injected fluid plug*

Using the notation introduced above, and the velocity U_P of the input fluid (taken to be constant), the volume V_P, the circulation K_P, the impulse P_P and the kinetic energy T_P of the plug fluid (the last two quantities are per unit mass, but we have set the density to unity here) can be written as

$$V_P = \tfrac{1}{4}\pi D^2 L, \tag{1a}$$

$$K_P = \tfrac{1}{2}U_P L, \tag{1b}$$

$$P_P = U_P V_P = \tfrac{1}{4}\pi U_P D^2 L, \qquad\qquad\qquad (1c)$$

$$T_P = \tfrac{1}{2}c U_P^2 V_P = \tfrac{1}{8}c\pi U_P^2 D^2 L. \qquad\qquad\qquad (1d)$$

Note that these forms imply that we have omitted any contributions to the impulse and energy associated with the interior fluid set into motion by the plug fluid. These interior motions were basic to the calculations of Taylor (1953) and Fohl (1968) referred to above. Here we assume, as argued by Gharib *et al.* (1998), that they are negligible compared to the direct effects of injecting a long plug of fluid.

Some further comments about the constant multipliers in (1*b*) and (1*d*) are needed. A simplistic estimate of the circulation based on the velocity integrated round a circuit passing through the centre of the injected plug of fluid leads to the form (1*b*), but without the factor 1/2. The correct form above is obtained by integrating the vorticity flux from a thin boundary layer with edge velocity equal to the piston speed. (See Shariff & Leonard (1992), equation (2.5) and the accompanying discussion. This factor could be increased or decreased a little from 1/2 by more subtle effects that will not be considered here – see also Didden (1979).) The factor $c(<1)$ in (1*d*) is the fraction of the nominal kinetic energy of the plug of fluid actually injected into the ring. While it seems reasonable to assume that K and P are strictly conserved during the formation of a ring, the possible loss of mean and rotational kinetic energy, due to turbulence for example, should be considered. If a parabolic velocity profile is assumed, rather than a uniform plug injection, then the following 'profile constants' must be introduced: (1*a*) is multiplied by 1/2, (1*c*) by 2/3 and (1*d*) by 1/2. The constant in front of (1*b*) will also increase when the vorticity is evaluated by integrating across the parabolic profile, but this calculation will not be considered in detail here.

2.2. *The family of vortex rings*

As noted above, we restrict attention here to the class of axisymmetric inviscid vortex rings discussed by Norbury (1973). These rings have vorticity ω proportional to the distance r from the axis of symmetry and propagate steadily through an unbounded ideal fluid. He classified these rings in terms of a parameter ε, a non-dimensional mean core radius, defined by the equation

$$\varepsilon = \frac{\text{area of core}}{\pi X^2}, \qquad\qquad\qquad (2)$$

where X is the length shown in the sketch of figure 1, i.e. $X = \tfrac{1}{2}(OB + OC)$. (The notation used by Norbury, α and L instead of ε and X, has been changed to avoid conflict with the symbols we have used for other quantities, following Gharib *et al.* (1998).) As ε increases, the core cross-sectional area increases, and the limiting members of this family are rings of small cross-section as ε tends to zero, and Hill's spherical vortex at $\varepsilon = \sqrt{2}$. In figures 2(*a*) and 2(*b*) we show the calculated shape of the core for various values of ε and also the corresponding dividing streamlines, separating fluid with circulation from the external potential flow, i.e. outlining the fluid carried along with the ring. For the Hill's vortex these two boundaries are on the same sphere.

In addition to the two boundaries reproduced in figure 2 (which define the volume of core fluid and the total volume of fluid carried along with the ring) Norbury (1973) calculated many other properties of the rings for discrete values of ε, including non-dimensional values of the propagation velocity W_R, the circulation K_R, the fluid impulse P_R and the kinetic energy T_R. He presented these properties in tabular form
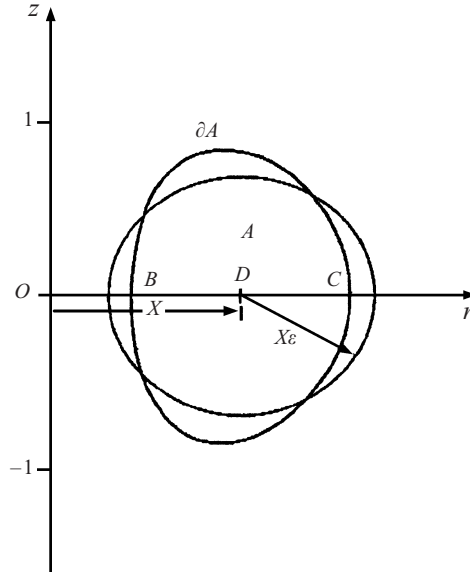
FIGURE 1. A definition sketch of a finite-core vortex ring, from Norbury (1973). The ring is specified by the radius $X = \frac{1}{2}(OB + OC)$, and the core A has area $\pi X^2 \varepsilon^2$.

(his tables 1 and 2) and his values will be used in matching them to those of the injected plug.

We will not reproduce the details of Norbury's method of calculation, but it is necessary to outline the scaling used to make the equations non-dimensional. All lengths have been scaled with X, defined in figure 1, and velocities with a reference velocity U defined by

$$U = \Omega X^2 \varepsilon^2, \tag{3}$$

where Ω, the vorticity constant defined by $\omega = \Omega r$, is a measure of the magnitude of the vorticity in the core. The propagation velocity W is related to U, and its non-dimensional value W_R is given by Norbury as a function of ε.

2.3. Matching

Having described the properties of the injected plug of fluid and of Norbury's class of vortex rings we now consider the consequences of matching the two, i.e. equating the expressions in (1) to the corresponding scaled relations in Norbury's analysis, using the constants tabulated by him. We emphasize again that no attempt has been made to describe the mechanism by which the plug rolls up to become a ring – we consider only the 'before' and 'after' stages of the process.

The three matched equations corresponding to (1b), (1c) and (1d) are

$$UXK_R = \tfrac{1}{2}U_P L, \tag{4b}$$

$$UX^3 P_R = \tfrac{1}{4}\pi U_P D^2 L, \tag{4c}$$
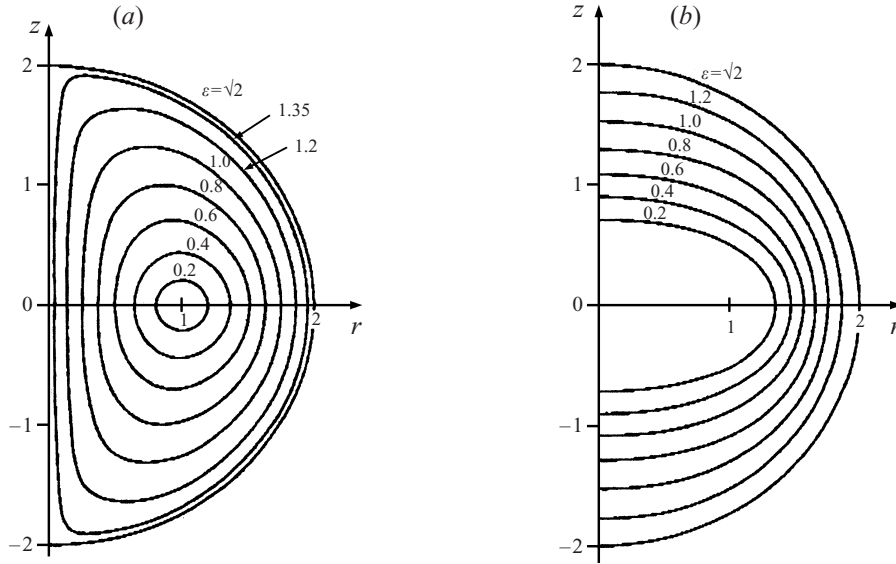
$$U^2 X^3 T_R = \tfrac{1}{8}\pi c U_P^2 D^2 L. \tag{4d}$$

FIGURE 2. (*a*) The boundary of the vortex core specified by the parameter ε, evaluated numerically by Norbury (1973) and shown for a range of values of ε. (*b*) The dividing streamlines, separating circulating fluid from the external potential flow, for the same values of ε shown in (*a*).

Combining (4*b*) and (4*c*) we find

$$\frac{X^2}{D^2} = \frac{1}{2}\pi\frac{K_R}{P_R};$$ (5)

and similarly (4*b*) and (4*d*) give

$$\frac{X}{D} = \frac{1}{2}\pi c\frac{K_R^2}{T_R}\frac{D}{L}.$$ (6)

Finally from (5) and (6) we have

$$\frac{L}{D} = \sqrt{\frac{\pi}{2}}c\frac{P_R^{1/2}K_R^{3/2}}{T_R}.$$ (7)

If we assume that the circulation, impulse and energy are conserved, there is a single ring in the class we are considering which is formed for a given plug aspect ratio. Figure 3 shows a plot of L/D against the parameter ε for $c = 1$. There is a maximum value of L/D above which a single ring cannot be formed. This limit, which corresponds to Hill's spherical vortex, is at $L/D = 7.83$. (This limit is about twice the value obtained experimentally by Gharib *et al.* It is reduced if some energy is lost during the formation process, but a comparison with their results, and discussion of the reasons for this difference, will be deferred to §3.)

However, according to the above formulation of the matching problem, a significant general conclusion can already be drawn from (4) (which have been used to derive (7)). If we fix the energy of the plug (and ring) and the diameter D, then from (4*d*) LU_P^2 is a constant. Hence if L is increased keeping the energy fixed, the plug velocity decreases, but from (4*b*) and (4*c*) it follows that both the circulation and impulse increase. This limitation on the aspect ratio of the plug thus implies that, for a fixed kinetic energy used to produce a ring, the particular member of a family of vortices
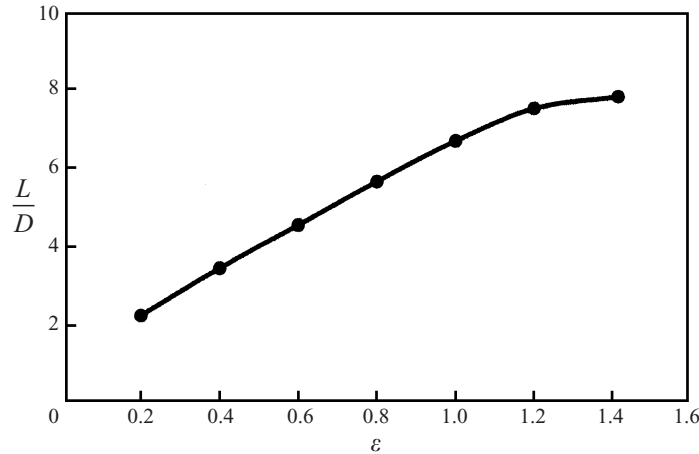
FIGURE 3. The aspect ratio L/D of the impulsively produced plug plotted against the core parameter ε for the corresponding ring, calculated using (7) and the data tabulated in table 2 of Norbury (1973).

corresponding to the maximum L/D is the one that has the maximum impulse and circulation. As the plug aspect ratio increases, the relative size of the core increases to accommodate the vorticity produced at the walls of the generator.

Using a similar technique, we can examine whether it is possible to incorporate all the plug fluid into the ring. In this case we match the volume of the plug to that of the ring – either the volume of the vorticity-containing core V_c or the total volume $V_c + V_e$ (where V_e is the volume of irrotational fluid carried along with the core, using Norbury's notation). Multiplying equations (5) and (6), which incorporate conservation of circulation, impulse and energy, and using (7) we obtain an expression for X^3. In this case, again using the non-dimensional tabulated values, and comparing the volume with (1a), we find

$$\frac{\text{Volume of core}}{V_P} = \frac{2V_c T_R}{c P_R^2}, \tag{8}$$

and a corresponding expression for the total volume. These two ratios are shown as functions of ε in figure 4, assuming that circulation, impulse and energy are conserved during formation. Note that for $\varepsilon > 0.42$ approximately, corresponding to $L/D > 3.5$, the total volume of fluid moving with the ring is smaller than that of the original plug, so that some of the ejected fluid cannot be transported with the ring. For all aspect ratios the core volume is less than that of the plug, and hence it is never possible to get all the plug fluid into the core of the rings in the family considered by Norbury. We should note, of course, that even in the case where the volume moving with the ring is larger than that of the plug volume, there is no guarantee that all the plug fluid will actually be incorporated. Some of the ambient fluid may be entrained while some of the injected fluid is left behind, as shown by Maxworthy (1972).

Similarly, a further property of interest can be calculated for this family of rings as a function of ε. The initial velocity of propagation compared to the ejection velocity can be written, when impulse and energy are conserved,
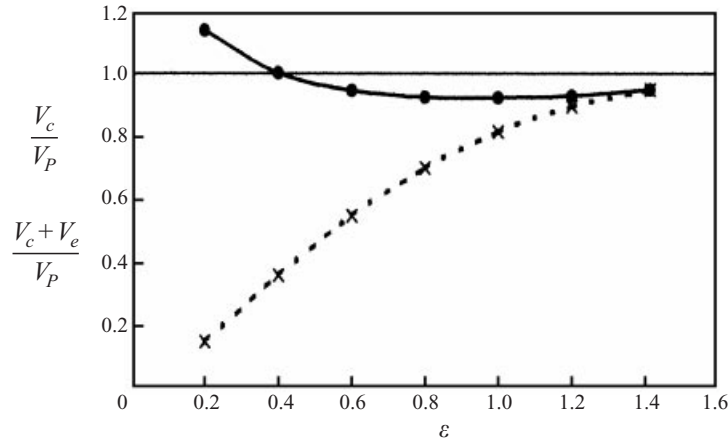
$$\frac{W}{U_P} = \frac{W_R P_R}{2T_R}. \tag{9}$$

FIGURE 4. The ratio of the volume of the ring core to the plug volume (dashed line), and the corresponding ratio of the total volume carried along by the ring to the plug volume (solid line), calculated from (8) for various values of ε using Norbury's data.
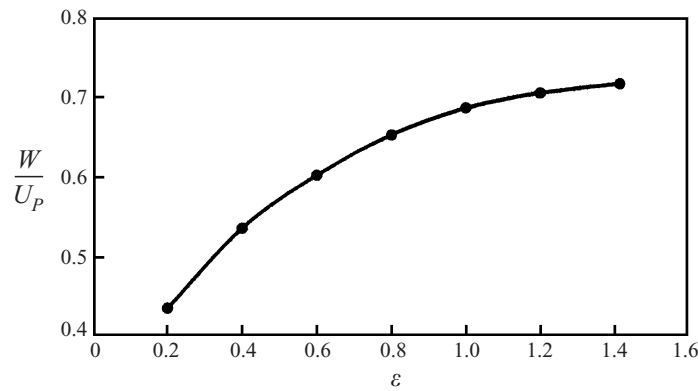

FIGURE 5. The ratio of the velocity of the ring to the plug velocity, calculated from (9) for various values of ε using Norbury's data.

This ratio is plotted in figure 5, and it is seen that the ring velocity is always less than that of the injected plug. This is a necessary feature of the formation process since otherwise the ring would leave plug fluid behind.

3. Comparisons with Gharib *et al.* (1998), and the effect of varying the parameters

So far, we have investigated the constraints on the L/D ratio obtained by matching two sets of parameters separately. Now we consider the implications of applying both of these constraints simultaneously. Viscosity has little time to act during the ring formation process at typical Reynolds numbers of these experiments, and vorticity is carried with the fluid. Consequently, the additional constraint imposed by matching the volume of the plug with that of the ring must be satisfied if the circulation is conserved. This constraint of volume conservation provides the final condition that selects the maximum plug length. The critical aspect ratio $L/D = 3.5$ is quite close to the value obtained experimentally by Gharib *et al.* (1998).

As mentioned above, it is never possible to get all of the plug fluid into the core of the ring (figure 4), so it may seem to be impossible to conserve the circulation. But the vorticity flux from the walls of the generator only penetrates some of the plug fluid – the exact fraction depends on the injection time divided by the viscous diffusion time – and so it is not necessary to incorporate all of the plug fluid into the core. The irrotational plug fluid can be carried outside the core. Of course, this requires some subtle rearrangements of the fluid during the ring formation, but since it is likely that the fluid near the edges of the plug does roll up into the core, such a matching seems feasible. Certainly the agreement between the theoretical and experimental values of the critical aspect ratio supports this possibility.

Gharib *et al.* (1998) attribute the critical plug aspect ratio to the maximum circulation that the ring can acquire. From our calculations (figure 4) we see that circulation, impulse and energy produced by the generator can be incorporated into the ring for values of the aspect ratio considerably larger than the critical value obtained experimentally. Only when the incorporation of the volume of the injected plug is considered is the experimental value predicted by the theory.

The effect of making different assumptions about the input conditions will be mentioned briefly. From (7), if energy is lost during the formation process ($c < 1$) then the maximum permissible L/D ratio is also reduced. At the same time, we see from (8) and figure 4 that this would make it easier to fit all the ejected fluid in the region moving with the ring, since this volume is increased relative to the plug volume. Insertion of the profile constants appropriate for a parabolic input velocity profile shows that the maximum plug length is reduced by a factor of 0.43, so that $(L/D)_{max}$, corresponding to the Hill's vortex, would be 3.39. The likely increase in the vorticity associated with a parabolic profile would reduce this value even more. (We note the numerical calculation by Rosenfeld, quoted by Gharib *et al.* (1998), which suggests that the reduction factor is about 0.25 when a parabolic profile is used.)

To make a further direct comparison between the experimental and our calculated results, we have plotted in figure 6 the dimensionless energy (cf. (7))

$$\alpha = \frac{T_R}{P_R^{1/2} K_R^{3/2}}, \tag{10}$$

which is the same form as given in equation (1) of Gharib *et al.* (1998). Data from figure 3 and the values of α calculated at discrete values of ε have been used to plot α as a function of L/D for the Norbury family, and the corresponding plot for the measurements with an impulsive 'plug' input have been transferred from Gharib *et al.*'s (1998) figure 15. The points and the line drawn through them represent the calculations, and the crosses are the experimental values. They lie virtually on the same curve, corresponding to (7), and there is only a small difference when experiments using other input conditions (i.e. motions of the piston) are considered.

The horizontal line drawn at $\alpha = 0.33$, is that taken by Gharib *et al.* (1998) to be the limiting non-dimensional energy α_{lim} below which the vortices become unstable. At this stage the limit $\alpha_{lim} = 0.33$ is an entirely experimental value; in order to justify it theoretically one would need to carry out a 'matching' calculation, based now on vorticity distributions such as that plotted in figure 14(*c*) of Gharib *et al.* (1998). The agreement shown in figure 6 suggests, however, that the differences between the experimental distributions of vorticity in the core and those assumed by Norbury are relatively unimportant in setting this limit.
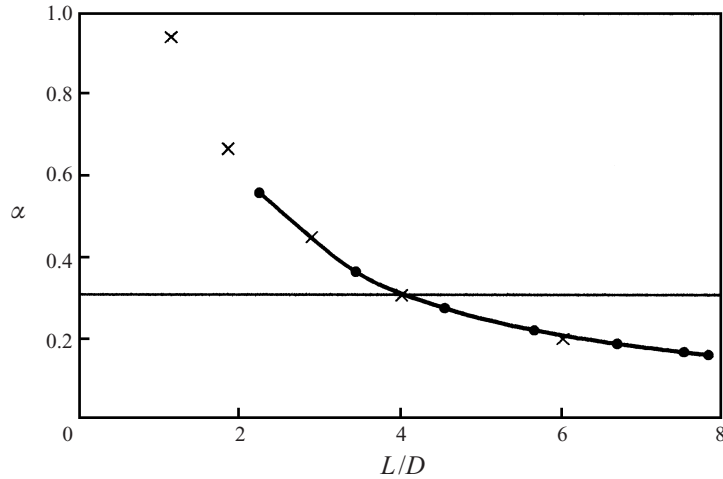
FIGURE 6. The dimensionless energy $\alpha = T_R P_R^{-1/2} K_R^{-3/2}$ plotted against the aspect ratio L/D. The points and the line refer to the values obtained in our matching calculation, and the crosses are from experimental data of Gharib *et al.* (1998).

4. Summary and discussion

Prompted by the experimental results recently published by Gharib *et al.* (1998), we have presented calculations aimed at understanding the physical mechanisms that determine the properties of vortices produced when fluid is ejected impulsively from a pipe. The method of calculation consists of matching the properties of the ejected fluid, the circulation, impulse, volume and kinetic energy, to the corresponding properties of the family of finite-core vortices examined theoretically by Norbury (1973). It is a 'before' and 'after' calculation, which does not consider the details of the formation process. We find that there is a maximum length-to-diameter ratio (L/D) of the plug such that a single vortex can form; for ratios above this a trail of vorticity-containing fluid is left behind. This limitation on the aspect ratio implies that, for fixed kinetic energy input, the member of a particular family of vortices corresponding to the maximum L/D is the one that has the maximum dimensionless core radius.

We regard these results as providing illumination on the physical principles govern-ing the formation of vortices. The Gharib *et al.* (1998) experiments show that in fact the limiting value of L/D is about 4. The value obtained by matching to the Norbury vortices is close to this value, and the differences may be attributable to the different vorticity distributions in the vortex cores – the measured profiles are more peaked than those assumed by Norbury. It is also suggested that the vorticity-containing fluid wraps into the vortex core, while irrotational fluid in the plug is carried along with the ring.

Our explanation of the maximum plug aspect ratio (or formation number in the terms used by Gharib *et al.* (1998)) is quite different from their explanation. Gharib *et al.* (1998) and Mohseni & Gharib (1998) argue that the critical value is determined by an energy constraint associated with the maximum energy carried by a vortex when the two cores touch. For rings with thicker cores they suggest that vorticity is left in the wake. However, it seems from their data (see Gharib *et al.* 1998, figure 14*c*) that even at this limiting stage most of the vorticity is carried in two well-separated cores. Our results show that the limit implied by matching the energy, impulse and

circulation for the Norbury class of rings (figure 3) has not been reached, and it is the volume constraint (figure 4) that is the determining factor.

These phenomena are relevant to two different situations in which a series of vortices is observed to form, and the phenomena have been related to a Strouhal number describing the rate of formation of vortices. The breakup of a circular jet (Crow & Champagne 1971) can be considered in terms of vortices with the optimal properties. Each vortex has formed from the maximum length of jet consistent with the conservation of circulation, impulse, volume, and energy. A less obvious extension is to the swimming of fish, which flap their tails and shed vortices at a rate that corresponds to a Strouhal number of about 0.3. Such vortices also have the 'optimal' property of having the maximum impulse (or thrust) for a given energy input, and they are as close as they can get without interfering.

The potential applications of the fish-swimming mechanism to propulsion of vehicles are being actively studied by ocean engineers who want to learn more about this mechanism and adapt it for the propulsion of vehicles (Triantafyllou, Triantafyllou & Grosenbaugh 1993; Triantafyllou & Triantafyllou 1995). The efficiency of propulsion by an oscillating foil has been measured as a function of frequency, and has been found to be a maximum at a Strouhal number of about 0.3, in the same range as swimming fish. But most of the thinking has been in terms of continuously flapping foils, and the vortex ring ideas presented above suggest other questions to be explored.

An important implication of the present paper is that the repetition of vortex production is not necessary for an individual vortex to have the 'optimal' characteristics. For example a fish often makes a single flap of its tail, or two flaps in quick succession, to change direction or swim a short distance. This can be compared to the formation of a single vortex ring by projecting just the right length of a plug from a pipe to produce the maximum impulse for a given energy input. The fish can then wait as long as it likes before creating another 'optimal' vortex, which will again have maximum efficiency in the sense that maximum thrust is produced for a given expenditure of energy. The concept of Strouhal number only arises when we consider how frequently this can be done before the vortices interfere with one another; it gives an upper limit to the frequency of vortex ring production, but there is no lower limit.

There is, of course, another form of propulsion used by marine animals that is directly related to the production of vortex rings by expelling fluid from a tube. Squid, salps and jelly fish propel themselves by a mechanism referred to by biologists as 'jet propulsion', but the pulsating nature of the flow has been clearly recognized. Siekmann (1963) has reported a theoretical study of a pulsating jet, and has applied it to some laboratory data and to the swimming of squid. These, and other applications of the optimal vortex ring to propulsion, will be examined in more detail in a future paper.

REFERENCES

BAIRD, M., WAIREGI, H. I. & LOO, H. J. 1977 Velocity and momentum of vortex rings in relation to formation parameters. *Can. J. Chem. Engng* **55**, 19–26.

BENJAMIN, T. B. 1976 The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics. In *Applications of Methods of Functional Analysis to Problems in Mechanics.* Lecture Notes in Mathematics, Vol. 503, pp. 8–28 (ed. P. Germain & B. Nayroles). Springer.

CROW, S. C. & CHAMPAGNE, F. H. 1971 Orderly structure in jet turbulence. *J. Fluid Mech.* **48**, 547–591.

DIDDEN, N. 1979 On the formation of vortex rings: rolling-up and production of circulation. *Z. Angew. Math. Phys.* **30**, 101–116.

FOHL, T. 1967 Optimization of flow for forcing stack wastes to high altitudes. *J. Air Pollution Control Assoc.* **17**, 730–733.

FOHL, T. 1968 The creation of vortex rings by impulsive acceleration of a sphere of fluid. *J. Appl. Phys.* **39**, 2470–2471.

GHARIB, M., RAMBOD, E. & SHARIFF, K. 1998 A universal time scale for vortex ring formation. *J. Fluid Mech.* **360**, 121–140.

GLEZER, A. 1988 The formation of vortex rings. *Phys. Fluids* **31**, 3532–3542.

GLEZER, A. & COLES, D. 1990 An experimental study of a turbulent vortex ring. *J. Fluid Mech.* **221**, 243–283.

MAXWORTHY, T. 1972 The structure and stability of vortex rings. *J. Fluid Mech.* **51**, 15–32.

MAXWORTHY, T. 1977 Some experimental studies of vortex rings. *J. Fluid Mech.* **81**, 465–495.

MOHSENI, K. & GHARIB, M. 1998 A model for universal time scale for vortex ring formation. *Phys. Fluids* **10**, 2436–2438.

NORBURY, J. 1973 A family of steady vortex rings. *J. Fluid Mech.* **57**, 417–431.

SAFFMAN, P. G. 1975 On the formation of vortex rings. *Stud. Appl. Maths* **54**, 261–268.

SHARIFF, K. & LEONARD, A. 1992 Vortex rings. *Ann. Rev. Fluid Mech.* **24**, 235–279.

SIEKMANN, J. 1963 On a pulsating jet from the end of a tube, with application to the propulsion of certain aquatic animals. *J. Fluid Mech.* **15**, 399–418.

TAYLOR, G. I. 1953 Formation of a vortex ring by giving an impulse to a circular disc and then dissolving it away. *J. Appl. Phys.* **24**, 104–105.

TRIANTAFYLLOU, G. S., TRIANTAFYLLOU, M. S. & GROSENBAUGH, M. A. 1993 Optimal thrust development in oscillating foils with application to fish propulsion. *J. Fluids Struct.* **7**, 205–224.

TRIANTAFYLLOU, M. S. & TRIANTAFYLLOU, G. S. 1995 An efficient swimming machine. *Sci. Am.* March, 40–48.

TURNER, J. S. 1957 Buoyant vortex rings. *Proc. R. Soc. Lond.* A **239**, 61–75.

TURNER, J. S. 1960 Intermittent release of smoke from chimneys. *Mech. Engng Sci.* **2**, 97–100.