Chapter 3

Stratification

A fluid in which the density ρ varies in space, i.e. $\rho = \rho(\mathbf{x})$ is said to be stratified. In a fluid at rest (2.26) reduces to

$$\boldsymbol{\nabla} p = \rho \mathbf{g} \tag{3.1}$$

Consequently, in a gravitational field

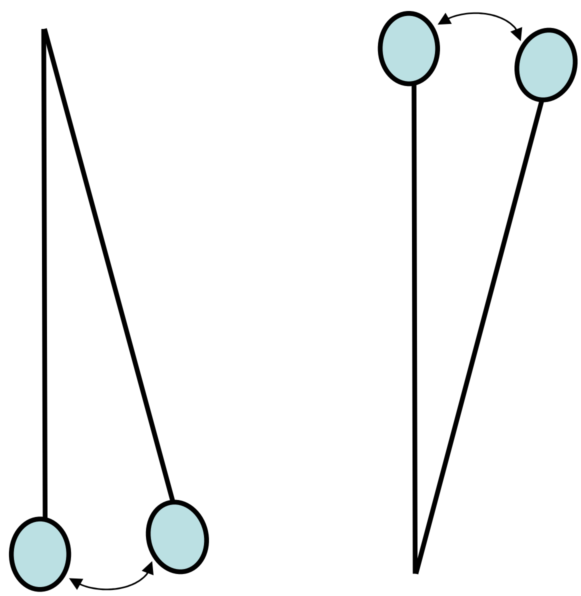
$$\frac{\partial \rho}{\partial z} = -g\rho, \tag{3.2}$$

where z is measured upwards, and the pressure must be constant on a horizontal surface i.e. $\rho = \rho(z)$. There are then two possibilities: the density either decreases or increases with height. We consider the two possibilities separately.

3.1 Stability

A standard approach in dynamics is to enquire about the stability of the equilibrium. This is achieved by considering a small departure (perturbation) from the equilibrium state. Since such perturbations always occur in nature due to random events that we have no control over, the response of the system tells us what is likely to happen in practice. A familiar example is that of a pendulum consisting of a mass on the end of a rigid rod (see figure 3.1). There are two equilibrium positions: one where the mass hangs directly below the pivot (figure 3.1 (a)) and one where it is directly above the pivot (figure 3.1 (b)). Common experience tells us that one of these equilibria is stable, while the other is unstable. When the mass hangs below the pivot as in figure 3.1 (a) a small perturbation is subject to a force that restores it to its equilibrium position. This equilibrium is said to be *stable*. On the other hand, when the mass is directly above the pivot, a small perturbation is amplified and the mass moves towards the stable equilibrium. This equilibrium is said to be *unstable*.

The difference between these two cases is that for the pendulum the downward gravitational force on the mass has a component that moves the mass



(*a*) (*b*)

Figure 3.1: A schematic showing the departure from equilibrium of a pendulum. (a) The stable equilibrium of a standard pendulum. (b) The unstable equilibrium of an inverted pendulum.

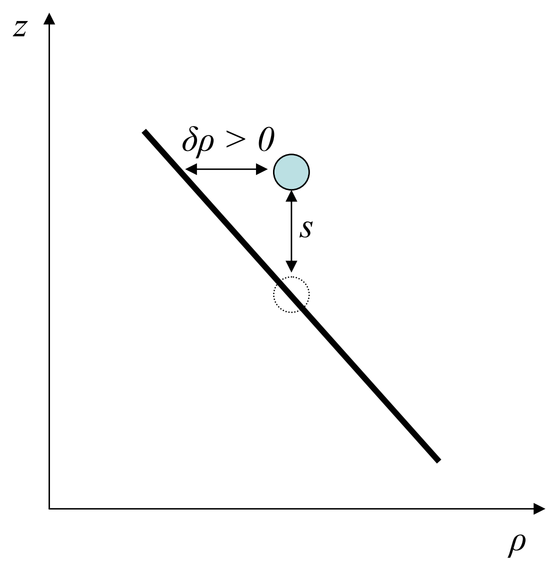


Figure 3.2: A schematic showing the density perturbation for a small parcel raised a distance s from its equilibrium position.

towards the equilibrium position. For the inverted pendulum this downward force moves the mass away from the equilibrium position. We will see in chapter 4that dimensional analysis shows that the pendulum oscillates with a frequency $\omega \propto \sqrt{l/g}$. This is a property of all stable equilibria – small departures cause the system to oscillate about the equilibrium position. The final return to equilibrium is caused by damping.

In natural systems these principles imply that the system is usually in a state close to a stable equilibrium and that random natural perturbations cause oscillations – usually in the form of waves – about this equilibrium. Hence waves play an important role in environmental flows.

3.2 Stable density stratification

We first consider the case where the density decreases with height as shown in figure 3.2. Since the pressure gradient is vertical, it is balanced by gravity through the hydrostatic relation (1.5). Thus the stationary state is an equilibrium state. Consider a small parcel of fluid of volume dV raised a (small) distance s above its initial position, without exchanging mass with its surroundings. Then, to a first approximation, the density of the parcel *exceeds* that of its surroundings by an amount $\delta\rho$ given by

$$\delta\rho = -s\frac{d\rho}{dz}. \tag{3.3}$$

Since the parcel is denser than its surroundings it experiences a downward buoyancy force $g\delta\rho dV$. This force tends to restore the parcel to its original position, and so the equilibrium is stable. The stratification is said to be *statically stable* since the equilibrium is from a state of rest. If the fluid is moving a different criterion determines whether the flow is stable or not.

This result has the important implication that natural bodies of fluid tend to be stably stratified. The warmest water in a lake or reservoir is at the surface, and the hottest air in a room is near the ceiling. It is also the reason why the temperature in the stratosphere increases with height. Thus the study of stably stratified fluids is fundamental to environmental fluid dynamics.

Newton's second law of motion gives the acceleration of the parcel as

$$\rho dV\frac{d^2s}{dt^2} = -\left(-sg\frac{d\rho}{dz}dV\right). \tag{3.4}$$

This equation can be re-arranged to

$$\frac{d^2s}{dt^2} + N^2 s = 0, \tag{3.5}$$

where

$$N \equiv \sqrt{-\frac{g}{\rho}\frac{d\rho}{dz}}, \tag{3.6}$$

is the *buoyancy frequency*[1] When the density decreases with height $\frac{d\rho}{dz} < 0$, and so the term under the square root in (3.6) is positive and N is real. In general $N = N(z)$ and the strength of the stratification increases with N.

When N is a constant, equation (3.5) is the equation for simple harmonic motion, and the general solution is

$$s(t) = A\cos Nt + B\sin Nt, \tag{3.7}$$

where A and B are arbitrary constants, whose values are set by the initial conditions. This solution shows that the parcel will oscillate vertically with frequency N. (In fact this result is only true for an infinitesimal parcel in an inviscid fluid - see chapter **??**.) The period of the oscillations is $\frac{2\pi}{N}$.

The buoyancy frequency N is the key parameter describing a stratified fluid. It has dimensions of T^{-1}, and increases with the magnitude of the density gradient. Thus the strength of the stratification is proportional to N, as is the oscillation of fluid parcels. In any system with a restoring force perturbations about an equilibrium state generate waves. In a stably stratified fluid

[1] The buoyancy frequency is also known as the Brunt-Väisälä frequency. David Brunt (1886 -1965), Welsh meteorologist. Vilho Väisälä (1889-1969) Finnish oceanographer.

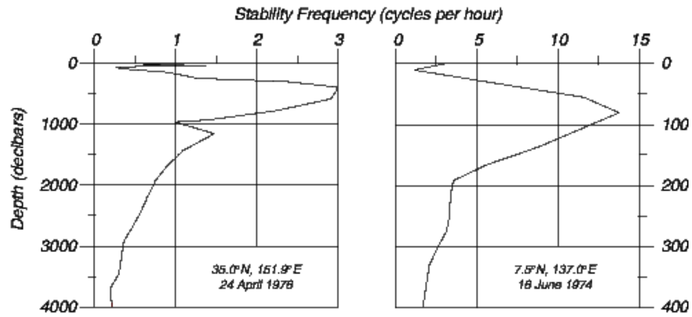


Figure 3.3: Observed stratification frequency in the Pacific. Left: Stability of the deep thermocline east of the Kuroshio. Right: Stability of a shallow thermocline typical of the tropics. Note the change of scales.

these waves are called internal gravity waves (IGW) and these are discussed in chapter **??**.

In the atmosphere typical values of N are $10^{-2}\mathrm{s}^{-1}$, and so the periods of the IGW are $\frac{2\pi}{N} \approx 10$ mins. Similar values are found in the ocean. In a room temperature variations are often around 5 K over a height of 2 m. This implies values of N of about $10^{-1}\mathrm{s}^{-1}$ and wave periods of 20 sec or so.

The inverse time scale N^{-1} is a measure of the response time of the stratification. If a process occurs on a time much shorter than N^{-1}, so that $Nt << 1$, then the stratification does not have time to respond and so the fluid behaves as though it is unstratified. So the propagation of say sound waves in a room, which take only a few milliseconds to travel across the room are not influenced by the stratification. So what you hear does not depend on how well ventilated the room is. On the other hand an air-conditioning system with a cycle time of, say, 10 mins, will generate motions that are influenced by the temperature variations within the room. We will see examples of this in chapter **??**.

Example 3.1 *The buoyancy frequency N of the atmosphere is calculated using the potential temperature. Consider the standard atmosphere shown in figure 1.13.*

(a) Troposphere

The observed temperature gradient is -6.5 Kkm^{-1}. The adiabatic lapse rate is $-\frac{g}{c_p} = -9.8Kkm^{-1}$. Hence the potential temperature gradient is +3.5 Kkm^{-1}. Then

$$N = \sqrt{\frac{g}{T_0}\frac{dT}{dz}}. \tag{3.8}$$

Hence, taking T_0 = 300K, $g = 9.8ms^{-2}$, N = 0.011s^{-1}.

(b) Stratosphere

In the lower part of the stratosphere, the observed temperature gradient is +1.0 Kkm^{-1}. The adiabatic lapse rate is $-\frac{g}{c_p} = -9.8Kkm^{-1}$. Hence the potential temperature gradient is +10.8 Kkm^{-1}. In this case N = 0.019s^{-1}.

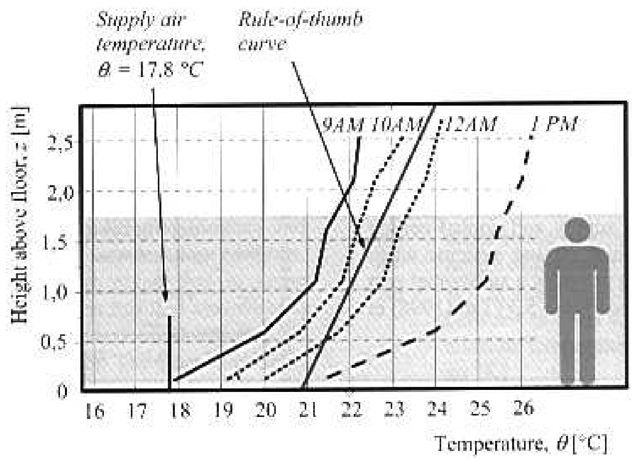


Figure 3.4: Temperature profiles measured at various times in a meeting room. From Skistad (2002)

Exercise 3.1 *Calculate N in the upper part of the stratosphere where the observed temperature gradient is +2.8 Kkm^{-1}.*

3.3 Unstable density stratification

The case where the density increases with height is shown in figure 3.6. The argument given in § 3.2 follows through as before except that now $N^2 < 0$ and the buoyancy frequency is imaginary. Write $M = iN$ and, in that case (3.5) has a solution of the form

$$s(t) = Ae^{Mt} + Be^{-Mt}, \tag{3.9}$$

where A and B are arbitrary constants. Since we can choose the sign of M (which is a real number), (3.9) implies that s grows exponentially with time. Hence the departure from equilibrium increases, and the stratification is unstable. It is said to be *statically unstable* because we are considering the stability of a fluid initially at rest.

This mathematical argument is consistent with our physical intuition. Figure 3.6 shows that when a parcel is raised above its initial position the density perturbation $\delta\rho < 0$ and so the parcel is buoyant. Consequently, it will continue to rise.

Exercise 3.2 *Show that the density perturbation of a parcel initially displaced downwards is positive and that it will continue to sink.*

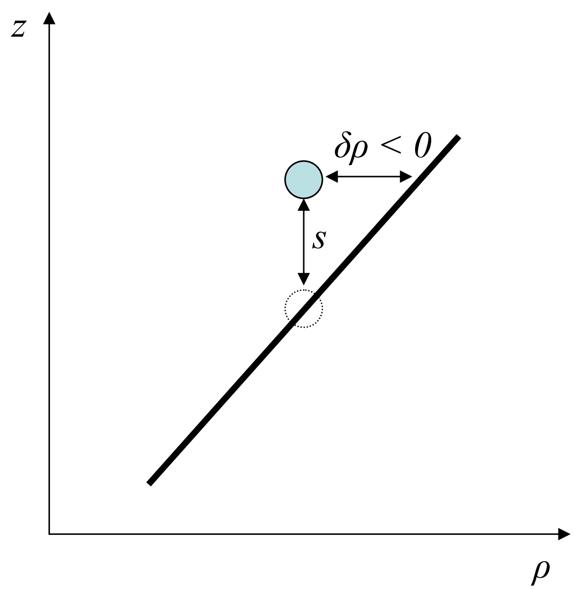


Figure 3.5: A schematic showing the density perturbation for a small parcel raised a distance s from its equilibrium position.

Exercise 3.3 *A stable linear density gradient is easily created in the laboratory. To fill a tank of volume V, fill two identical buckets each to a volume $\frac{1}{2}V$. One bucket B_1 contains water of the lowest density ρ_1 required and the other B_2 contains fluid with density ρ_2, the highest. The two buckets are connected by a pipe at the bottom so that the level in both buckets is always the same.*

The tank is filled by withdrawing fluid from bucket B_2, and this fluid is added to the tank via a floating sponge to reduce mixing. B_2 is continually stirred so that it is well mixed. As fluid is withdrawn, fluid from B_1 flows into B_2 so that the fluid levels in each are the same.

Set up an equation for the density ρ_2 as a function of the volume of fluid in the tank, and show that, for a tank of constant horizontal cross-section, this implies that the density decreases linearly with height.

Carry out this procedure in the laboratory, with $\rho_1 = 1000 kgm^{-3}$ and $\rho_2 = 1100 kgm^{-3}$, and a depth of about 0.3m. Calculate the buoyancy frequency N. Observe the motion of a neutrally buoyant sphere (ping pong ball) released from its equilibrium position. What is the frequency of the oscillations? What causes the motion to be damped so quickly?

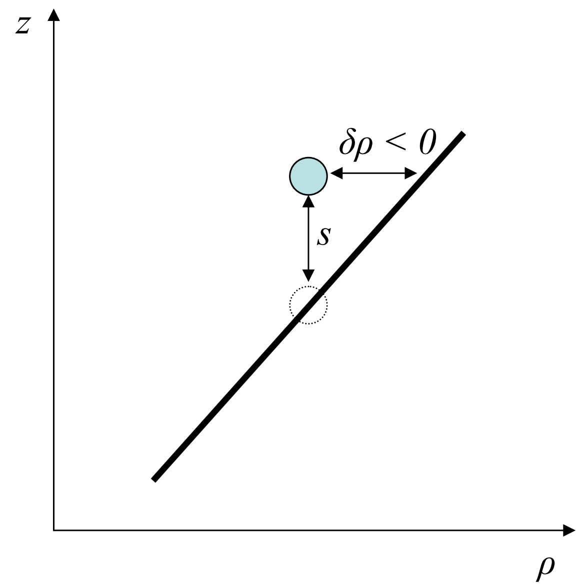


Figure 3.6: A schematic showing the density perturbation for a small parcel raised a distance s from its equilibrium position.

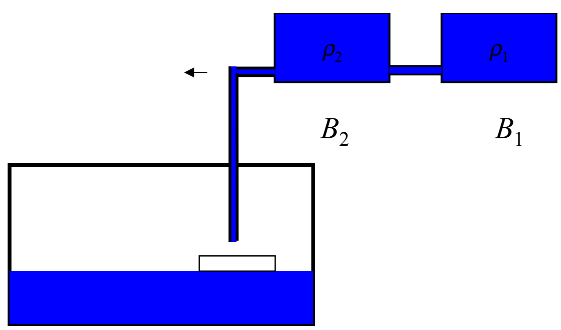


Figure 3.7: The 'double-bucket' method for the production of a linear density gradient.

3.4 The Boussinesq approximation

The Boussinesq approximation recognizes the fact that in many natural and industrial flows the variations in density are small. In a perfect gas differences in density are caused by differences in temperature. If fluid of density ρ and temperature T Kelvin is heated so that its temperature rises by an amount ΔT, its density changes by an amount $\Delta\rho$, where (see Problem 2.2)

$$\frac{\Delta\rho}{\rho} = -\frac{\Delta T}{T}. \tag{3.10}$$

In many cases $\Delta T << T$, so that $\Delta\rho << \rho$. For example, differences between the internal and external temperatures of a building seldom exceed 20K. In that case, the variation in density is less than 10%. Similar small density differences occur in other atmospheric and oceanic flows. On the other hand large density differences can occur. One example is in a fire where temperatures can easily exceed 1000K and the density of the heated gases is much less than the ambient air.

The Navier-Stokes equations (2.26), which express the conservation of momentum in a fluid, naturally involve the fluid density ρ. On the left hand side of (2.26) density appears in the inertia of the fluid. On the right hand side it occurs in both the viscous and body force terms. The Boussinesq approximation takes the density as constant in both the inertia and viscous terms but allows for variations in density in the body force (gravity) term.

Physically, it is clear what this approximation implies. First, by allowing variations in density associated with gravity, it means that buoyancy forces between fluids of different densities are included in the equations. Thus less dense fluid is subjected to forces which make it rise while heavy fluid sinks. These density variations are crucial to the study of buoyancy driven flows. If they are ignored the gravitational field plays no role in the fluid motion.

Second, and most important, variations in the fluid inertia associated with density differences are ignored. So in a Boussinesq flow the inertia of the flow is assumed to be the same whether it is a few percent denser than the surrounding fluid or a few percent lighter.

Thirdly, by ignoring the change in viscosity with density it is assumed that variations in fluid properties with density are small and can be ignored.

A full mathematical derivation of the Boussinesq is complex and the interested reader is referred to Spiegel & Veronis (1960), who discuss in detail the implications for fluids including the effects of viscosity and diffusion. Here we restrict ourselves to an analysis of an incompressible and inviscid fluid.

Suppose that the fluid has a density $\rho(\mathbf{x}, t)$, which may be written as the sum of a *constant* density ρ_0 and a perturbation $\rho^*(\mathbf{x}, t)$, such that

$$\rho(\mathbf{x}, t) = \rho_0 + \rho^*(\mathbf{x}, t), \tag{3.11}$$

where $\rho^*(\mathbf{x}, t) << \rho_0$.

As discussed at the start of the chapter, when a fluid is at rest in a gravitational field, lines of constant density are horizontal (normal to the gravity field – see (3.1)). Then the density may be written as

$$\rho^*(\mathbf{x},t) = \overline{\rho}(z) + \rho'(\mathbf{x},t), \tag{3.12}$$

where z is the vertical (antiparallel to gravity) coordinate, $\overline{\rho}$ is the background vertical density variation and $\rho'(\mathbf{x},t)$ is the density perturbation due to fluid motion.

The corresponding pressure field is

$$p(\mathbf{x},t) = \overline{p}(z) + p'(\mathbf{x},t), \tag{3.13}$$

the corresponding hydrostatic density field $\overline{p}(z)$ is given by

$$\frac{d\overline{p}}{dz} = -g\overline{\rho}, \tag{3.14}$$

In addition the pressure consists of a linear variation $-g\rho_0 z$ associated with the hydrostatic component due to the (uniform) reference density and $p'(\mathbf{x},t)$ is the pressure perturbation corresponding to the fluid motion.

Substitution of (3.12) and (3.13) into (2.26) and division by the density gives

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0 + \overline{\rho} + \rho'}\nabla p + \mathbf{g} + \nu\nabla^2\mathbf{u}. \tag{3.15}$$

Now subtract the hydrostatic pressure field using (3.14) and obtain

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0 + \overline{\rho} + \rho'}\nabla p' + \mathbf{g}\frac{\rho'}{\rho_0 + \overline{\rho} + \rho'} + \nu\nabla^2\mathbf{u}. \tag{3.16}$$

The Boussinesq approximation assumes that all density variations are small compared to ρ_0 so that $\overline{\rho} + \rho' << \rho_0$, but the limit $g\frac{\rho'}{\rho_0} = g'$ is finite. Then the Boussinesq equations become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla p' + \mathbf{g}' + \nu\nabla^2\mathbf{u}. \tag{3.17}$$

The buoyancy force enters the Boussinesq equations (3.17) through the reduced gravity $\mathbf{g}'$, defined by

$$g' \equiv g\frac{\rho'}{\rho_0}. \tag{3.18}$$

For the two layer flow shown in figure 3.8, the reduced gravity is given

$$g' = g\frac{\rho_2 - \rho_1}{\rho_2}. \tag{3.19}$$

For a Boussinesq fluid, the choice of either ρ_1 or ρ_2 in the denominator of (3.19). However, for non-Boussinesq flows the choice of the density in the denominator is important.

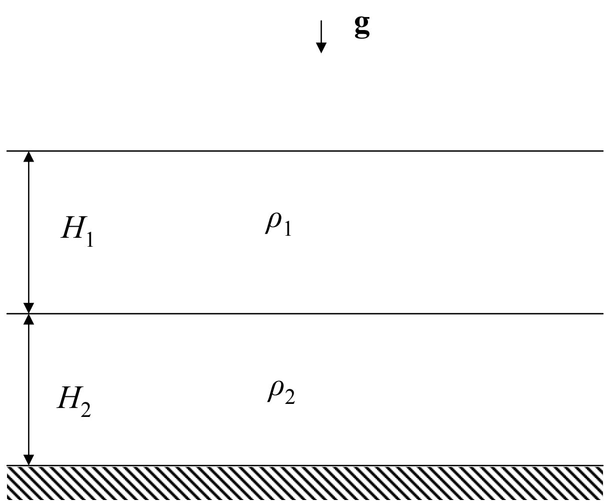


Figure 3.8: A two layer stratification.

3.5 Baroclinic vorticity

We now consider the consequences of density variations within the fluid. Noting that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \times \boldsymbol{\omega} - \frac{1}{2} \nabla |\mathbf{u}|^2 , \tag{3.20}$$

the curl of (2.26) yields the vorticity equation for $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ in the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \frac{1}{\rho^2} \nabla \rho \times \nabla p + \nu \nabla^2 \boldsymbol{\omega}. \tag{3.21}$$

The effect of buoyancy is shown in the middle term on the left-hand-side of (3.21),

$$\Gamma \equiv \frac{1}{\rho^2} \nabla \rho \times \nabla p. \tag{3.22}$$

This term is non-zero whenever surfaces of constant pressure and density are non-parallel. In a stationary fluid under gravity the pressure is hydrostatic and constant pressure surfaces are horizontal. Hence, as discussed in § **??**, if the density field contains horizontal variations, vorticity will be generated, and flow will occur. When the background density field varies only in the

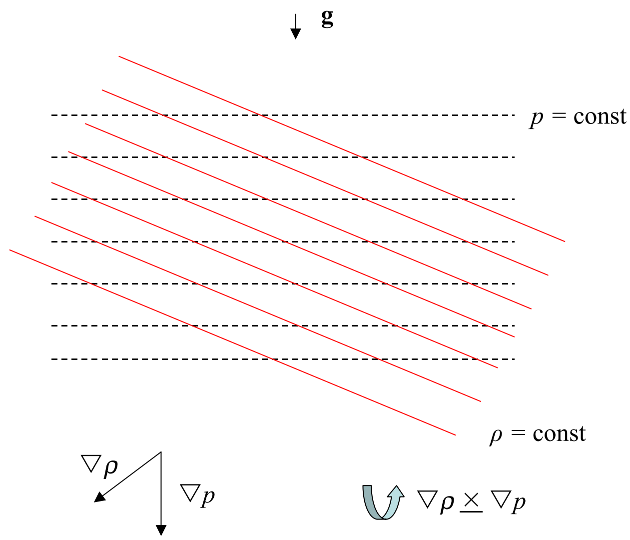


Figure 3.9: The baroclinic torque generated when lines of constant density (isopycnals) are at an angle to lines of constant pressure (isobars).

vertical and is such that the density increases in the direction of gravity, buoyancy forces provide a restoring force and damped oscillations (internal gravity waves) occur. Such a density field is said to be *statically stable* . On the other hand, if the density decreases in the direction of gravity the motion is amplified and convection ensues. Such a stratification is said to be *statically unstable*.

The term $\frac{1}{\rho^2}\nabla\rho\times\nabla p$ is zero if $p = p(\rho)$ and such a fluid is called 'barotropic' . A trivial example is an unstratified fluid with constant density. If $\frac{1}{\rho^2}\nabla\rho\times\nabla p \neq 0$ the fluid is called 'baroclinic' and the flow results from the baroclinic generation of vorticity. In non-stationary fluids, such as those in a rotating frame of reference, surfaces of constant pressure are not necessarily perpendicular to gravity, and so the terminology has a more general usage.

An important feature of Γ is that *whenever* it is non-zero flow is generated. This is the principle behind placing heaters around the perimeter of a room. Since the temperature and, therefore, the density varies in the horizontal, vorticity is generated (figure 3.5).

Problem 3.1 *Solve the differential equation (3.5) subject to the initial conditions that the parcel is released from rest at a height s_0 at $t = 0$.*

Problem 3.2 *Calculate N in a room assuming that the temperature increases linearly from 18^{o}C at the floor to 22^{o}C at the ceiling. The floor to ceiling height is 3m. Calculate the period of internal gravity waves in the room.*

Problem 3.3 *A revolving door is fitted to the room in Exercise 3.2. Calculate the minimum rotation rate of the door before the disturbance in the room is affected by the interior stratification. Is this limit likely to be achieved in practice?*

Problem 3.4 *Show that under the Boussinesq approximation the buoyancy frequency is given by*

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz}. \tag{3.23}$$

Consider an exponential density profile $\rho(z) = \rho_0 e^{-\beta z}$, where $\beta > 0$ is a constant. Show that N as defined by (3.6) is a constant. By considering a linear approximation to the exponential profile, discuss the implications on the height over which motion can occur within the validity of the Boussinesq approximation.