Chapter 4

Dimensional analysis

4.1 Independence of units

The governing equations of physics relate quantities with the same dimensions. The dimensions of any quantity may be expressed in terms of a fundamental set of units. We shall use the SI system, in which the fundamental set that concerns us is mass (kilogram – kg), length (meter – m), time (second – s) and temperature (Kelvin – K).

Dimensional analysis can be used to predict the functional forms of relations between physical quantities and also to produce nondimensional groups that govern flow characteristics.

As the simplest possible example, consider the system of a mass on a string. Without knowing anything about the dynamics of the system, the parameters that come into play are the length of the string l (m), the acceleration due to gravity, g (ms^{-2}), and the mass of the pendulum m (kg). The only quantity with the dimensions of time that can be constructed is $(l/g)^{1/2}$. Hence, if the pendulum performs periodic motion, which it will for small enough amplitudes, the frequency of the motion must be proportional to $(g/l)^{1/2}$ (for very small amplitudes, the frequency is $(g/l)^{1/2}/2\pi$).

4.2 Buckingham's Π theorem

The *Buckingham* Π *theorem* states that the relation among n dimensional quantities $q_1, \ldots, q_n$,

$$g(q_1, \ldots, q_n) = 0, \tag{4.1}$$

can be represented as

$$G(\Pi_1, \ldots, \Pi_{n-m}) \tag{4.2}$$

where the Π_i are all the dimensionless combinations of the q_n that can be formed. A formal procedure for using this theorem is given in **?**.

Example 4.1 *Consider the two layer fluid shown in figure 3.8. The dimensional parameters are the densities ρ_1 and ρ_2, the depths H_1 and H_2, gravity **g** and the viscosity ν. Suppose we are concerned with oscillations with frequency ω.*

If we assume that viscous effects are unimportant, then flow in the system depends on 6 independent variables. Since the dimensions of these variables include mass M, length L and time T, the properties of the system can be described by $6 - 3 = 3$, dimensionless variables.

The choice is somewhat arbitrary, but possible choices are

$$\gamma = \frac{\rho_1}{\rho_2},$$

$$D = \frac{H_1}{H_2}$$

and

$$\Omega = \omega\sqrt{\frac{H_1}{g}}.$$

Formally we can write

$$\Omega = \mathsf{F}(\gamma, D).$$

However, this is somewhat mindless.

Example 4.2 *A more appropriate way to describe the flow is to note that restoring force across the interface is due to the density* difference. *Hence we use instead of γ, the dimensionless group*

$$\frac{\rho_2 - \rho_1}{\rho_2} = 1 - \gamma.$$

The noting that this is the only parameter involving mass, we can combine it with gravity to get the reduced gravity

$$g' = g\frac{\rho_2 - \rho_1}{\rho_2}.$$

Then, since g' is the only variable with the dimensions of time we conclude that

$$\omega = \sqrt{\frac{g'}{H_1}}F(D).$$

This is exactly the same result as in the previous example, but is written in a more revealing way.

4.3 Equations of motion in dimensionless form

In addition, the equations of motion can be nondimensionalized. We take characteristic values for quantities, such as L for length, U for velocity, and so on, which we substitute into the equations. Simplifying the equations yields nondimensional groups which, along with boundary conditions, determine the behavior of the system.

Probably the most important case in fluid mechanics is the Navier–Stokes equations. Taking L and U as above, as well as time scale T, pressure scale P, and density scale ρ_0, gives

$$\frac{U}{T}\frac{\partial \boldsymbol{u}}{\partial t} + \frac{U^2}{L}(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{P}{\rho_0 L}\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{\nu U}{L^2}\nabla^2 \boldsymbol{u}. \tag{4.3}$$

If there is no exterior imposed time scale, then $T = L/U$. In addition, the pressure force must balance the inertial terms, so $P \sim \rho_0 U^2$. Then

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{Re}\nabla^2 \boldsymbol{u}, \tag{4.4}$$

where the Reynolds number is defined by

$$Re \equiv \frac{UL}{\nu}. \tag{4.5}$$

Hence in general, the behavior of a viscous flow with no body forces is determined entirely by its Reynolds number and geometry. Two flows with the same Reynolds number are *kinematically similar*, while two flows with the same geometry are *geometrically similar*: if both hold, the flows are *dynamically similar*.

For large Reynolds number, (4.4) suggests that the last term is negligible. This leads to the Euler equations, which are known to be an incomplete description of fluid motion. This apparent paradox may be reconciled with observations by the presence of a thin *boundary layer* next to solid surfaces, in which the influence of viscosity cannot be neglected.

For a solid boundary at $y = 0$, the term u_{yy} will be much larger than the u_{xx} term, because variations are much more rapid in the y-direction. If the boundary layer has width δ, the viscous term may be balanced with the advective term to give $U^2/L \sim \nu U/\delta^2$, which shows that the boundary layer width is

$$\delta = L Re^{-1/2}. \tag{4.6}$$

Example 4.3 Nuclear blast front. *Consider an intense detonation. Shortly after the blast the spread of the front is characterized only by the amount of energy E released and the density of the fluid, ρ. How does the front evolve? How could the energy released be computed from pictures of the blast?*

It is necessary to make two assumptions.

- *that the blast wave is spherical and characterized by a single scale – its radius R. Figure 4.1 shows that this is indeed the case (of course the presence of the ground makes it a hemisphere).*
- *that the air moves adiabatically. This is reasonable because the motion is fast, so there is little time to transfer heat.*

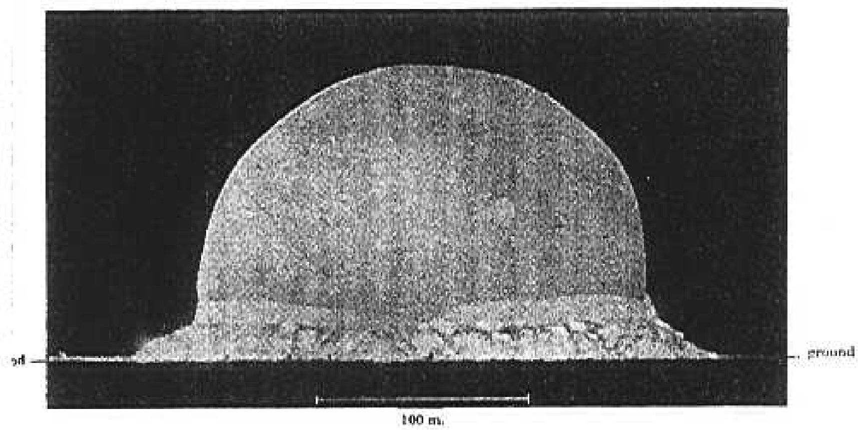

Figure 4.1: The ball of fire at 15ms, showing the sharpness of its edge. From Taylor (1950).

Then the radius R of the detonation wave depends on E, ρ and the time t. These 4 variables depend on M, L and T, so there is a single nondimensional parameter.

Noting that [E] = ML^2T^{-2}, then $R = C\left(\frac{Et^2}{\rho}\right)^{1/5}$. The dimensionless parameter is $C = C(\gamma)$, where $\gamma = \frac{c_p}{c_V}$ is the ratio of specific heats for air.

Figure 4.2 confirms the dimensional analysis. Taylor (1950)[1] estimated the yield of the 1945 bomb to be 16.8 kilotons, in fair agreement with the figure of 20 kilotons announced by President Truman, and based on measurements of air velocity and temperatures.

4.4 Application to water waves

A classical fluid dynamical problem with environmental consequences is the motion of waves on the surface of water. Waves are described by their wave length λ and phase speed c_p, which is the speed of the wave crests. Alternatively, we can use the wave frequency $\omega = \frac{c_p}{\lambda}$, as a parameter.

Assuming that the viscosity and surface tension of the water is unimportant, c_p depends on λ, gravity **g** and the water depth H. Dimensional analysis then implies that

$$c_p = \sqrt{gH}f\left(\frac{\lambda}{H}\right), \tag{4.7}$$

[1]G.I. Taylor 1885 – 1975. Considered by many to be the greatest fluid dynamicist. This analysis is typical of the way he was able to get simple and accurate answers to very complex problems.

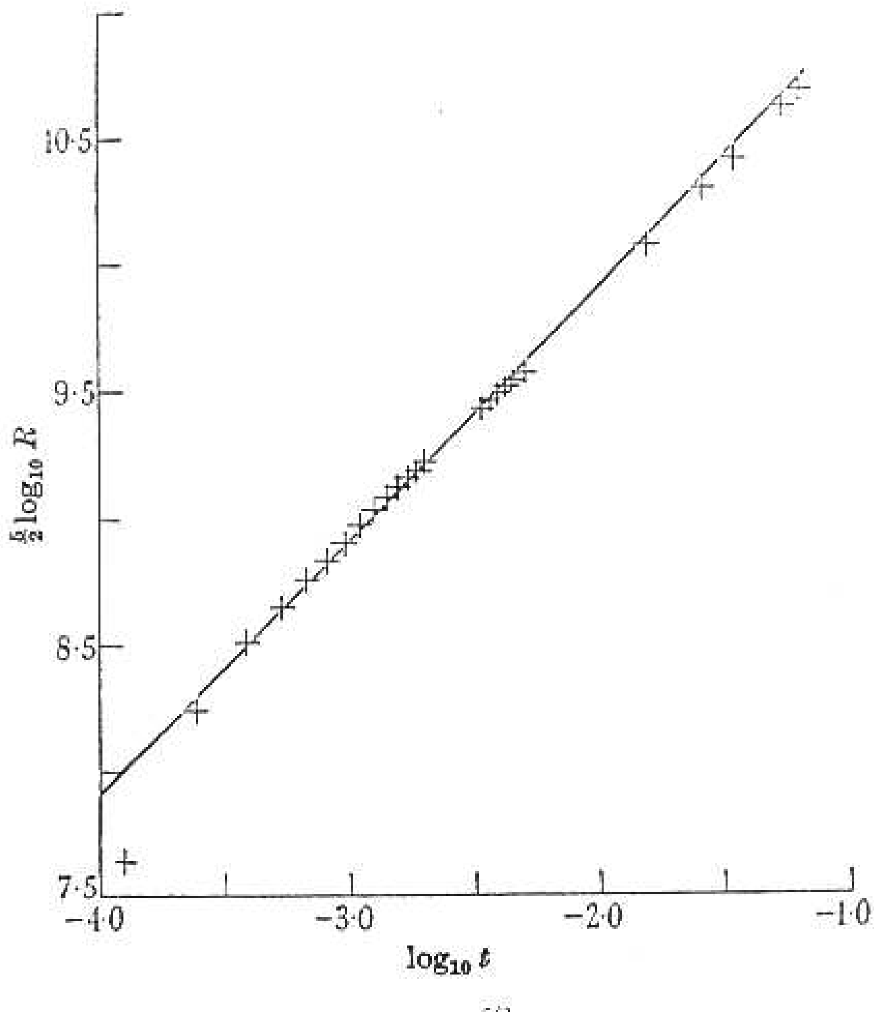

Figure 4.2: Logarithmic plot showing $R^{5/2}$ is proportional to t. From Taylor (1950).

where $f\left(\frac{\lambda}{H}\right)$ is a dimensionless function.

4.4.1 Deep water waves

When the water is much deeper than the wavelength of the waves, i.e $\lambda << H$, we expect the speed to be independent of the depth. In that case (4.7) implies that $f\left(\frac{\lambda}{H}\right) \propto \sqrt{\frac{\lambda}{H}}$ and the speed is given by

$$c_p = K_1\sqrt{g\lambda}, \tag{4.8}$$

where K_1 is a dimensionless constant.

Thus the speed of deep water waves depends on their wavelength, and such waves are called *dispersive*. Long waves travel faster than short waves. This implies that when a storm generates waves in the deep ocean, the long waves arrive at the coast first followed by the shorter waves. By observing the different wavelengths (or, more easily, the different frequencies) of waves at successive times it is possible to calculate the distance to the storm.

4.4.2 Shallow water waves

When waves arrive at a beach the other limit $\lambda >> H$ applies. In that limit we expect the wave speed to be independent of λ, and so $f\left(\frac{\lambda}{H}\right) = K_2$ a dimensionless constant and

$$c_p = K_2\sqrt{gH}. \tag{4.9}$$

In this case the wave speed is independent of the wavelength and such waves are called *nondispersive*. Since waves on shallow water have wavelengths much larger than the depth, they are also referred to as *long waves*.

Equation (4.9) is the reason that it is possible to go surfing! Since the wave speed decreases as the depth decreases, waves approaching a beach at an angle are refracted parallel to the beach (see figure 4.3). As can bed seen from the figure, the portion of the wave crest near the beach travels slower then the portion further away, allowing the latter part to catch up and align the wave crest parallel to the beach.

In order to determine the full form of the function $f\left(\frac{\lambda}{H}\right)$ it is necessary to carry out a theoretical analysis. This analysis is beyond the scope of this course, but can be found in many textbooks e.g. Lighthill (1978), chapter 3. The result is

$$f\left(\frac{\lambda}{H}\right) = \sqrt{\frac{\lambda}{2\pi H}\tanh\left(\frac{2\pi H}{\lambda}\right)}. \tag{4.10}$$

Thus

$$c_p = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi H}{\lambda}\right)}, \tag{4.11}$$

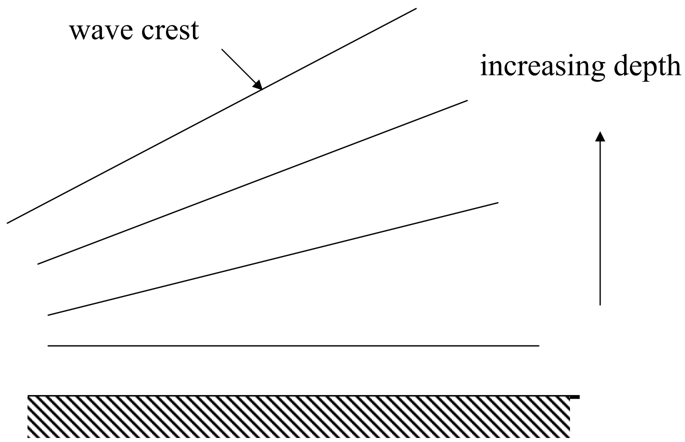

Figure 4.3: The refraction of waves approaching a beach. Since the speed is greater in deeper water, the crests curve around parallel to the shore.

which is known as the dispersion relation for water waves. (Usually the dispersion relation relates the frequency to the wavelength, but this is an equivalent relation.) Taking the appropriate limits for $\frac{\lambda}{H}$, we find that $K_1 = \frac{1}{\sqrt{2\pi}}$ and $K_2 = 1$. Figure 4.4 is a plot of the wave speed as a function of the wave length. We see that c_p increases with wave length, and the fastest waves are waves in shallow water, with a wave speed $c_p = \sqrt{gH}$.

4.4.3 Hydraulics of a river

If we throw a rock into a stationary pond waves will propagate away from the point of impact – the fastest waves will be long waves that travel with speed $\sqrt{gH}$. Suppose instead that we throw the rock into a river flowing with speed U. Waves will then travel downstream with maximum speed $U + \sqrt{gH}$, while waves will travel upstream with a maximum speed $U - \sqrt{gH}$. Thus when $U > \sqrt{gH}$, no waves will travel upstream. A river flowing at a speed greater than the long wave speed is said to be *supercritical*, since no information can propagate upstream. This is similar to a supersonic flow, where the speed is faster than the speed of sound waves; so for a jet travelling at a supersonic speed, the sound is left behind and the approach of the jet can not be heard.

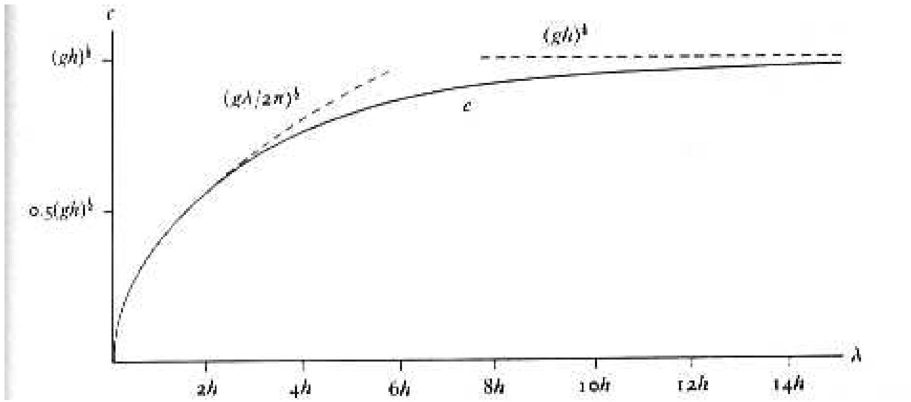

Figure 4.4: The wave speed c given by linear theory for waves of varying wavelength λ on water of uniform depth. Note the transition from the deep-water value $\sqrt{\frac{g\lambda}{2\pi}}$ to the long wave value $\sqrt{gH}$. From Lighthill (1978).

4.4.4 Froude number

The character of a river flow can be determined by the Froude number $F \equiv \frac{U}{\sqrt{gH}}$. Supercritical flow is given by $F > 1$ and subcritical flow by $F < 1$. In subcritical flow information can propagate both upstream and downstream, while it can only propagate downstream in a supercritical flow.

Suppose a river flow is supercritical and at some time an obstacle partially blocking the flow is placed at some location. Since information can only propagate downstream, the presence of the obstacle is undetectable to the oncoming river ahead of the obstacle. Clearly if the obstacle is large enough to reduce the river flow this will lead to a increase in the water level and this is unsustainable.

The way the river responds is to create a hydraulic jump, across which the flow changes from supercritical upstream to subcritical downstream as shown in figure 4.5. Downstream of the jump, since the flow is supercritical, information can propagate upstream and so the presence of the obstacle is known up to the jump. As the jump propagates upstream the river level changes in response to the obstacle.

4.4.5 Flow over a weir

Consider flow over a weir as shown in figure 4.7. Suppose that the depth of the water is $d(x)$ and the (uniform) speed is $u(x)$. Then the volume flow rate Q per unit width is

$$Q = u(x)d(x). \tag{4.12}$$

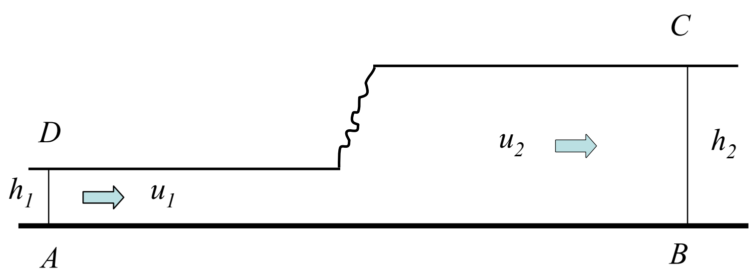

Figure 4.5: A hydraulic jump. The control volume is the region $ABCD$ including the free surface from C to D.

Figure 4.6: Turbulent bore on the River Severn, UK.

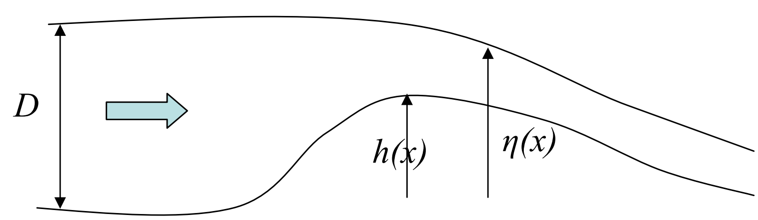

Figure 4.7: The flow over a weir. The slope of the weir is supposed sufficiently small for the flow to be hydrostatic.

Obviously, if the flow is steady conservation of mass implies that Q is independent of the downstream location x. Apply Bernoulli's equation along the free surface from the point A in the reservoir, where the flow is slow and the water depth is D, to a point B over the weir. If $\eta(x)$ is the height of the surface over the weir,

$$p_A + g\rho D = p_A + g\rho\eta + \tfrac{1}{2}\rho u^2. \tag{4.13}$$

Thus

$$u^2 = 2g(D - \eta). \tag{4.14}$$

Given the height of the weir is $h(x)$,

$$d = \eta - h. \tag{4.15}$$

Hence using (4.12) and (4.15) in (4.14) we get

$$u^3 - u\left[2g(D-h)\right] - 2gQ = 0. \tag{4.16}$$

Differentiate (4.16) with respect to x, noting that Q is constant, to get

$$u'\left(3u^2 - \left[2g(D-h)\right]\right) - 2guh' = 0. \tag{4.17}$$

At the top of the weir, $h' = 0$ and $h = h_m$, and at that point

$$u^2 = \frac{2}{3}g(D - h_m). \tag{4.18}$$

The Froude number F based on the velocity and depth at the top of the weir is

$$F = \frac{u}{\sqrt{g(\eta - h_m)}}. \tag{4.19}$$

From (4.14)

$$g(\eta - h_m) = g(d - h_m) - \tfrac{1}{2}u^2 = \frac{2}{3}g(D - h_m), \tag{4.20}$$

and so by (4.18) we see that, at the crest of the weir, $F = 1$. Thus the flow is *critical* at the crest. The crest is said to be a *control* point for the flow.

Obviously, the flow is subcritical in the reservoir, so what happens downstream of the crest? At the crest of the weir

$$u^3 = gQ, \tag{4.21}$$

so that (4.16) may be rewritten as

$$h = D - \frac{u^2}{2g} - \frac{Q}{u}, \tag{4.22}$$

and the turning points occur at the values of u given by (4.21).

The curve $h(u)$ is shown in figure 4.8. Upstream in the reservoir corresponds to the point R and as the flow approaches the crest, it follows the arrow to the right. At the crest C, the solution has two branches. Either the speed can decrease again downstream and the flow remains subcritical (the solution reverses direction back towards R), or the speed can increase and the solution is on a different, supercritical branch.

Problem 4.1 *From a consideration of the Navier-Stokes equations show that a reasonable estimate of the viscous drag on a sphere, of radius a and moving with speed U, is $\frac{\nu U}{a^2}$. Hence show that a reasonable equation for the motion of a sphere, displaced vertically a small distance s from its equilibrium position in a viscous fluid with buoyancy frequency N, is*

$$\frac{d^2s}{dt^2} + \frac{\nu}{a^2}\frac{ds}{dt} + N^2 s = 0.$$

Solve this equation for a sphere released from rest from a position s_0 above its equilibrium position. Using reasonable estimates for N, ν and a from the laboratory experiment, discuss whether this is a good model for your observations.

Problem 4.2 *Waves of small wavelength, such as ripples on the sea surface, ate influenced by the surface tension T of the free surface. The surface tension T is the force per unit length of surface and for water has the value T = 0.074Nm^{-2}.*

Use dimensional analysis to show that

$$\frac{T}{g\rho\lambda^2}$$

is a dimensionless group. Hence show that the wave speed c_p can be written as

$$c_p = \sqrt{gH} f\left(\frac{\lambda}{H}, \frac{T}{g\rho\lambda^2}\right).$$

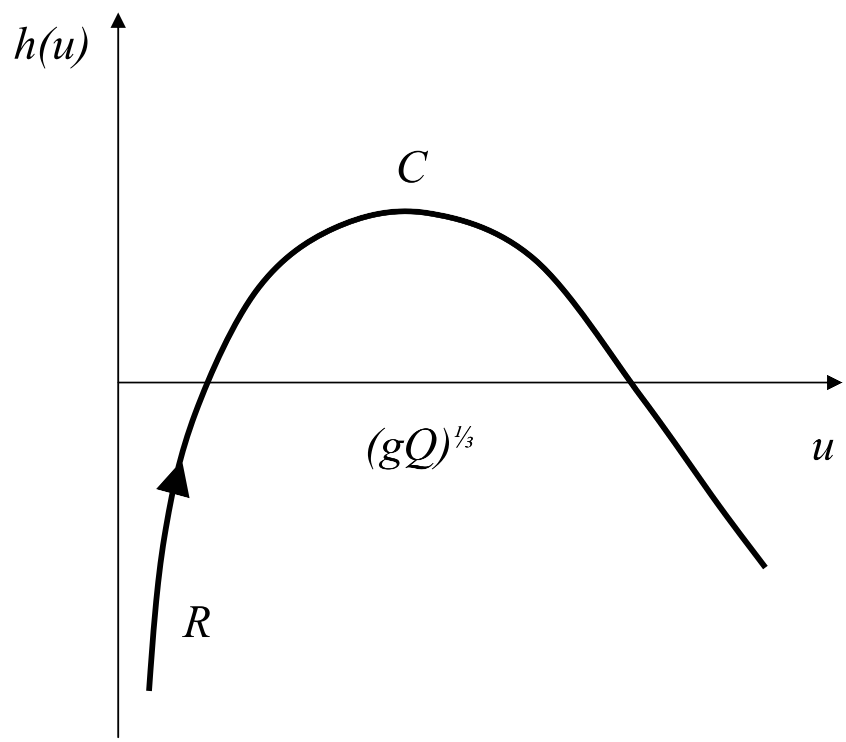

Figure 4.8: The depth h plotted against the flow speed u. The reservoir is denoted by the point R and the crest of the weir by C.

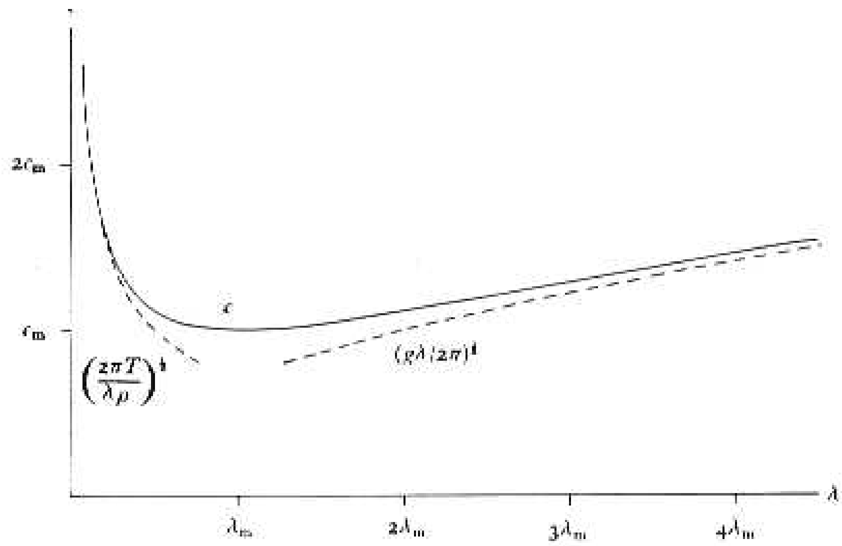

Figure 4.9: The wave speed c for ripples on deep water. From Lighthill (1978)).

In deep water, and for small surface tension show that, to a first approximation, the wave speed is given by

$$c_p = \sqrt{\frac{g\lambda}{2\pi}}\left(1 + K\frac{T}{g\rho\lambda^2}\right),$$

where K is a dimensionless constant. Compare this result with the exact solution given in figure 4.9.

Problem 4.3 *Apply conservation of mass flux and momentum flux to the control volume around the hydraulic jump shown in figure 4.5, and calculate the speed and depth of the flow on the downstream side of the jump. Show that $F > 1$ on the upstream side and $F < 1$ on the downstream side. By calculating the energy fluxes into and out of the control volume show that energy is dissipated in the jump.*