UNIVERSITY OF CALIFORNIA SAN DIEGO

Stochastic Analysis of Fluid Flows in Domains, whose Rough Surfaces are modeled as Random Fields

A dissertation submitted in partial satisfaction of the
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in

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by

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The Dissertation of Yawo Semanu Ezunkpe is approved and is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To my family.

EPIGRAPH

As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Albert Einstein

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ABSTRACT OF THE DISSERTATION

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This dissertation deals with flows impacted by wall roughness and/or uncertain flow-domain geometry. Specifically, it focuses on stochastic analysis of fluid flows in domains whose rough surfaces are modeled as random fields. More broadly, this work addresses some of the unresolved theoretical and practical questions concerning differential equations defined on random domains. It has significant impact on geophysical and biological flows, and can be extended to other areas where surface roughness affects fluid flows, such as nanoscale devices.

In the first part of this thesis, we present the background and the mathematical tools used in our study. They were presented in a manner to help engineers, engineering students and

practitionners to grasp the concepts.

The second part of this work discusses the stochastic modeling of Stokes flow in a channel with rough walls. The adopted approach consists of regarding the rough surface as a random field characterized by its statistical moments, a mapping of the stochastic domain of definition onto a deterministic domain, and stochastic homogenization of the resulting differential equations with random coefficients. This enables one to obtain closed-form expressions for the effective or apparent viscosity in terms of the statistical moments characterizing the wall roughness to fluid viscosity, and the Poiseuille number. The most important consequence of this analysis is a rigorous explanation of why Stokes flows drastically change their behavior depending on whether the flow takes place in a micro or macro channel. The results were validated using Comsol multiphysics software to simulate flow through domain bounded at the top by smooth wall and at the bottom by a sinusoidal wall with various amplitudes and different periods.

The third part deals with the application of the proposed approach to technology and life science. In technology, we investigate the impacts of the roughness of the boundary surfaces on the average flow thermal properties.

In the fourth and final part, we dicuss our findings and describe the future direction of this work. This elucidates the mechanical effects that take place at the stochastic solid/fluid interface in biological systems (blood/endothelial lining). This result has important implications for biology, physiology and medicine, and micro/nano technology because these interfaces are incredibly complex and difficult to quantify deterministically.

Chapter 1

Introduction

The word “perfection” does not belong to this world and the idea that a model can be built to describe perfectly or to explain with 100% accuracy any physical phenomenon is not realistic. It is at least good to make an approximation or to predict within a margin of error working model. We should not dig far into the litterature to find a corroborative statement from Albert Einstein when he addressed the ”Pruissian Academy of Sciences in January of 1921. “ *As far as the laws of mathematics refer to reality, they are not certain; and as far as they refer they are certain, they do not refer to relality*.” It is therefore clear that using Stochastic analysis would be the appropriate way to build model and it is trivial the title of this thesis. Fluid mechanics in the course of its several decades of evolution, have focused attentively on the flow field characterization upon the velocity fields and the dynamic pressure for a specific fluid and for a given geometry. The majority of thermo-fluid problems in mainly engineering fields have encountered major difficulties because of the complexity of the geometries of the flow domains. Despite the availability of the technology, recents models and approaches that evolve over the time still to be improved. The main difficulty always resides in the shapes of the fluid/solid interface which often, lack a symmetry. Moreover, some of these problems are time dependent and higher dimension problems make them even harder to deal with, see more complicated.

The irregulariry of the surface texture known as roughness was considered as a defection which occurs on a surface of material. Often considered as a result of an accident it is therfore assumed to cause an imperfection of the material. Recently, it could be intentionally introduced. Wall roughness affects a number of physical, biological, and chemical phenomena. In the natural (macro-scale) environments, the surfaces of rock-fractures are highly irregular and non-smooth, therefore impacting both the extraction of natural resources (such oil, gas, and water), and the propagation of solute plumes[**?**, see, e.g.]brown1987fluid,brown1989transport, zimmerman1991lubrication. At the micro-scale, with the miniaturization of traditional devices in electro-mechanical systems, the roughness of the surfaces has an important role for the design [**?**]. Similarly, in computer technologies the hard drive head is suspended within a rough domain, and concurrently dealing with rough surfaces becomes of paramount importance [**?**, **?**]. In bio-engineering, the effects of the endothelial tissues as determined by the irregularity of the surfaces of the vessels can greatly alter the blood flows [**?**]. Such a state of affair notwithstanding, only recently researchers have become more and more interested into studying the influence of rough walls upon laminar flow. The main reason is that in channels of conventional size (the relative roughness's height is less or equal than 5%) the impact of the roughness could be neglected [**?**, see e.g.]]moody1944friction,webb1994principl. However, this is not the case in numerous situations of practical interest, where instead a correlation between the surface texture and the flow variables has been clearly highlighted [**?**, see, e.g.]]brown1987fluid, brown1989transport, kandlikar2005characterization.

The main difficulty related to modeling of Stokes flow in rough channels is about the proper conceptualization of the geometry of the roughness. One approach consists into representing the surface by simple geometrical shapes (e.g. sinusoidal, sawtooth, and cell). However, such a methodology results in most of the cases too simplistic, and incapable to mimic the complex structure of the roughness [**?**, **?**, **?**, **?**, **?**]. Alternatively, the fractal approach has been proposed by [**?**]. Nevertheless, all these studies have highlighted that the pressure drop is largely influenced by the roughness of the wall, and a few perturbation approaches have been used to come up with a better characterization of the flow field [**?**, see, e.g.]]plouraboue2004conductances,Tavakol2017ExtendedLT.

Likewise, the Poiseuille number Po is also significantly affected by the wall's roughness. In particular, it was shown that Po may increase up to 2–3 times the Poiseuille number in case of smooth surface [**?**, see]and references therein]kandlikar2005characterization. Another parameter that is strongly influenced by the roughness of the wall is the friction factor [**?**]. However, the above well defined geometrical shapes are not adequate to mimic the increase of the friction factor due to the lack of available data. Finally, measured values are biased by experimental errors, therefore rendering the roughness uncertain, and concurrently the corresponding Stokes flow equations stochastic [**?**, **?**, **?**, **?**].

In Chapter **??**, we lay out some important concepts in mathematics definitions and properties especially for vector space, tensors calculus,and random variables. The third chapter **??** deals with the study of the impact of wall roughness on the average behaviour Stokes flow in channel with walls treated as random fields. This is followed by application in technology where we invistigate the heat transfer in domain bounded by rough walls in chapter **??**. And finally in the conclusion **??**, we present our recommendation and the future direction of this work.

Chapter 2

Mathematical Review

2.1 Objective

During a faculty meeting at Yale university the famous Mathematician Josiah Willard Gibbs was asked about the importance of mathematics in undergrad curriculum. He responded with four words: "Mathematics is a language". We stretch his view in here to give the priliminaries on properties of vectors and tensors which are some of the specific languges often spoken in mechanics of continuum medium. While we are aware of the plethora of books, where these two concepts are well-detailled, we must admit that based on personal expreience, that there are several reasons to give an a short overview in this section. Most importantly, it is beleived that most of the mathematics textbooks cover these concepts in a way that they are unlikely to be in the firm grasp of the majority of practitioners and engineering students or other technology majors.

Our hope is to lay out the basic these concepts in a way to

- make it easier for the reader to understand the foregoing material without any mathematical hindered,
- help refresh the readers on the concepts and notations,
- avoid a confusion on the notation

2.2 Vectors and Vector Space

In mechanics of continium medium such fluid mechanics, the manipulation of vectors is of great importance. Vectors can defined as quantities with magnitude, direction and has an origin. Elegantly, we can define a vector as directed segment. Vectors are governed by rules. Lets consider a set of vectors $\mathscr{E}$ associated with two operations addition $(+)$ and multiplication $(\cdot)$. Let assume that $\mathscr{E}$ is closed under the addition and multiplication by a scalar. i.e if $\mathbf{x},\mathbf{y} \in \mathscr{E}$, $\mathbf{x}+\mathbf{y} \in \mathscr{E}$ and for $\alpha \in \mathbb{R}, \mathbf{y} \in \mathscr{E}$, $\alpha \cdot \mathbf{x} \in \mathscr{E}$. $\mathscr{E}$ is said to constitute a vector space over the field of the real number $\mathbb{R}$ if $\mathscr{E}$ of vector elements in arbitrary order $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$... is such that the following properties for addition in $\mathbb{R}$ and multiplication by scalar are satisfied:

$$\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x} \quad \forall \mathbf{x},\mathbf{y} \in \mathscr{E} \quad \text{commutativity of elelments} \tag{2.1}$$

$$\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z} \quad \forall \mathbf{x},\mathbf{y},\mathbf{z} \in \mathscr{E} \quad \text{associativity of elements} \tag{2.2}$$

$$\mathbf{x}+\mathbf{0} = \mathbf{0}+\mathbf{x} = \mathbf{x} \quad \text{identity element of addition zero vector } \mathbf{0} \tag{2.3}$$

$$\forall \mathbf{x} \in \mathscr{E}, \quad \exists -\mathbf{x} \in \mathscr{E} \quad \text{such that} \quad \mathbf{x}+(-\mathbf{x}) = \mathbf{0} \quad \text{inverse element of addition} \tag{2.4}$$

$$\exists 1 \in \mathbb{R} \quad \text{s.t} \quad \forall \mathbf{x} \in \mathscr{E}, \quad \mathbf{1}\cdot\mathbf{x} = \mathbf{x} \quad \text{identity of multiplication} \tag{2.5}$$

$$\forall \alpha,\beta \in \mathbb{R}, \quad \forall \mathbf{x} \in \mathscr{E} \quad (\alpha+\beta)\cdot\mathbf{x} = \alpha\cdot\mathbf{x}+\beta\cdot\mathbf{x} \quad \text{distributivity + onto } \cdot \tag{2.6}$$

$$\forall \alpha \in \mathbb{R}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathscr{E} \quad \alpha \cdot (\mathbf{x}+\mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y} \quad \text{distributivity of multiplication onto addition} \tag{2.7}$$

$$\forall \alpha, \beta \in \mathbb{R}, \quad \forall \mathbf{x} \in \mathscr{E} \quad \alpha(\beta \cdot \mathbf{x}) = (\alpha\beta) \cdot \mathbf{x} \quad \text{associativity of scalar elements} \tag{2.8}$$

The element $\mathbf{0}$ is called *null vector*. By the mean of these eight properties, we defined the concept of the vector space. We might give an example but the goal here is to refresh the readers minds. However if we define $\mathbf{x}(x_1, x_2, x_3, ...x_n)$ and $\mathbf{y}$ $(y_1, y_2, y_3, ...y_n)$ in n-dimensional space, we have $\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, x_3+y_3, ..., x_n+y_n)$ and $\alpha \cdot \mathbf{x} = (\alpha \cdot x_1, \alpha \cdot x_2, \alpha \cdot x_3, ...\alpha \cdot x_n)$

Let us insert here the definition of the linear independence and a dimension of a space vector. Consider n vectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, ...\mathbf{x_n}$ in a vector $\mathscr{E}$. If $\alpha_{\mathrm{i}}\mathbf{x_i} = \mathbf{0}$ (Einstein notation adopted here over n) implies that $\alpha_i = 0$ for all $i = 1, 2, 3.....n$, then $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, ...\mathbf{x_n}$ are said to be linearly independent. One can say that the dimension of $\mathscr{E}$ is n and we note $\mathscr{E}_\mathbf{n}$

2.2.1 Basis

For an arbitrary n-dimensional vector $\mathbf{u} \in \mathscr{E}_\mathbf{n}$ suppose that there exists linearly independent set of n vectors $\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}, \varepsilon_\mathbf{3}, ..., \varepsilon_\mathbf{n}$. One can therefore xpress in vertue of the definition of the dimension of vector space that

$$\mathbf{u} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \alpha_{\mathrm{i}}\varepsilon_\mathbf{i} \quad \text{or} \quad \mathbf{u} = \alpha_{\mathrm{i}}\varepsilon_\mathbf{i} \quad \text{Einstein notation} \tag{2.9}$$

. Any set of linearly independent vectors $(\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}, \varepsilon_\mathbf{3}, ...\varepsilon_\mathbf{n})$ constitutes a basis in a vector space $\mathscr{E}_\mathbf{n}$. It follows that any vector in the vector space $\mathscr{E}_\mathbf{n}$ can be expressed as unique linear combination the basis vectors. The more practical and familar example is the cartesian coordinate with basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

2.2.2 Scalar Multiplication of vectors

In this section, we are introducing the notion of scalar multiplication or dot product known also as inner product.This operation is mapped as follow:

$$\mathscr{E} \times \mathscr{E} \longrightarrow \mathbb{R}$$
$$(\mathbf{x},\mathbf{y}) \mapsto \mathbf{x}\cdot\mathbf{y} \tag{2.10}$$

and can be interpreted as given two vectors **x** and **y** in the vector space, the dot product is a scalar (we often write))$\mathbf{x}\cdot\mathbf{y} \in \mathbb{R}$. It has the folowing properties:

1. $\forall \mathbf{x},\mathbf{y} \in \mathscr{E}, \mathbf{x}\cdot\mathbf{y} = \mathbf{y}\cdot\mathbf{x}$
2. $\forall \mathbf{x},\mathbf{y},\mathbf{z} \in \mathscr{E}, \forall \alpha \in \mathbb{R}, (\alpha\mathbf{x})\cdot(\mathbf{y}+\mathbf{z}) = \mathbf{x}\cdot(\alpha\mathbf{y}) + \alpha(\mathbf{x}+\mathbf{z})$
3. $\forall$**x**, $\mathbf{x}\cdot\mathbf{x}$**=** x^2 where $x^2 \geq 0$

In vertue of these aforementioned properties, we can write that with the coordinate basis vectors ε_i that $\mathbf{x}\cdot\mathbf{y} = \mathbf{x^i y^j \varepsilon_j \varepsilon_i}$. We can also relate dot product concept to the magnitude or Euclidean norm of a vector as $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

2.2.3 Orthogonality and Orthonormality

Two vectors are said to be orthogonal if their dot product is equal to zero. This means in mathematical notation that $\mathbf{x}\cdot\mathbf{y} = \mathbf{0}\ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}$. This could also mean that the two vectors are perpendicular if we define the cosine of the angle between the two vectors to be the ratio of the dot product to the product of their magnitude, i.e. $\cos\theta = (\mathbf{x}\cdot\mathbf{y})/(\|\mathbf{x}\|\|\mathbf{y}\|)$ which will be equal to zero. If in addition to the fact that the dot product between the two vectors are zero, we have $\|\mathbf{x}\| = \mathbf{1} = \|\mathbf{y}\|$ then $\mathbf{x},\mathbf{y}$ are said to be orthonormal. In more generalize way one, a basis $(\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}, ..., \varepsilon_\mathbf{n})$ in *mathcalE* is an orthonormal basis if $\varepsilon_\mathbf{i}\cdot\varepsilon_\mathbf{j} = \delta_\mathbf{ij}$, where δ_{ij} is the Kronecker delta defined as follow $i = j \Rightarrow \delta_{ij} = 1$ and 0 otherwise.

2.2.4 Contravariant and Covariant Components of vectors:(Reciprocal of a Basis)

Let assume that the vector **u** is an arbitrary vector in the vector space $\mathscr{E}_{\mathbf{n}}$. Recall $(\varepsilon_{\mathbf{1}}, \varepsilon_{\mathbf{2}}, ..., \varepsilon_{\mathbf{n}})$ and arbitrary basis in $\mathscr{E}_{\mathbf{n}}$. **u** is said to have a contravariant component if there exists a set of number α^i such that $\mathbf{u} = \alpha^{\mathrm{i}}\varepsilon_{\mathbf{i}}$. If there exists another basis $\varepsilon^{\mathbf{1}}, \varepsilon^{\mathbf{2}}, ..., \varepsilon^{\mathbf{n}}$ of $\mathscr{E}_n$ such that **u** can be written as linear combination of these basis vectors , i.e $\exists \alpha^i \in \mathbb{R}$ such that $\mathbf{u} = \alpha_{\mathbf{i}}\varepsilon^{\mathbf{i}}$, then α^i and α_i are referred to as contravariant and covariant components of the vector **u** respectively. It results that $\varepsilon_i \cdot \varepsilon^j = \delta_i^j$. One can easily obtain this result from $\mathbf{u} = \alpha^{\mathbf{i}}\varepsilon_{\mathbf{i}}$ and $\mathbf{u} = \alpha_{\mathbf{i}}\varepsilon^{\mathbf{i}}$. Forgoing, we will be using Einstein notation to indicate the summation over the repeated indices otherwise will be idicated. We also using $\{\varepsilon_i\}$ to denote the arbitrary basis vectors $\varepsilon_{\mathbf{1}}, \varepsilon_{\mathbf{2}}, ..., \varepsilon_{\mathbf{n}}$ and $\{\varepsilon^i\}$ to denote the set of vectors $\varepsilon^{\mathbf{1}}, \varepsilon^{\mathbf{2}}, ..., \varepsilon^{\mathbf{n}}$ uniquely defined in terms of $\{\varepsilon_i\}$.

2.3 Tensors

The concept of tensor is one of the important tools in mechanic of contnuum medium. Since we have used intensively tensorial notation in the work presented in the future sections, we judge it meaningful to shed light on some of the terms and properties used. The notion of tensor is sometime mixed up with the notion of matrix, therefore we present the concept of tensor from vectors and vector space standpoint to allow the engineers and practitioners who will be reading this work. Suppose that there exists $r, s \in \mathbb{N}$ and a vector space with dimenstion r$\times$s. We can define a tensor as a product of two vector spaces $\mathscr{E}_r$ and $\mathscr{E}_s$ of dimensions r and s respectively which we denote by $\mathscr{E}_r \otimes \mathscr{E}_s$. In simple language, Tensors could be seen as higher–order vectors with different components that depend upon their orders and the dimension of their space. We limit our work to tensor description in Euclidean space since that is where most fluid mechanics problems take place.

2.3.1 Tensor Algebra

Consider A^i_{jk}, B^i_{jk} and C^i_{jk} three tensors defined in Euclidean space. It is shown that the sum of two tensors is also a tensor and that is

$$C^i_{jk} = A^i_{jk} + B^i_{jk}. \tag{2.11}$$

It is also shown that the product of two tensors is a tensor with higher order. For given P^i_{jkl} there exists two variants A^i_j and B_{kl} such that

$$P^i_{jkl} = A^i_j B_{kl}. \tag{2.12}$$

2.3.2 Covariant and Contravariant metric Tensors

Consider the covariant basis $\{\varepsilon_{\mathbf{i}}\}$ and the contravariant basis $\{\varepsilon^{\mathbf{i}}\}$ defined in the previous section. The covariant metric g_{ij} is defined as the pairwise dot product of the covariant basis i.e. $g_{ij} = \varepsilon_{\mathbf{i}} \cdot \varepsilon_{\mathbf{j}}$ and it plays a central role in tensor calculus. The reader will see in the forgoing work how intensively this concept is used. In other literatures, it referred to as fundamental tensor. it also possesses the property of symmetry that is $g_{ij} = g_{ji}$ given the fact that the dot product is comutative. it is an indication of measurement such as quatifying lenghts, areas and volumes as one can see in the term " Metric". For example, if given two arbitrary vectors **u** and **v**, the dot product of these two vectors can be expressed in term of the covariant metric tensor that is

$$\mathbf{u} \cdot \mathbf{v} = \mathrm{g_{ij} u^i v^j} \quad \text{and} \quad \|\mathbf{u}\| = \sqrt{\mathrm{g_{ij} u^i u^j}}. \tag{2.13}$$

The contravariant metric tensor (often referred to as metric tensor) is the conjugate metric tensor or inverse metric tensor of the covariant metric tensor. It can be written as $g^{ij} = \varepsilon^i \cdot \varepsilon^{\mathbf{j}}$ or $g^{ik} g_{kj} = \delta^i_j$. Itis trivial to see that this is basically the inverse matrix of the $\{g_{ij}\}$. As it counterpart covariant metric tensor, Contravariant metric tensor g^{ij} is symmetric i.e. $(g^{ij} = g^{ji}$

for all $i, j \in \mathbb{N}$. It can be shown that covariant basis view as matrix form is a positive definite matrix. It also can be proven that $\{g^{ij}\}$ is positive definite.

2.3.3 Raising and Lowering of Indices

The raising and lowering of indices is a very important *juggling mind exercise* in tensor calculus. generally speaking, since it is possible to write any basis vector as linear combination of another basis vectors over the same space, we have

$$\varepsilon^{\mathbf{i}} = \mathrm{g}^{\mathrm{ij}} \varepsilon_{\mathbf{j}} \tag{2.14}$$

. In general manner, one can write

$$T^i = T_j g^{ji}, \tag{2.15}$$

and the raising of the index is basically the contraction of an arbitrary tensor with the contravariant metric tensor. similarly, the lowering of index is a contraction of a variant with the covariant metric tensor.That is,

$$\varepsilon_{\mathbf{i}} = \mathrm{g}_{\mathrm{ij}} \varepsilon^{\mathbf{j}}, \tag{2.16}$$

which in general way writes

$$T_i = T^j g_{ji}. \tag{2.17}$$

Here we give a scenario where for a tensor A^i_j, the upper index is lowered and the lower index is raised. we have

$$A^k_l = A^i_j g_{il} g^{jk} \tag{2.18}$$

2.3.4 Christoffel symbols and Covariant Derivative

It is worthy to point out that Christoffel symbol in any guesses is not a tensor. It captures something very fundamental about the coordinates systems and how they are defined in space. it captures the rate of change of the metric tensor in a way that it is always referred to as connecting element. if we denote by ξ^i the curvilinear coordinate components, the variation of the basis from one point to another is given by the $\partial\varepsilon_\mathbf{i}/\partial\xi^\mathbf{j}$. Since $\varepsilon_\mathbf{i}$ depend upon position, the partial derivative can be expressed in terms of the set of the basis vectors and a set of other quantities called Christoffel symbols of second kind Γ^k_{ij} which results in dim^3 coefficients.exemple if we are dealing with two dimensional space, we will have $2^3 = 8$ coefficients and if it is in dimension three, then we will have $3^3 = 27$ coefficients. We then have

$$\frac{\partial \varepsilon_\mathbf{i}}{\partial \xi^j} = \Gamma^k_{ij}\varepsilon_\mathbf{k} \tag{2.19}$$

. If we dot both sides of this expression by $\varepsilon^\mathbf{k}$, we obtain that

$$\Gamma^k_{ij} = \frac{\partial \varepsilon_\mathbf{i}}{\partial \xi^j}\varepsilon^\mathbf{k}. \tag{2.20}$$

Using the product rule, one can can show that

$$\Gamma^k_{ij} = \frac{\partial(\varepsilon^\mathbf{k}\cdot\varepsilon_\mathbf{i})}{\partial\xi_\mathrm{j}} - \frac{\partial\varepsilon^\mathbf{k}}{\partial\xi_\mathrm{j}}\cdot\varepsilon_\mathbf{i} \tag{2.21}$$

Therefore it is trivial to note that

$$\Gamma^k_{ij} = -\frac{\partial\varepsilon^\mathbf{k}}{\partial\xi^j}\cdot\varepsilon_\mathbf{i} \tag{2.22}$$

An alternative and useful expression of the Christoffel symbols of second kind is

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(\frac{\partial g_{mi}}{\partial\xi_j} + \frac{\partial g_{mj}}{\partial\xi_i} - \frac{\partial g_{ij}}{\partial\xi_m}\right). \tag{2.23}$$

2.4 Random Variables

One cannot talk about stochastic analysis without talking about the basic concepts involved in the study of this process. Therefore it is important to define some frequently used concepts and give some useful properties. Starting with the Random event, we can asy that a random event is an outcome of an experiment of observation, a random variable is a quantity taking on different values depending on the observations. However when the random functions is merely time dependent, $U(t)$ is said to be random process and random field if $U(\mathbf{x})$ is solely space- dependent. For example, erradic behavior of surfaces or boundaries causing fluctuation in the velocity, temperature and pressure in a flow field domain. A random function is a function $U(\mathbf{x},\mathrm{t})$ of space $\mathbf{x}$ and or of time t that varies randomly with $(\mathbf{x},\mathrm{t})$. Another important concept is a Statistical Ensemble of a random function can be seen as the set of all possible realizations $u(\mathbf{x},\mathrm{t})$ of $U(\mathbf{x})$. However it is noted here that any realization could be a subset of more than one ensemble. More importantly, the moments of random functions are the useful tool to characterize the random variable. Among the frequently used moments one can note the mean or mathematical expectation which is the first moment, the variance and autocovariance which are the second order moment.

$$\langle U(\mathbf{x})\rangle = \int \mathrm{u}p(\mathrm{u};\mathbf{x})\mathrm{du} \tag{2.24}$$

The autovariance is defined by:

$$C_U(\mathbf{x_1},\mathbf{x_2}) = \langle \mathrm{U}'(\mathbf{x_1})\mathrm{U}'(\mathbf{x_2})\rangle = \iint \mathrm{u_1u_2}p(\mathrm{u_1},\mathrm{u_2})\mathrm{du_1du_2} - \langle \mathrm{u_1}\rangle\langle \mathrm{u_2}\rangle. \tag{2.25}$$

The variance is also defined by:

$$\sigma_U^2(\mathbf{x}) = \mathbf{C_U}(\mathbf{x},\mathbf{x}) = \int \mathrm{u}^2 p(\mathrm{u};\mathbf{x})\mathrm{du} - \langle \mathrm{u}(\mathbf{x})\rangle^{\mathbf{2}} \tag{2.26}$$

The autocorrelation coefficient can be written as:

$$\rho_U(\mathbf{x_1},\mathbf{x_2}) = \frac{\mathrm{C_U}(\mathbf{x_1},\mathbf{x_2})}{\sigma_\mathrm{U}(\mathbf{x_1})\sigma_\mathbf{U}(\mathbf{x_2})} \tag{2.27}$$

Moreover, a given spatial random function is said to be stationary or homogeneous if given $\mathbf{x}$ (a space coordinates), $U(\mathbf{x})$ is insensitive to the vectors shift. Also said to be second order stationary if the following three properties are valid.

$$\langle U(\mathbf{x})\rangle = \langle \mathbf{U} \rangle = \text{constant} \tag{2.28}$$

$$C_U(\mathbf{x}, \mathbf{x}+\mathbf{r}) = \mathbf{C_U}(\mathbf{r}) = \mathbf{C_U}(-\mathbf{r}) \implies \text{symmetry} \tag{2.29}$$

$$\sigma_U^2(\mathbf{x}) = \mathbf{C_U}(\mathbf{0}) = \text{constant} \tag{2.30}$$

which, in another words means that the covariance does not depend on the individual position in space but depend upon the shift $\mathbf{r}$.

Chapter 3

Stokes Flow in channel with randomly varying rough walls

3.1 Objective

Surface roughness is a key property affecting fluid flow in bounded domains. Its effect on bulk flow is usually quantified by means of anempirical roughness coefficient which is introduced into models that treat bounding surfaces as smooth. We present a new approach, which treats the irregular geometry of rough walls as a random field, whose statistical properties (mean, standard deviation, and spatial correlation) are inferred from measurements. The subsequent stochastic mapping of a random flow domain onto its deterministic counterpart and stochastic homogenization of the transformed Stokes equations yield an expression for the roughness coefficient in terms of the wall's statistical parameters. The analytical nature of our solutions allows us to handle random surfaces with short correlations lengths, which cannot be treated by numerical stochastic simulations.

3.2 Review Stokes flow

We consider an infinitesimal fluid element to possess a well-defined density ρ, pressure p, and the velocity vector $\mathbf{u}(\mathbf{x},\mathrm{t})$ at a position $\mathbf{x}$ and at instant t. The substantial derivative D/Dt

is given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \text{or} \quad \frac{\partial}{\partial \mathrm{t}} + \mathrm{u_i}\frac{\partial}{\partial \mathrm{x_i}}. \tag{3.1}$$

3.2.1 Mass conservation

For the fluid particle considered, we assume it has a cubic form with sides dx_1, dx_2, dx_3 and volume element is $dx_1dx_2dx_3$. The rate of linear deformation in each of the direction of the coordinate system is

$$\frac{\partial u_1}{\partial x_1}dx_1, \quad \frac{\partial u_2}{\partial x_2}dx_2, \quad \frac{\partial u_3}{\partial x_3}dx_3. \tag{3.2}$$

By multiplying each of the linear deformation term by the area of by their perspective faces which summmed up to zero, that is

$$\frac{\partial u_1}{\partial x_1}dx_1(dx_2dx_3) + \frac{\partial u_2}{\partial x_2}dx_2(dx_1dx_3) + \frac{\partial u_3}{\partial x_3}dx_3(dx_1dx_2) = 0. \tag{3.3}$$

It is trivial that the elementary volume element $dx_1dx_2dx_3$ is not zero, therefore by dividing the latter equation by the volume element, one gets

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \equiv \mathrm{div}\mathbf{u} = 0, \tag{3.4}$$

which is the continuity equation (conservation of mass) of a fluid with a constant density. The Mass conservation states that the velocity cannot vary (increase or decrease) in all three direction at the same time together; thus

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \quad \text{or} \quad \frac{\partial \rho}{\partial \mathrm{t}} + \frac{\partial \mathrm{u_i}}{\partial \mathrm{x_i}} = 0. \tag{3.5}$$

which for an impressible fluid reduced to

$$\frac{D\rho}{Dt} = 0 \implies \nabla \cdot \mathbf{u} = \mathbf{0} \tag{3.6}$$

3.2.2 Equation of Motion

Assuming that the infinitesimal cubic fluid particle considered earlier has its origin coincides with the origin of the coordinate system and whose sides are parallel to the principal axis of the coordinate system. Let assume that there pressure force acting on it has different components according to the normal surface they are applied to p_1, p_2 and p_3. Eventually, since the fluid has a viscosity, one may consider the shear stress denoted τ and has six components τ_{12}, τ_{21}, τ_{31}, τ_{13}. τ_{32}, τ_{23} for which the first subscript denotes the normal of the plane it is acting on and the second the direction. When Newton's second law of motion is applied to the fluid elelment in a control volume under assumption of continuous medium, we can write that

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \tau + F \quad \text{or} \quad \rho \frac{\mathrm{Du_i}}{\mathrm{Dt}} = \frac{\partial \tau_{\mathrm{ij}}}{\partial \mathrm{x_j}} + \mathrm{F_i}, \tag{3.7}$$

where the substantial or material derivative known also as convective derivative is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \text{or} \quad \frac{\partial}{\partial \mathrm{t}} + \mathrm{u_i} \frac{\partial}{\partial \mathrm{x_i}} \tag{3.8}$$

In Newtonian fluid, the strain rate or shear rate tensor **e** is produced by the deformation of the fluid. It is the symmetric term of the velocity gradient tensor $\nabla\mathbf{u}$. The relationship between σ_{ij} and the deformation rate isinstantaneous, linear, local and isotropic and is given by the expression

$$\sigma = -p\mathbf{I} + \mathbf{2}\mu\mathbf{e} \tag{3.9}$$

$$\mathbf{e} = \frac{1}{2}\left[(\nabla\mathbf{u}) + (\nabla\mathbf{u})^{\top}\right] \quad \text{or} \quad \mathrm{e_{ij}} = \frac{1}{2}\left(\frac{\partial \mathrm{u_i}}{\partial \mathrm{x_j}} + \frac{\partial \mathrm{u_j}}{\partial \mathrm{x_i}}\right) \tag{3.10}$$

In the recent expression, we denote the transpose by $^{\top}$. One must deduce that for an incompressible fluid, the divergence free is applied and therefore, the trace of the rate of the deformation results in zero. This translates mathematically by

$$Tr(\mathbf{e}) = \mathbf{0} \implies \mathrm{e}_{11} + \mathrm{e}_{22} + \mathrm{e}_{33} = 0 \quad \text{or} \quad \mathrm{e}_{\mathrm{ii}} = 0, \quad \text{summation} \quad \text{applied} \tag{3.11}$$

From the Newton's constitutive relations, we can write that the shear stress tensor is proportional to the strain rate tensor and that is $\tau = 2\mu\mathbf{e}$, the fluid equation can be written as

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{3.12}$$

Realizing that in case of incompressibility, that the continuity equation leads to divergence free $\nabla \cdot \mathbf{u} = \mathbf{0}$, that is:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0. \tag{3.13}$$

In vertue of the mass conservation, we can write in contract form using ∇^2 in component form that:

$$\frac{Du_1}{Dt} = -\frac{\partial}{\rho\partial x_1}(p + \gamma h) + \nu\nabla^2 u_1 \tag{3.14}$$

$$\frac{Du_1}{Dt} = -\frac{\partial}{\rho\partial x_2}(p + \gamma h) + \nu\nabla^2 u_2 \tag{3.15}$$

$$\frac{Du_1}{Dt} = -\frac{\partial}{\rho\partial x_3}(p + \gamma h) + \nu\nabla^2 u_3 \tag{3.16}$$

These equations are known as equations of Navier–Stokes for incompressible flow, constant viscosity with general form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \mu\nabla^2\mathbf{u} + \mathbf{f} - \nabla \mathrm{p} \tag{3.17}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{3.18}$$

Very often the force is related to the potential function ϕ such that $\mathbf{f} = \nabla\phi$ and incorporated in the modified pressure term. Strictly speaking, the assumption of uniform movement of viscous fluid prohibits the possibility of local variation which will result in absence of fluctuation in the velocity. This in fact translates to the motion of the fluid particles in layers known as laminae, owing the term or expression of ”laminar flow”. Aditionally, if the acceleration of the fluid particle due to the time variation of the velocity term vanishes for example if there is no sudden acceleration or an absence of pulsatile regime etc.. which means that the flow is time independent and characterized as contant flow in time; that is Steady state. Also in case that the convective acceleration of the fluid particle due to a spatial variation of the velocity vector (note that this is independently of whether the flow is steady or unsteady) is zero(inertial force is negligible) vis -a-vis of the viscous force and the pressure force, i.e very Low Reynolds number from (dimensional analysis), we have a simplified form or Navier-Stokes equations known as Stokes equation. The equations of motion maybe written as

$$\nabla\cdot\sigma = \mathbf{0} \quad \text{or} \quad \frac{\partial\sigma_{\mathbf{ik}}}{\partial\mathbf{x_k}} = \mathbf{0} \implies \nabla\cdot(\tau - \mathbf{pI}) = \mathbf{0} \quad \text{or} \quad \frac{\partial\tau_{\mathrm{ik}}}{\partial \mathrm{x_k}} - \delta_{\mathrm{ik}}\frac{\partial \mathrm{p}}{\partial \mathrm{x_k}} = 0 \tag{3.19}$$

where σ is the Cauchy stress tensor. which can be written in more convenient way relating the pressure to the velocity for a Newtonian fluid as

$$\mu\nabla^2\mathbf{u} - \nabla\mathrm{p} = 0 \quad \text{or} \quad \mu\frac{\partial^2 \mathrm{u_i}}{\partial \mathrm{x_k}\partial \mathrm{x_k}} - \frac{\partial \mathrm{p}}{\partial \mathrm{x_i}} = 0 \tag{3.20}$$

One of the most intriguing fact about the Laminar Stokes flow is that the pattern of the stream lines in the horizontal plane (x_1x_2) in this case will remain invariant across the flow. Despite the variation in the speed in the x_3 direction, it’s improtant to point out that the hydrostatic pressure is normally distributed (underlining the fact that they are normal to the boundary surfaces), therefore the pressure gradient is independent of x_3. The stationary Stokes flows are basically flows which ocur at very low Reynolds number which could be seen as a balance of the pressure gradient and the viscous friction independently of the time. It is shown that taking divergence of the Stokes equations leads to an expression of the harmonic function of the pressure. Foregoing, we consider only the Stokes flows described in above **??**. Note that a typical Stokes flow problem leads to incompressible, Newtonian with constant density and viscosity in the flow domain associated withe the boundary conditions. some properties such as linearity, reversibility and the uniqueness related to Stokes flows help a lot in solving Stokes flows problems. The linearlity is used in decomposing and writing the Stokes problems as linear combination of subproblems, making it easy to find an individual solution. Also the fact that the Stokes flow is reversible could be viewed as a scenario where it is very difficult to mix two fluids otherwise turbulent induced. This leads to the uniqueness of the solution

3.3 Problem Formulation

We consider steady, isothermal and fully developed Stokes flow of an incompressible fluid with viscosity μ and density ρ. The flow takes place in an infinite horizontal channel and is driven by pressure gradient ∇p. Flow velocity $\mathbf{u} \equiv (\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})^\top$ is described by equations of conservation of momentum and mass,

$$\mu \nabla^2 \mathbf{u} = \nabla \mathbf{p}, \qquad \nabla \cdot \mathbf{u} = \mathbf{0}, \qquad \mathbf{x} = (\mathbf{x}, \mathbf{y}, \mathbf{z})^\top \in \mathscr{D}. \tag{3.21}$$

If the channel's aperture were constant, b, and impermeable walls were perfectly smooth, these equations would give rise to a parabolic velocity profile, i.e., the Hagen-Poiseuille law. Our focus, instead, is on a channel $\mathscr{D} = \{\mathbf{x} : |\mathbf{x}| < \infty, |\mathbf{y}| < \infty, \mathbf{z_l} < \mathbf{z} < \mathbf{z_u}\}$, which is bounded at the top and bottom by rough surfaces $z_u \equiv z_u(x,y)$ and $z_l \equiv z_l(x,y)$, respectively. The small-scale erratic variability of these surfaces is modeled by treating them as random fields, $z_u \equiv z_u(x,y,\omega)$ and $z_l \equiv z_l(x,y,\omega)$, where ω is a realization or "coordinate" in the sample space Ω.

We assume these fields and, hence, the random channel aperture $w(x,y,\omega) \equiv z_u(x,y,\omega) - z_l(x,y,\omega)$ to be second-order stationary (statistically homogeneous), i.e., to have constant ensemble mean $\langle w \rangle = b$ and variance σ_w^2 and a correlation function $C_w(r)$ that depends only on the distance between two points $r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ rather than on two points separately. We also assume the random fields $z_u(x, y, \omega)$ and $z_l(x, y, \omega)$ to be differentiable with respect to x and y, so that no-slip and no-flow boundary conditions are defined in each realization ω.

3.4 Domain Transformation

NOTE: We would like to notify the reader that some slight modifications have been done to the conventional notations to avoid confusion and to make sense of some expression. Hence cartesian coordinate is denoted $(x_i \equiv x, y, z)$ respectively for $i = 1, 2, 3$; The curvilinear coordinate system $(x^i \equiv \xi_i)$ for $i = 1, 2, 3$ in the same way the contravariant component $(u^i \equiv v_i)$ for $i = 1, 2, 3$. Instead of solving the Stokes equations on a random domain $\mathscr{D}$ we map the latter onto a deterministic domain $\mathscr{A}$. The central idea is the need to provide an algorithm in order to develop expressions that are consistent and valid in both domains. This allows us to take advantage of the existing methods of solving differential equations in a deterministic domain. Specifically, we introduce a transformation of coordinates,

We introduce the dimensionless quantities:

$$\hat{x} = \frac{x}{b}, \qquad \hat{y} = \frac{y}{b}, \qquad \hat{z} = \frac{z}{b}. \tag{3.22}$$

$$\xi_1 = \hat{x}, \qquad \xi_2 = \hat{y}, \qquad \xi_3 = \frac{\hat{z} - \hat{z}_l}{\hat{z}_u - \hat{z}_l} \tag{3.23}$$

and

$$\hat{\mathbf{u}} = \frac{\mu}{b^2 \rho g}\mathbf{u} \quad \text{and} \quad \hat{\mathrm{h}} = \frac{\mathrm{h}}{\mathrm{b}} = \frac{\mathrm{p}}{\mathrm{b}\rho\mathrm{g}}$$

which maps the random domain $\mathscr{D}$ onto $\mathscr{A} = \{\xi : |\boldsymbol{\xi_1}| < \infty, |\boldsymbol{\xi_2}| < \infty, \mathbf{0} \leq \boldsymbol{\xi_3} \leq \mathbf{1}\}$. where we denote the Cartesian coordinate reference by (x, y, z) and the coordinate reference frame of the new domain ξ_i. For any transformation or domain mapping, one needs to compute the Jacobian which is need to scale the element from one domain to another.

$$\mathbf{J} \equiv \begin{pmatrix} \frac{\partial \hat{x}}{\partial \xi_1} & \frac{\partial \hat{x}}{\partial \xi_2} & \frac{\partial \hat{x}}{\partial \xi_3} \\ \frac{\partial \hat{y}}{\partial \xi_1} & \frac{\partial \hat{y}}{\partial \xi_2} & \frac{\partial \hat{y}}{\partial \xi_3} \\ \frac{\partial \hat{z}}{\partial \xi_1} & \frac{\partial \hat{z}}{\partial \xi_2} & \frac{\partial \hat{z}}{\partial \xi_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \zeta_1 & \zeta_2 & w \end{pmatrix}, \qquad \zeta_1 = \frac{\partial \hat{z}}{\partial \xi_1}, \quad \zeta_2 = \frac{\partial \hat{z}}{\partial \xi_2}, \tag{3.24}$$

$$\frac{\partial \hat{z}}{\partial \xi_1} = (1-\xi_3)\frac{\partial \hat{z}_l}{\partial \xi_1} + \xi_3 \frac{\partial \hat{z}_u}{\partial \xi_1}, \quad \frac{\partial \hat{z}}{\partial \xi_2} = (1-\xi_3)\frac{\partial \hat{z}_l}{\partial \xi_2} + \xi_3 \frac{\partial \hat{z}_u}{\partial \xi_2} \tag{3.25}$$

and has a determinant w which is nonzero. The guiding principle of this transformation is the fact that the fundamental physics laws are independent of the choice of coordinate system under any circumstances. For the classic orthogonal coordinate system, if we denote by ds^2 the element of the differential line, one can write that:

$$ds^2 = dxdx + dydy + dzdz \quad \text{or} \quad ds^2 = (dx)^2 + (dy)^2 + (dz)^2, \tag{3.26}$$

which will be later used to define the metric elements. The basis vectors of the new coordinate frame are defined as $\varepsilon_\mathbf{i} = \partial \mathbf{x}/\partial \xi_\mathrm{i}$. The covariant metric elements $g_{ij} = \varepsilon_\mathbf{i} \cdot \varepsilon_\mathbf{j}$ are calculated. Given tha the determinant g of the metric tensor is related to the determinant w of the jacobian matrix by the expression $w = \sqrt{g}$. It is trivial that the inverse of metric tensor called here conjugate tensor g^{ij} and is explicitly defined as $g^{ij} = \varepsilon^\mathbf{i} \cdot \varepsilon^\mathbf{j}$. They are symmetric and very important quantities because both of these quantities relate contravariant quantities and covariant quantities in the same reference system.

3.4.1 Stokes equations transformed

In reality, Stokes equations can be expressed in any coordinate system as long as the coordinate system is well defined.We know that the strain rate in local coordinate system is

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv e_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right) \tag{3.27}$$

Note here that in cartesian coordinate system, the covariant derivative is reduced to a simple partial differentiation. To tansform the viscous term in the equation (**??**) to obtain its expression in the new domain, we need to note first that $\tau_{ij} - p\delta_{ij}$in the new coordinate system becomes $\tau^{ij} - g^{ij}p$. From the relation $\tau_{ij} = 2\mu e_{ij}$,we can write the expression of τ^{ij} once we rewrite e_{ij} in the new coordinate frame which is E^{ij}. We have :

$$E^{ij} = \frac{1}{2}(g^{ik}\nabla_k v_j + g^{jk}\nabla_k v_i), \tag{3.28}$$

where $\nabla_k v$ represents the covariant derivative of the contravriant velocity v with respect to ξ_k. We have:

$$\nabla_k v_i = \frac{\partial v_i}{\partial \xi_k} + \Gamma^i_{jk} v_j. \tag{3.29}$$

Here and foregoing, the usual notation $v_{i;k}$ which refers to the covariant derivative of the contravariant velocity $\nabla_k v_i$ is used. We take the covariant derivative of E^{ij} with respect to ξ_l. The commutativity property of covariant differentiation in the Euclidean space allows us to simplify the obtained result especially. Next we use the metrinillic property which states that the covariant derivatives of the covariant metric and the contravariant metric vanishe , see [**?**].

$$E^{ij}_{;l} = \frac{1}{2}\left(g^{ik}(v_{j;k})_{;l} + g^{jk}(v_{i;k})_{;l}\right). \tag{3.30}$$

Finally, we let $l = j$ and consequently (**??**) simplifies to

$$E^{ij}_{;j} = \frac{1}{2} g^{ik} (v_{i;k})_{;j}, \tag{3.31}$$

because $g^{ik}(v_{j;k})_{;j} = 0$ for divergence free which in this case is translates to $v_{j;j} = 0$. The implication of contracting l to j is to recover the complete expression of the viscous stess tensor for incompressible and Newtonian fluid in this new coordinate system. We note that. Since the (**??**) is reduced to evaluating the expression of $(v_{i;k})_{;j}$, which is found to be written as : Using the definition of covariant derivative of a a given tensor T^i_j tensor given as :

$$(T^i_j)_{;k} = \frac{\partial T^i_j}{\partial x^i} + \Gamma^i_{km} T^m_j - \Gamma^m_{kj} T^i_m, \tag{3.32}$$

we can further expand the expression of $(v_{i;k})_{;j}$ to obtain

$$\begin{aligned} (v_{i;k})_{;j} = \frac{\partial^2 v_i}{\partial \xi_j \xi_k} - \Gamma^m_{kj} \frac{\partial v_i}{\partial \xi_m} + \Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + \Gamma^i_{kr} \frac{\partial v_r}{\partial \xi_j} + \Gamma^i_{kr} \Gamma^r_{jm} v_m \\ + \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r + \Gamma^i_{jm} \Gamma^m_{kr} v_r - \Gamma^n_{jk} \Gamma^i_{nr} v_r - \Gamma^n_{rj} \Gamma^i_{kn} v_r, \end{aligned} \tag{3.33}$$

This expression (**??**) can be re-arranged and grouped in way to further simplification by indices juggling.

$$\begin{aligned} (v_{i;k})_{;j} = \left(\frac{\partial^2 v_i}{\partial \xi_j \xi_k} - \Gamma^m_{kj} \frac{\partial v_i}{\partial \xi_m} \right) + \left(\Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + \Gamma^i_{kr} \frac{\partial v_r}{\partial \xi_j} \right) \\ + \left(\frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r + \Gamma^i_{jm} \Gamma^m_{kr} v_r - \Gamma^n_{jk} \Gamma^i_{nr} v_r \right) - \left(\Gamma^n_{rj} \Gamma^i_{kn} v_r - \Gamma^i_{kr} \Gamma^r_{jm} v_m \right) \end{aligned} \tag{3.34}$$

We then proceed to simplify (**??**) to give a more convenient and shorter expression.

$$2E^{ij}_{;j} = \nabla^2 v_i + 2 g^{jk} \Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + g^{jk} \left(\frac{\partial \Gamma^i_{kr}}{\partial \xi_j} + \Gamma^i_{jm} \Gamma^m_{kr} - \Gamma^n_{jk} \Gamma^i_{nr} \right) v_r. \tag{3.35}$$

By adding and substracting $\frac{\partial \Gamma^j_{kr}}{\partial \xi_i}$, we notice that from Riemann - Christoffel theorem on Euclidean space, one can write that

$$\frac{\partial \Gamma^j_{kr}}{\partial \xi_i} - \frac{\partial \Gamma^j_{ki}}{\partial \xi_r} + \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} + \Gamma^i_{jm}\Gamma^m_{kr} - \Gamma^n_{jk}\Gamma^i_{nr} = \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} \tag{3.36}$$

because

$$\frac{\partial \Gamma^j_{kr}}{\partial \xi_i} - \frac{\partial \Gamma^j_{ki}}{\partial \xi_r} + \Gamma^j_{im}\Gamma^m_{kr} - \Gamma^n_{ik}\Gamma^j_{nr} = 0 \tag{3.37}$$

Therefore,

$$E^{ij}_{;j} = \frac{1}{2}\left\{\nabla^2 v_i + 2g^{jk}\Gamma^i_{jr}\frac{\partial v_r}{\partial \xi_k} + g^{jk}\frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r\right\}. \tag{3.38}$$

In Vertue of the viscous stress tensor to the strain rate relation: $\tau^{ij}_{;j} = 2\mu E^{ij}_{;j}$, where μ is the fluid viscosity. We can write that

$$g^{ij}\frac{\partial \hat{h}}{\partial \xi_j} = \mu\left(\nabla^2 v_i + 2g^{jk}\Gamma^i_{jr}\frac{\partial v_r}{\partial \xi_k} + g^{jk}\frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r\right). \tag{3.39}$$

where

$$\nabla^2 = g^{jk}\frac{\partial^2}{\partial \xi_j \partial \xi_k} - g^{jk}\Gamma^m_{jk}\frac{\partial}{\partial \xi_m}, \tag{3.40}$$

noted form (**??**) is the general expression of the laplacian in general curvilinear system which is derived in [**?**] (pp.110) for more details. We further decompose the right hand side of this general expression of Stokes equations as sum of Laplacian in the orthogonal coordinate system along the principal axes and a force term:

$$g^{ij}\frac{\partial \hat{h}}{\partial \xi_j} = \nabla^2_{\perp} v_i + \left(2g^{jk}\Gamma^i_{jr}\frac{\partial v_r}{\partial \xi_k} + g^{jk}\frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r - g^{jk}\Gamma^m_{jk}\frac{\partial v_i}{\partial \xi_m}\right) + \left(g^{jk}\frac{\partial^2 v_i}{\partial \xi_j \partial \xi_k}\right)_{k \neq j}, \tag{3.41}$$

where

$$\nabla^2_{\perp} v_i = g^{11}\frac{\partial^2 v_i}{\partial \xi_1^2} + g^{22}\frac{\partial^2 v_i}{\partial \xi_2^2} + g^{33}\frac{\partial^2 v_i}{\partial \xi_3^2}. \tag{3.42}$$

We introduce the conservative force f_i given by

$$f_i = \left(2g^{jk}\Gamma^i_{jr}\frac{\partial v_r}{\partial \xi_k} + g^{jk}\frac{\partial \Gamma^i_{kr}}{\partial \xi_j}v_r - g^{jk}\Gamma^m_{jk}\frac{\partial v_i}{\partial \xi_m}\right) + \left(g^{jk}\frac{\partial^2 v_i}{\partial \xi_j \partial \xi_k}\right)_{k\neq j} \tag{3.43}$$

which can be related to the potential Φ via

$$g^{ij}\frac{\partial \hat{h}}{\partial \xi_j} - g^{ij}\frac{\partial \Phi}{\partial \xi_j} = \nabla^2_\perp v_i. \tag{3.44}$$

The Stokes equations are then simplified to:

$$\begin{cases} g^{ij}\frac{\partial \hat{h}}{\partial \xi_j} - \frac{\partial \Phi}{\partial \xi_j} = \nabla^2_\perp v_i, \\ \frac{1}{g^{1/2}}\frac{\partial (g^{1/2} v_i)}{\partial \xi_i} = 0 \end{cases} \tag{3.45}$$

We define the dimensionless quantity $\partial H/\partial \xi_j = g^{ij}\partial \hat{h}/\partial \xi_j - g^{ij}\partial \Phi/\partial \xi_j$ the modified pressure gradientoverall flow field along the jth direction in the curvilinear system.. Thus the mass-momentum conservation equations write:

$$\begin{cases} \frac{\partial H}{\partial \xi_j} = g^{11}\frac{\partial^2 v_j}{\partial \xi_1^2} + g^{22}\frac{\partial^2 v_j}{\partial \xi_2^2} + g^{33}\frac{\partial^2 v_j}{\partial \xi_3^2} \\ \frac{\partial (w v_j)}{\partial \xi_j} = 0, \end{cases} \tag{3.46}$$

and since we are using dimensionless quantities, the μ appears implicitly in the $\mathbf{v}$. The mapping introduced earlier is a one-to-one mapping and it is worth pointing out that by changing coordinate system through a mapping, flow field remains invariant under no other additional external influence. Thus the velocity field and the pressure force remain invariant under this transformation. Also, to reduce the order to infinitesimal, we now consider the truncation of the terms in note-8 using the expression $\zeta_1 \& \zeta_2$ in Appendix in the limit where the deformation on $\hat{z}_u$ and on $\hat{z}_l$ are very small owing the finite linear term of ζ_1, ζ_2. Thus we can write metric with an approximation of $g^{33} \approx 1/w^2$, in components form

$$\begin{cases} \frac{\partial H}{\partial \xi_j} = \frac{\partial^2 v_j}{\partial \xi_1^2} + \frac{\partial^2 v_j}{\partial \xi_2^2} + \frac{1}{w^2}\frac{\partial^2 v_j}{\partial \xi_3^2} & \text{(momentum)}, \\ \frac{\partial (w v_j)}{\partial \xi_j} = 0 & \text{(mass} - \text{conservation)}. \end{cases} \tag{3.47}$$

3.5 Ensemble Average

The first step to take in computing the moment of the differential equation is to perform the Reynolds decomposition of all the intrinsic random variables:

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}', \qquad \mathrm{H} = \langle \mathrm{H} \rangle + \mathrm{H}',$$
$$w = \langle w \rangle + w', \qquad \text{where} \qquad \langle w \rangle = 1,$$

where brackets $\langle \cdot \rangle$ represent the ensemble average or mathematical expectation of the intrinsic variables, while the primed quantities indicate the zero-mean fluctuations.

Substituting these decomposed expressions in the momentum and continuity equations while neglecting the higher order moment terms (third, fourth..), and taking the ensemble average, we obtain:

$$\begin{cases} \nabla^2 \langle \mathbf{v} \rangle + \sigma_\mathrm{w}^2 \nabla_\mathrm{h}^2 \langle \mathbf{v} \rangle + 2\langle \mathrm{w}' \nabla_\mathrm{h}^2 \mathbf{v}' \rangle = \nabla \langle \mathrm{H} \rangle + \sigma_\mathrm{w}^2 \nabla \langle \mathrm{H} \rangle + 2\langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ \nabla \cdot \langle \mathbf{v} \rangle + \nabla \cdot \langle w' \mathbf{v}' \rangle = 0. \end{cases} \tag{3.48}$$

We subtract equations (**??**) from (**??**), and multiply the obtained expressions by $w'(\xi)$ and then take the ensemble average. The expressions obtained are:

$$\begin{cases} 2\sigma_\varepsilon^2 \nabla_h^2 \langle \mathbf{v} \rangle + \langle \mathbf{w}' \nabla_\mathbf{h}^\mathbf{2} \mathbf{v}' \rangle + \dfrac{\partial^2 \mathrm{C}_{\mathrm{w}\mathbf{v}}}{\partial \xi_3^2} = 2\sigma_\mathrm{w}^2 \nabla \langle \mathrm{H} \rangle + \langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ \langle \nabla \cdot \mathbf{v} \rangle + \nabla \cdot \mathbf{C_{wv}} = \mathbf{0}. \end{cases} \tag{3.49}$$

Similar algebraic technique is perfomed; the result is being multiplied by $w'(\zeta)$ at another location ζ and we then take the expectation to obtain the covariance C_w and cross-covariance $C_{w\mathbf{v}}$:

$$\begin{cases} 2C_w \nabla_h^2 \langle \mathbf{v} \rangle + \nabla^\mathbf{2} \mathbf{C_{wv}} = 2\mathrm{C_w} \nabla \langle \mathrm{H} \rangle + \nabla \mathrm{C_{wH}}, \\ \nabla \cdot \mathbf{C_{wv}} = 0. \end{cases} \tag{3.50}$$

Putting together and re-writing these expressions (**??**) – (**??**) as a system of equations gives:

$$\begin{cases} \nabla^2 \langle \mathbf{v} \rangle + \boldsymbol{\sigma}_\mathbf{w}^\mathbf{2} \nabla_\mathbf{h}^\mathbf{2} \langle \mathbf{v} \rangle + \mathbf{2} \langle \mathbf{w}' \nabla_\mathbf{h}^\mathbf{2} \mathbf{v}' \rangle = \nabla \langle \mathrm{H} \rangle + \sigma_\mathrm{w}^2 \nabla \langle \mathrm{H} \rangle + 2 \langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ 2\sigma_w^2 \nabla_h^2 \langle \mathbf{v} \rangle + \langle \mathbf{w}' \nabla_\mathbf{h}^\mathbf{2} \mathbf{v}' \rangle + \dfrac{\partial^2 \mathrm{C}_{\mathrm{w}\mathbf{v}}}{\partial \xi_3^2} = 2\sigma_\mathrm{w}^2 \nabla \langle \mathrm{H} \rangle + \langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ 2C_w \nabla_h^2 \langle \mathbf{v} \rangle + \nabla^\mathbf{2} \mathbf{C_{wv}} = 2\mathrm{C_w} \nabla \langle \mathrm{H} \rangle + \nabla \mathrm{C_{wH}}, \\ \langle \nabla \cdot \mathbf{v} \rangle + \nabla \cdot \mathbf{C_{wv}} = \mathbf{0}, \\ \nabla \cdot \mathbf{C_{wv}} = 0. \end{cases} \tag{3.51}$$

3.6 Perturbation

As often used to analyze problems involving non linear cases, perturbation technique is also used in stochastic analysis. the central idea is to Taylor-expanded the functions and operators involved about their expected values. Perturbation approach are applicable when the parametric variations are 'weak' or 'small'. In other words, given that the aperture function w, velocity function and the pressure head function are assumed to be expanded in asymptotic series in very small parameters $\sigma_w^2 \ll 1$, we represent each intrinsic variable as an infinite sum as follows:

$$\begin{aligned} \mathbf{v} &= \sum_{k=0}^{\infty} \sigma_w^{2k} \mathbf{v}^{(\mathbf{k})}, & \mathrm{H} &= \sum_{\mathrm{k}=0}^{\infty} \sigma_{\mathrm{w}}^{2\mathrm{k}} \mathrm{H}^{(\mathrm{k})}, \\ \langle \mathbf{v} \rangle &= \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \sigma_{\mathbf{w}}^{\mathbf{2k}} \langle \mathbf{v}^{(\mathbf{k})} \rangle, & \langle \mathrm{H} \rangle &= \sum_{\mathrm{k}=0}^{\infty} \sigma_{\mathrm{w}}^{2\mathrm{k}} \langle \mathrm{H}^{(\mathrm{k})} \rangle, \end{aligned} \tag{3.52}$$

where the superscript (k) indicates terms of the k-th order in σ_w^2 Replacing and collecting the same order terms of σ_w, one gets for zero order (σ_w^0):

$$\begin{cases} \nabla^2 \langle \mathbf{v}^{(\mathbf{0})} \rangle = \nabla \langle \mathrm{H}^{(0)} \rangle, \\ \nabla \cdot \langle \mathbf{v}^{(\mathbf{0})} \rangle = 0. \end{cases} \tag{3.53}$$

We have for second order i.e (σ_w^2):

$$\begin{cases} \nabla^2 \langle \mathbf{v}^{(\mathbf{1})} \rangle + \nabla_{\mathrm{h}}^2 \langle \mathbf{v}^{(\mathbf{0})} \rangle + \mathbf{2} \langle \mathrm{w}' \nabla_{\mathrm{h}}^2 \mathbf{v}' \rangle = \nabla \langle \mathrm{H}^{(1)} \rangle + \nabla \langle \mathrm{H}^{(0)} \rangle + 2 \langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ 2 \nabla_h^2 \langle \mathbf{v}^{(\mathbf{0})} \rangle + \langle \mathbf{w}' \nabla_{\mathbf{h}}^{\mathbf{2}} \mathbf{v}' \rangle + \dfrac{\partial^{\mathbf{2}} \mathbf{C}_{\mathbf{wv}}}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}} = 2 \nabla \langle \mathrm{H}^{(0)} \rangle + \langle \mathrm{w}' \nabla \mathrm{H}' \rangle, \\ 2 C_w \nabla_h^2 \langle \mathbf{v}^{(\mathbf{0})} \rangle + \nabla^{\mathbf{2}} \mathbf{C}_{\mathbf{wv}} = \mathbf{2} \mathbf{C}_{\mathbf{w}} \nabla \langle \mathbf{H}^{(\mathbf{0})} \rangle + \nabla \mathbf{C}_{\mathbf{wH}}, \\ \nabla \cdot \langle \mathbf{v}^{(\mathbf{1})} \rangle = 0. \end{cases} \tag{3.54}$$

Rearranging (**??**) & (**??**) for convenience,we have

$$\begin{cases} \nabla^2\langle \mathbf{v^{(0)}}\rangle = \nabla\langle \mathbf{H^{(0)}}\rangle, \\ \nabla\cdot\langle \mathbf{v^{(0)}}\rangle = 0, \\ \nabla^2\langle \mathbf{v^{(1)}}\rangle + \nabla^2_{\mathbf{h}}\langle \mathbf{v^{(0)}}\rangle + \mathbf{2}\langle \mathbf{w'}\nabla^2_{\mathbf{h}}\mathbf{v'}\rangle = \nabla\langle \mathbf{H^{(1)}}\rangle + \nabla\langle \mathbf{H^{(0)}}\rangle + \mathbf{2}\langle \mathbf{w'}\nabla\mathbf{H'}\rangle, \\ 2\nabla^2_h\langle \mathbf{v^{(0)}}\rangle + \langle \mathbf{w'}\nabla^2_{\mathbf{h}}\mathbf{v'}\rangle + \dfrac{\partial^2 \mathbf{C_{wv}}}{\partial\xi_3^2} = \mathbf{2}\nabla\langle \mathbf{H^{(0)}}\rangle + \langle \mathbf{w'}\nabla\mathbf{H'}\rangle, \\ 2C_w\nabla^2_h\langle \mathbf{v^{(0)}}\rangle + \nabla^2\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle \mathbf{H^{(0)}}\rangle + \nabla\mathbf{C_{wH}}, \\ \nabla\cdot\langle \mathbf{v^{(1)}}\rangle = 0. \end{cases} \tag{3.55}$$

Multiplying the 4th equation by minus two then adding it to the third equation of (**??**) to eliminate some terms in order to solve the system. The resulting system writes:

$$\begin{cases} \nabla^2\langle \mathbf{v^{(0)}}\rangle = \nabla\langle \mathbf{H^{(0)}}\rangle, \\ \nabla\cdot\langle \mathbf{v^{(0)}}\rangle = 0, \\ \nabla^2\langle \mathbf{v^{(1)}}\rangle - \mathbf{3}\nabla^2_{\mathbf{h}}\langle \mathbf{v^{(0)}}\rangle - \mathbf{2}\dfrac{\partial^2 \mathbf{C_{wv}}}{\partial\xi_3^2} = \nabla\langle \mathbf{H^{(1)}}\rangle - \mathbf{3}\nabla\langle \mathbf{H^{(0)}}\rangle, \\ 2C_w\nabla^2_h\langle \mathbf{v^{(0)}}\rangle + \nabla^2\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle \mathbf{H^{(0)}}\rangle + \nabla\mathbf{C_{wH}}, \\ \nabla\cdot\langle \mathbf{v^{(1)}}\rangle = 0. \end{cases} \tag{3.56}$$

For a fully developed and unidirectional Poiseuille flow, we have:

$$\nabla^2_h\langle \mathbf{v^{(0)}}\rangle = \mathbf{0} = \nabla\langle \mathbf{H^{(1)}}\rangle,$$

$$\langle \mathbf{v^{(0)}}\rangle \equiv (\langle \mathbf{v_1^{(0)}}\rangle, \mathbf{0}, \mathbf{0}) = \frac{\mathbf{J}}{\mathbf{2}}\xi_3(\mathbf{1}-\xi_3), \qquad -\mathbf{J} \equiv \nabla\langle \mathbf{H}\rangle = \frac{\partial\langle \mathbf{H^{(0)}}\rangle}{\partial\xi_1}.$$

Substituting the latter in the system from **??**, we have :

$$\begin{cases} \nabla^2\langle \mathbf{v^{(0)}} \rangle = -\mathbf{J}, \\ \nabla^2\langle \mathbf{v^{(1)}} \rangle = \mathbf{3J} + \mathbf{2}\dfrac{\partial^2 \mathbf{C_{wv}}}{\partial \xi_3^2}, \\ \nabla^2 C_{w\mathbf{v}} = -2JC_w + \nabla C_{wH}, \\ \nabla \cdot \mathbf{C_{wv}} = 0. \end{cases} \tag{3.57}$$

To obtain the expression for $\nabla^2\langle \mathbf{v^{(1)}} \rangle$ in the second equation of **??**, it is necessary to solve the third equation in terms of J and C_w. In particular, for the special case where $C_{wH} = 0$, we have:

$$\begin{cases} \nabla^2 C_{w\mathbf{v}} = -2C_w J \quad |\xi_1| < \infty, \quad |\xi_2| < \infty, \quad \text{and} \quad 0 \leq \xi_3 \leq 1, \\ C_{w\mathbf{v}}(\xi_1, \xi_2, 0) = 0, \\ C_{w\mathbf{v}}(\xi_1, \xi_2, 1) = 0. \end{cases} \tag{3.58}$$

3.7 Green's Function

Let $\mathbf{F}$ be a smooth function in $\mathbb{R}^n$ with $n \geq 1$, the divergence theorem states that :

$$\int_\Omega \nabla \cdot \mathbf{F}\mathbf{dx} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{N}\mathrm{dS} \tag{3.59}$$

where the $\Omega \subset \mathbb{R}^d$ and $\partial\Omega$ is the boundary . $\mathbf{N}$ denotes the unit normal vector orthogonal to the tangent to the boundary contour. And in vertue of the Green's identities (first and second),

$$\int_\Omega v \frac{\partial u}{\partial N} dS = \int_\Omega (v\nabla^2 u + \nabla u \cdot \nabla v) dx, \quad \text{first identity} \tag{3.60}$$

$$\int_{\Omega}(v\frac{\partial u}{\partial N}-u\frac{\partial v}{\partial N})dS=\int_{\Omega}(v\nabla^2 u-u\cdot\nabla^2 v)dx, \quad \text{second identity} \tag{3.61}$$

The fundamental solution for :

$$\begin{cases} -\nabla^2 u = f, \quad \mathbf{x}\in\Omega \\ u = g, \quad \mathbf{x}\in\partial\Omega \end{cases} \tag{3.62}$$

is given by:

$$u(\mathbf{x}) = -\int_{\Omega}\mathrm{G}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{x})\mathbf{dy}-\int_{\partial\Omega}\mathrm{g}(\mathbf{y})\frac{\partial\mathrm{G}(\mathbf{x},\mathbf{y})}{\partial\mathbf{N_y}}\mathbf{dS_y} \quad \mathbf{x}\in\Omega \tag{3.63}$$

where the green's function satisfies

$$\begin{cases} -\nabla^2 G(\mathbf{x},\mathbf{y}) = \delta^{\mathrm{d}}(\mathbf{x}-\mathbf{y}) \quad \mathrm{y}\in\Omega \\ G(\mathbf{x},\mathbf{y}) = 0 \quad \mathrm{y}\in\partial\Omega \end{cases} \tag{3.64}$$

Let G be the Green's function associated with eq.**??** which is defined as follows:

$$\begin{cases} \nabla^2 G = \delta(\xi-\zeta) \quad |\xi_{\mathbf{1}}|<\infty, \quad |\xi_{\mathbf{2}}|<\infty, \quad \text{and} \quad \mathbf{0}\le\xi_{\mathbf{3}}\le\mathbf{1}, \\ G(\xi_1,\xi_2,0,\zeta_1,\zeta_2,\zeta_3)=0, \quad |\xi_1|<\infty, \quad |\xi_2|<\infty, \\ G(\xi_1,\xi_2,1,\zeta_1,\zeta_2,\zeta_3)=0, \quad |\xi_1|<\infty, \quad |\xi_2|<\infty. \end{cases} \tag{3.65}$$

The Green's function is given by

$$\begin{aligned} G(\xi,\zeta) =& \frac{1}{4\pi}\sum_{-\infty}^{\infty}\frac{1}{\sqrt{(\xi_1-\zeta_1)^2+(\xi_2-\zeta_2)^2+(\xi_3-\zeta_3-2n)^2}} \\ & -\frac{1}{\sqrt{(\xi_1-\zeta_1)^2+(\xi_2-\zeta_2)^2+(\xi_3+\zeta_3-2n)^2}}. \end{aligned}$$

[?]. Let $\rho_w(\|\zeta-\eta\|)$ be the correlation function of $C_w(\zeta,\eta)$:

$$C_w(\zeta,\eta)=\boldsymbol{\sigma_w^2\rho_w}(\|\zeta-\eta\|). \tag{3.66}$$

From (**??**)–(**??**),

$$\nabla^2\langle \mathbf{v}^{(\mathbf{1})}\rangle = 3\mathrm{J}-4\mathrm{J}\int_\Omega \rho_{\mathrm{w}}(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta. \tag{3.67}$$

In particular, up to the first order we have:

$$\nabla^2\langle\mathbf{v}\rangle=\boldsymbol{\nabla^2}\left(\sum_{\mathbf{k=0}}^{\mathbf{1}}\boldsymbol{\sigma}_{\mathbf{w}}^{\mathbf{2k}}\langle\mathbf{v}^{(\mathbf{k})}\rangle\right)=\boldsymbol{\nabla^2}\langle\mathbf{v}^{(\mathbf{0})}\rangle+\boldsymbol{\sigma}_{\mathbf{w}}^{\mathbf{2}}\boldsymbol{\nabla^2}\langle\mathbf{v}^{(\mathbf{1})}\rangle. \tag{3.68}$$

It follows from (**??**) that

$$\nabla^2\langle\mathbf{v}\rangle=-\mathrm{J}+3\sigma_{\mathrm{w}}^2\mathrm{J}-4\sigma_{\mathrm{w}}^2\mathrm{J}\int_\Omega\rho_{\mathrm{w}}(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta, \tag{3.69}$$

where $\sigma_w^2\left(3J-4J\int_\Omega\rho_w(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta\right)$ is the correction term.

By analogy to the classical Stokes flow, one can express the dimensionless effective viscosity as

$$\mu_{\mathrm{eff}}=\frac{1}{1-\sigma_w^2\left(3-4\int_\Omega\rho_w(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta\right)}, \tag{3.70}$$

where

$$\int_\Omega\rho_w(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta=\int_{\mathbf{0}}^{\mathbf{1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{\rho_w}(\|\zeta-\eta\|)\frac{\partial^{\mathbf{2}}\mathbf{G}(\xi,\eta)}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}}\mathbf{d}\eta_{\mathbf{1}}\mathbf{d}\eta_{\mathbf{2}}\mathbf{d}\eta_{\mathbf{3}}. \tag{3.71}$$

and

$$\frac{\partial^2 G}{\partial \xi_3^2} = -\frac{1}{4\pi}\sum_{-\infty}^{\infty} \frac{1}{\left((\xi_3+\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{3}{2}}} - \frac{1}{\left((\xi_3-\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{3}{2}}} - \frac{3\,(2\xi_3+2\eta_3-4n)^2}{4\left((\xi_3+\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{5}{2}}} + \frac{3\,(2\xi_3-2\eta_3-4n)^2}{4\left((\xi_3-\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{5}{2}}}. \tag{3.72}$$

We obtain the general form of the effective viscosity:

$$\mu_{\text{eff}} = \frac{1}{1-\sigma_w^2\left(3-4\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_w(\|\zeta-\eta\|)\frac{\partial^2 \mathbf{G}(\xi,\eta)}{\partial \xi_3^2}\mathbf{d}\eta_1\mathbf{d}\eta_2\mathbf{d}\eta_3\right)}. \tag{3.73}$$

We also check if the variation of the μ_{eff} is sensitive to the variation in ξ_3 direction by computing the average effective viscosity which is given by:

$$\langle\mu_{\text{eff}}\rangle = \int_0^1\left(\frac{1}{1-\sigma_w^2\left(3-4\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_w(\|\zeta-\eta\|)\frac{\partial^2 \mathbf{G}(\xi,\eta)}{\partial \xi_3^2}\mathbf{d}\eta_1\mathbf{d}\eta_2\mathbf{d}\eta_3\right)}\right)d\xi_3. \tag{3.74}$$

3.8 Results and Discussion

Figure **??** shows the effective viscosity versus the relative roughness height σ_w over the range $0 \le \sigma_w \le 0.35$ which is relevant to applications. We observe that in the hypothetical limit of ideal smooth wall geometry, the corresponding contribution to adjust the scaled viscosity vanishes.This means that the viscosity remains unaffected, *i.e* $\mu_{\text{eff}} \to 1$, and the mean flow rate does not drop rapidly. However, in the presence of small relative roughness, the effective properties start changing from their raw state. We observe a slow increase in the viscosity and slow decrease in the mean flow rate. As the roughness becomes important, a significant change in the effective properties of the flow is expected. Figure **??** shows how the correlation length l_c affects the effective properties of the flow. Decreasing the correlation length l_c while keeping the relative roughness height σ_w constant, increases the effective viscosity. This emphasizes the importance of the correlation length. To better understand the effect of the surface roughness, in other words the impact of the correlation length on the effective properties, we choose two additional correlation functions ρ_w to be a delta function for a very short correlation length and $\rho_w = 1$ for very long correlation length. The closed-form expression (**??**) shows that the presence of the roughness on the wall adjusts the fluid viscosity μ. It is reasonable to say that a rough wall induces viscosity, $\mu_{ind} \geq 0$. We can, therefore, write $\mu_{eff} = \mu + \mu_{ind}$. The minimum value of the effective viscosity is 1 which occurs for $\sigma_w^2 = 0$ which implies $\mu_{ind} = 0$; this corresponds to a roughness-free surface and this is in agreement with the classical Poiseuille flow in the Stokes regime.

In Figure **??**, we present the effective viscosity ratio μ_{eff}/μ as a function of the relative roughness height σ_w for different correlation functions ρ_w. The results suggest that the correlation length does have a significant effect on the effective properties of the flow regime. This result also shows that the effective properties do not change as functions of the position in the channel, but instead are only sensitive to the relative roughness height and correlation length.

We also note that the average effective viscosity $\langle \mu_{\text{eff}} \rangle$ (given in Appendix **??**) can be used to verify the spatial dependence of the effective viscosity. The first order correction to the viscosity is still a fluid property which depends on the roughness, but does not depend upon the position, as it does in the theory of heterogeneous porous media. This lack of spatial dependence is due to the fact that we have considered an infinite domain with flow generated by a uniform pressure gradient; this will differ for a source-type flow. This result is analogous to findings from the stochastic theory of groundwater flow [**?**, e.g.,].

The relationship between the wall characteristic elements and the dimensionless effective properties of the flow allows us to examine the effects of the wall on the flow resistance. The coefficient of the fluid resistance (CFR) per unit length is defined as the ratio of the average pressure gradient over the effective flow rate, i.e.

$$R_F = \frac{\Delta p}{Q}. \tag{3.75}$$

A simplified expression given by $CFR = 12\mu_{eff}$ (Appendix **??**) enables one to measure the friction between the outer layer of the fluid and the wall surfaces. It is important to note that the friction between fluid layers can be neglected, because there is no mixing in the fluid. In Figure **??**, it is observed that the fluid must overcome more resistance when the area decreases; the latter is represented by the presence of roughness on the walls which narrows the cross section area.

The expression for the coefficient of the fluid resistance (CFR) given above enables us to further investigate the effect of the friction factor associated with the rough channel. The friction factor f_{K_w} given by (**??**) provides an explicit relation between the relative roughness heights and the Reynolds number in laminar Poiseuille flow in plane channels with randomly varying aperture. In contrast, in the classical theory of laminar flow regime, the friction factor does not depend on the relative roughness since rough channels or pipes exhibit friction factor independent of the relative roughness; instead it depends on the Reynolds number similarly to the smooth channel case. Figure **??** shows the friction factor f_{K_w} for a rough channel as a function of Reynolds numbers for given relative roughness heights σ_w and correlation length l_c. Based on our results, we observe that the Poiseuille number is not constant, but it is intrinsically linked to the wall's properties. However, we recover the expression $\frac{24}{Re}$ for the case $K_{\sigma_w} = 1$, where K_{σ_w} is defined to be equal to μ_{eff}/μ. As the channel height increases to the limit of the ideal aperture (small relative roughness heights), the Poiseuille number Po approaches the conventional value 24 as shown in the expression of f_{K_w}. We remark that the relative roughness is a factor in determining the friction factor especially in the case of highly viscous laminar creeping motion ($R_e < 1$). In the latter case, the friction factor decreases rapidly compared to the case of a laminar flow with strong Reynolds number dependence motion ($1 < R_e \leq 100$). This is shown in the Figure **??**. Note that the Poiseuille number is related to the dimensionless boundary shear stress by the Reynolds number. Decreasing the effective Reynolds number implies an increase in the Poiseuille number as the roughness height increases. Figure **??** shows the velocity profile as function of the channel aperture.It is shown that the profile stay parabolic for the average velocity, but it decreases as the walls become rougher. The figure**??** shows the effective conductivity as function of the relative roughness heights. The graph shows the comparison of our model to the empirical models obtained from studying the flow over single fracture of Lomize (195) and for Louis (1969).

The last three graphs show that the induced viscosity is more important near the walls than far away from the walls. However, the sensitivity in the change of the effectivity with respect to the aperture is not important because of the infinite domain considered for this study.

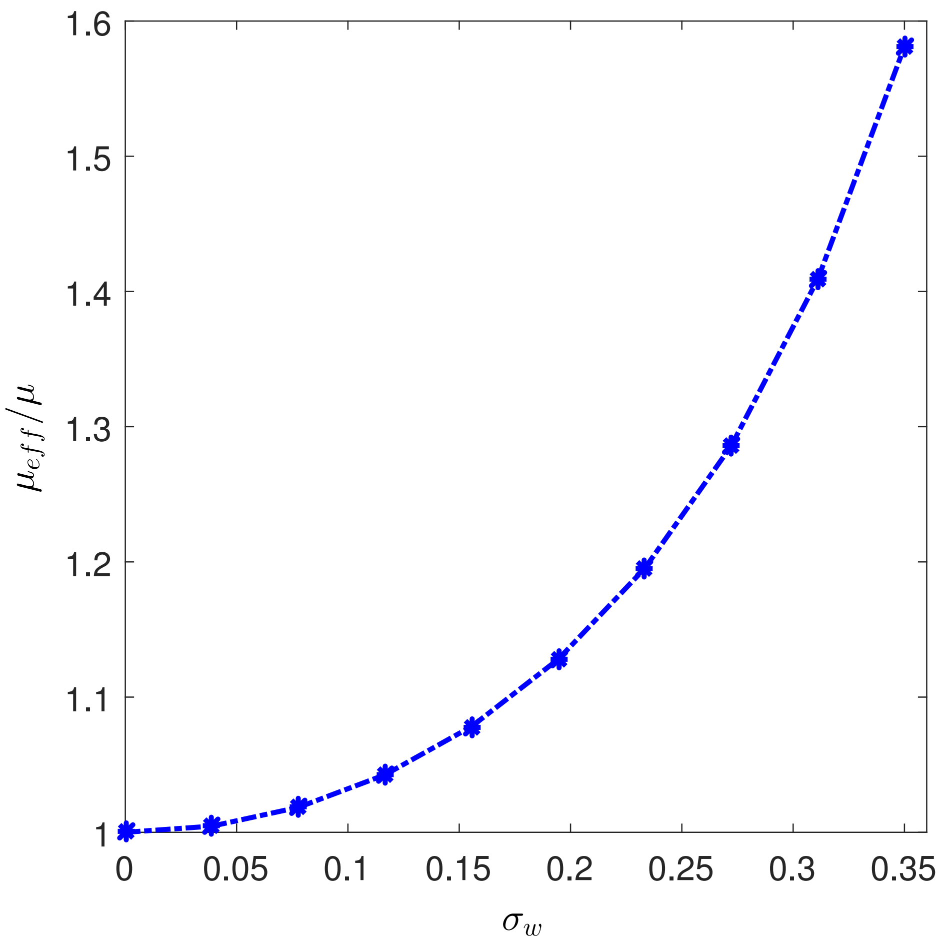

Figure 3.1. Effective viscosity ratio μ_{eff}/μ as a function of the relative roughness height σ_w for very short correlation length l_c =0.1.

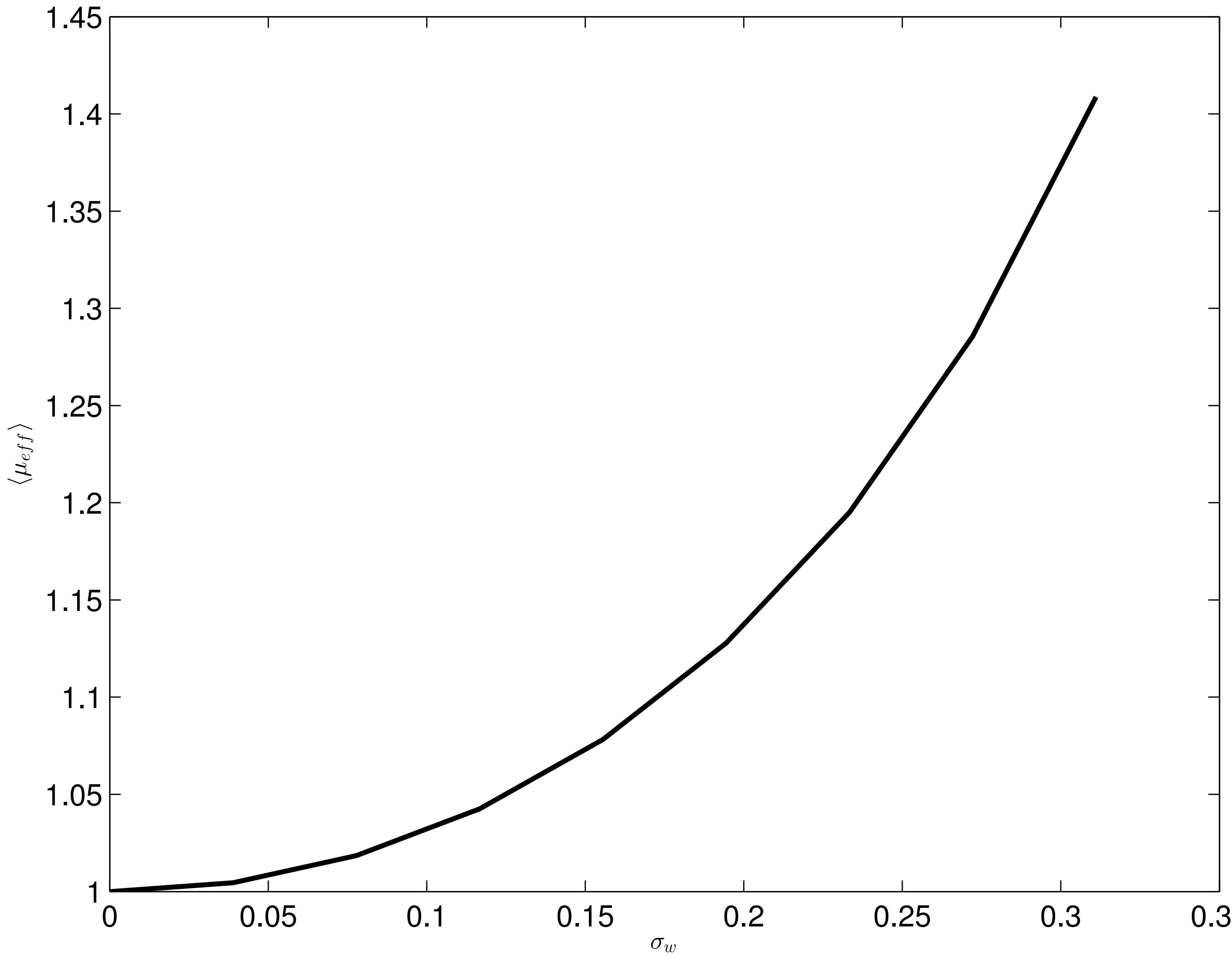

Figure 3.2. Mean effective viscosity as a function of relative roughness heights σ_w

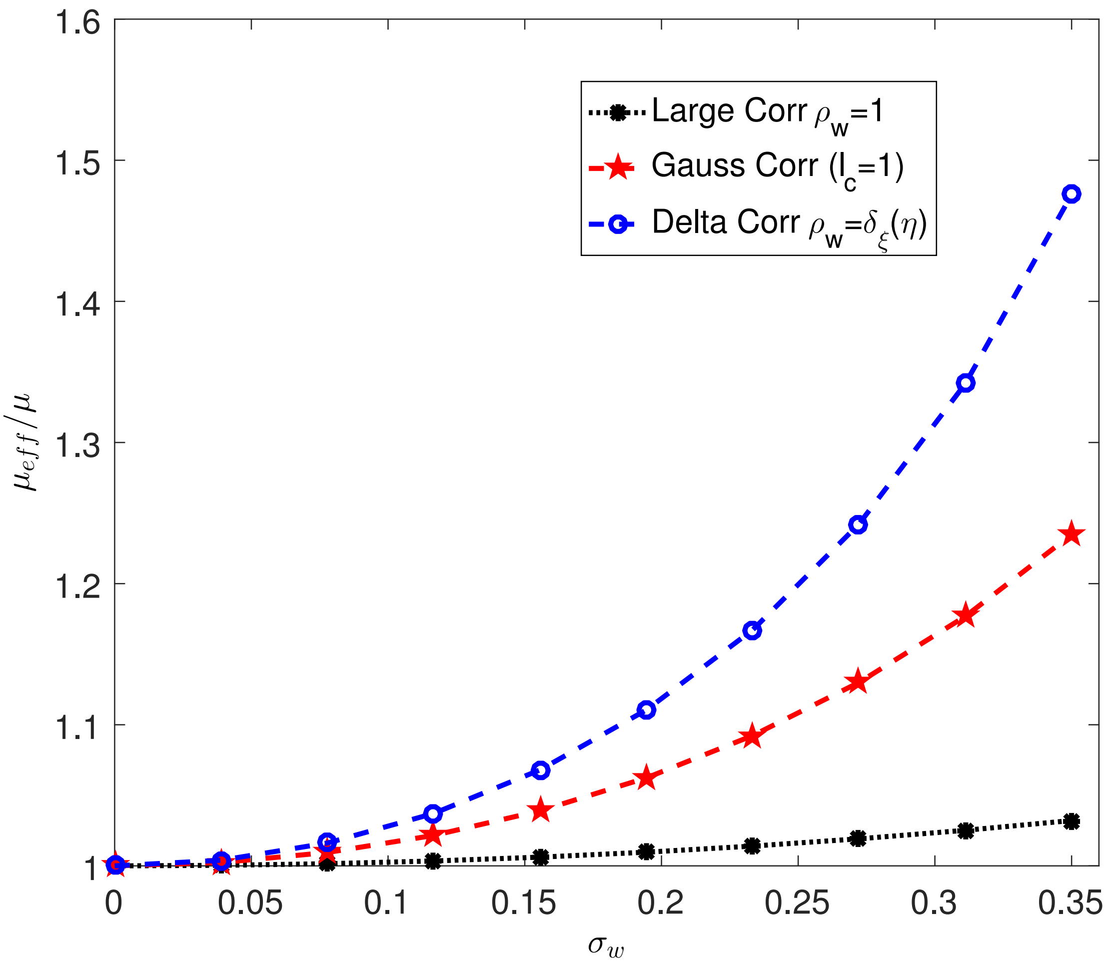

Figure 3.3. Effective viscosity ratio μ_{eff}/μ as a function of the relative roughness height σ_w for different correlation functions ρ_w.

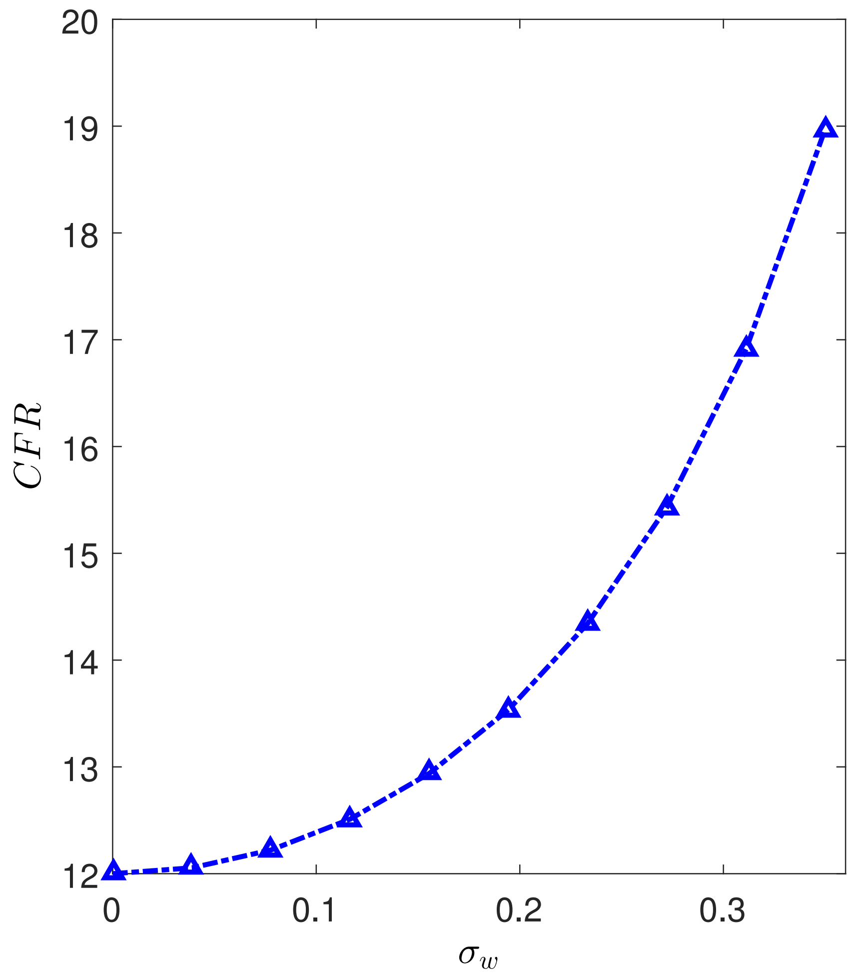

Figure 3.4. Flow resistance coefficient (CFR) per unit length as a function of the relative roughness height σ_w for a given small correlation length l_c= 0.1.

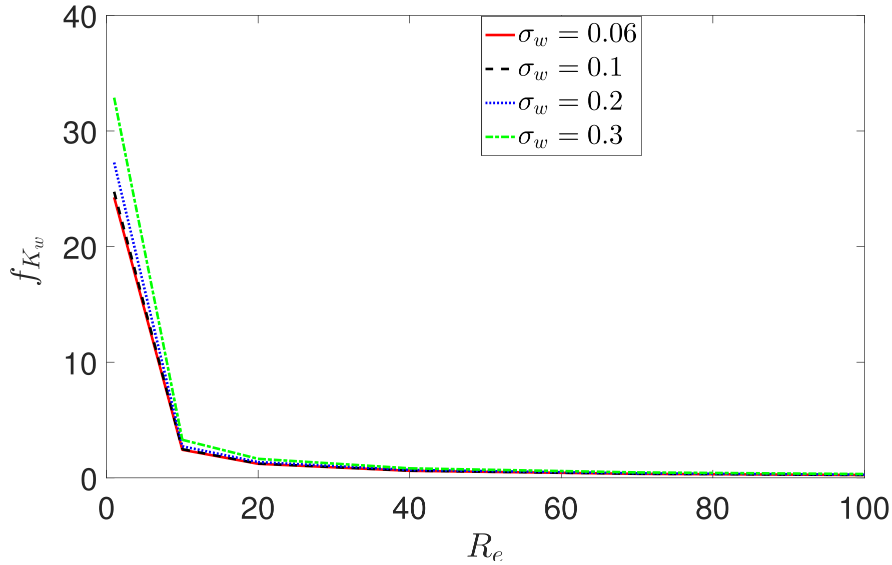

Figure 3.5. Friction factor f_{K_w} for a rough channel as a function of Reynolds numbers for given relative roughness heights σ_w and correlation length $l_c = 0.1$.

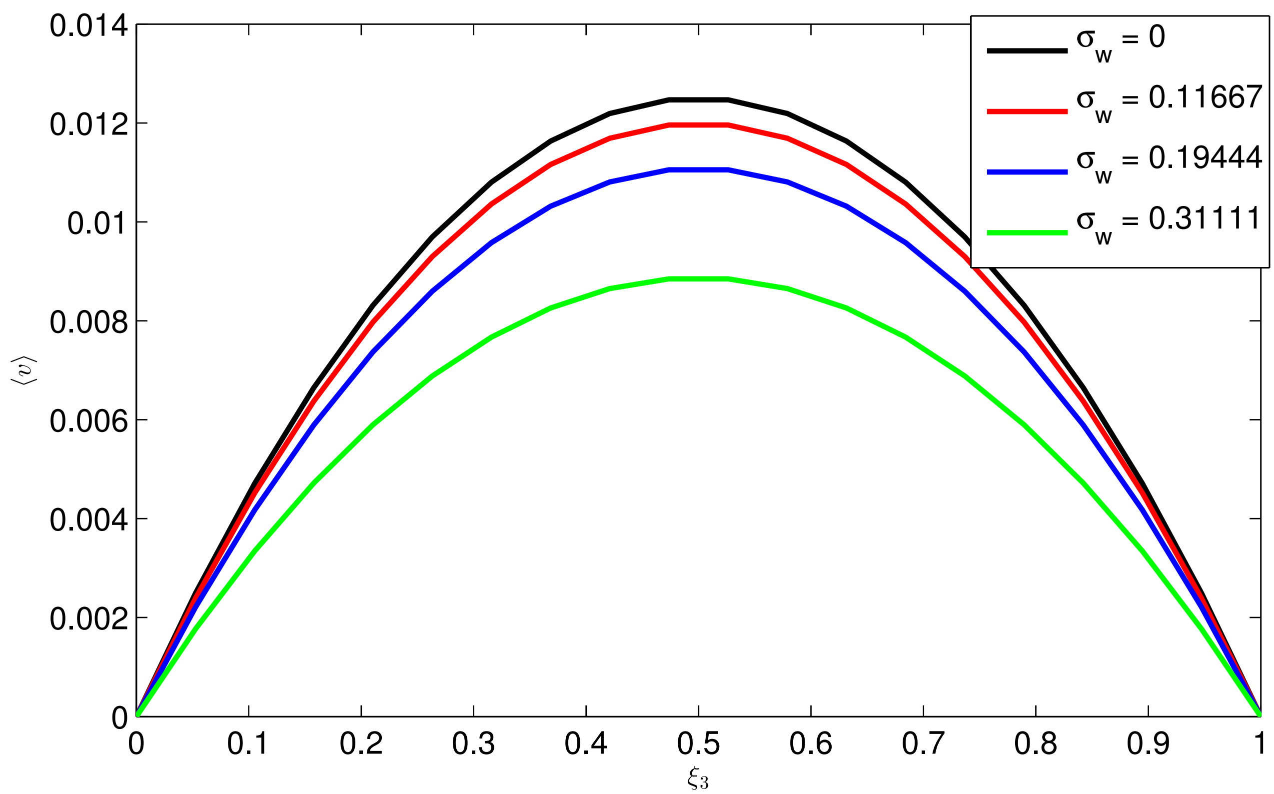

Figure 3.6. Velocity profiles as a function of the aperture ξ_3 for different relative roughness heights σ_w at a given small correlation length l_c= 0.1.

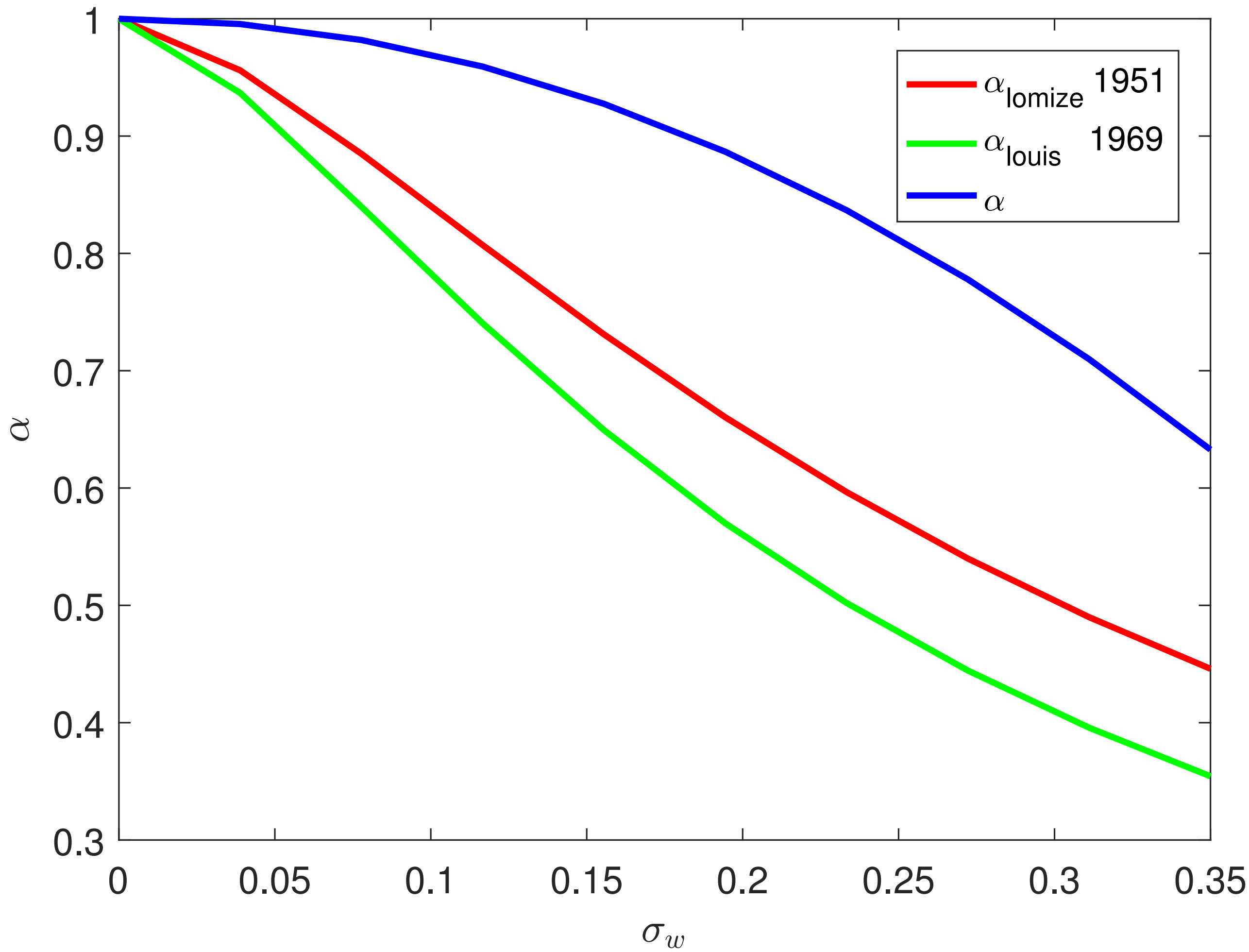

Figure 3.7. Effective conductivity ratio as a function of relative roughness heights σ_w

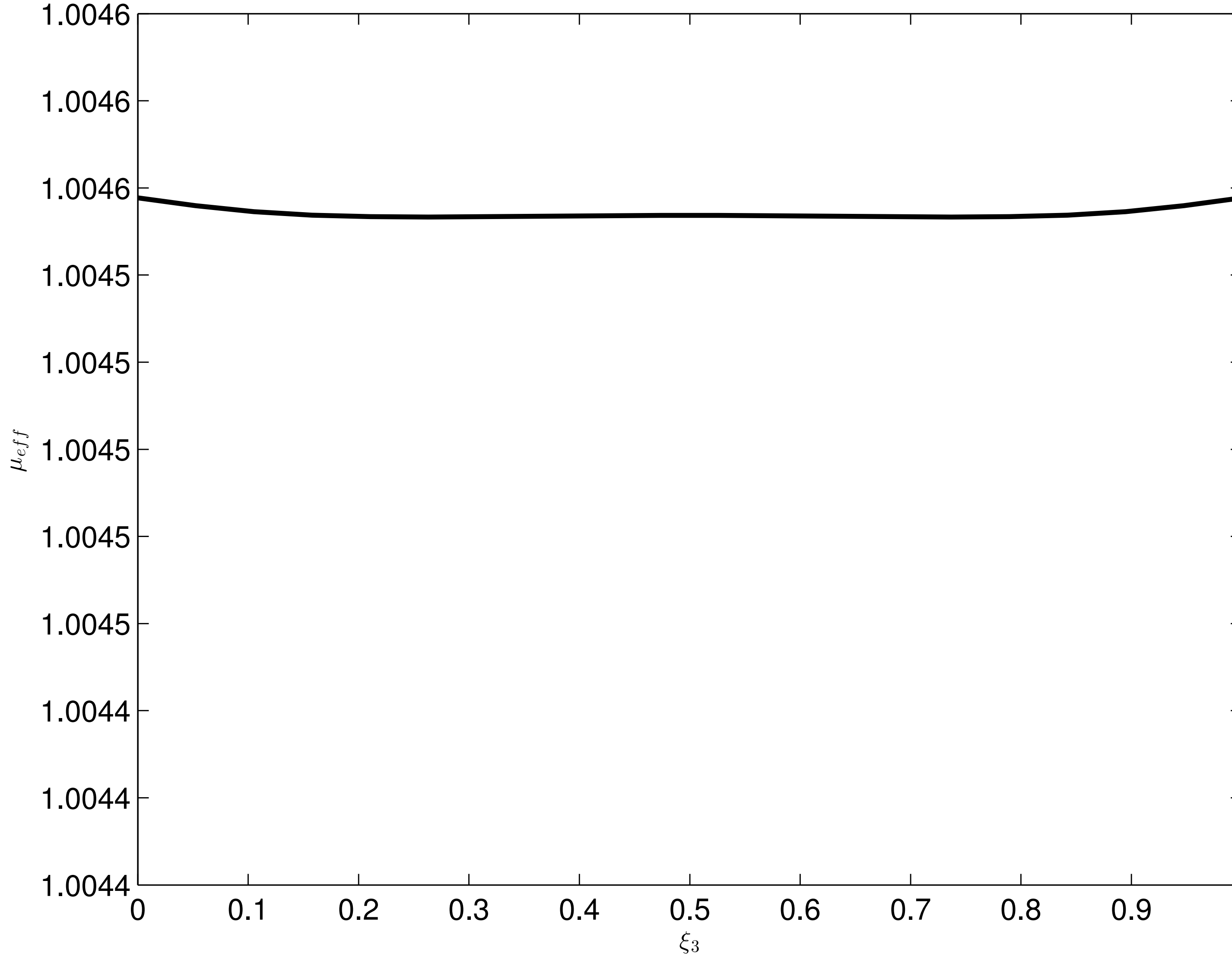

Figure 3.8. Effective viscosity profile as a function of the aperture ξ_3 for given roughness heights σ_w

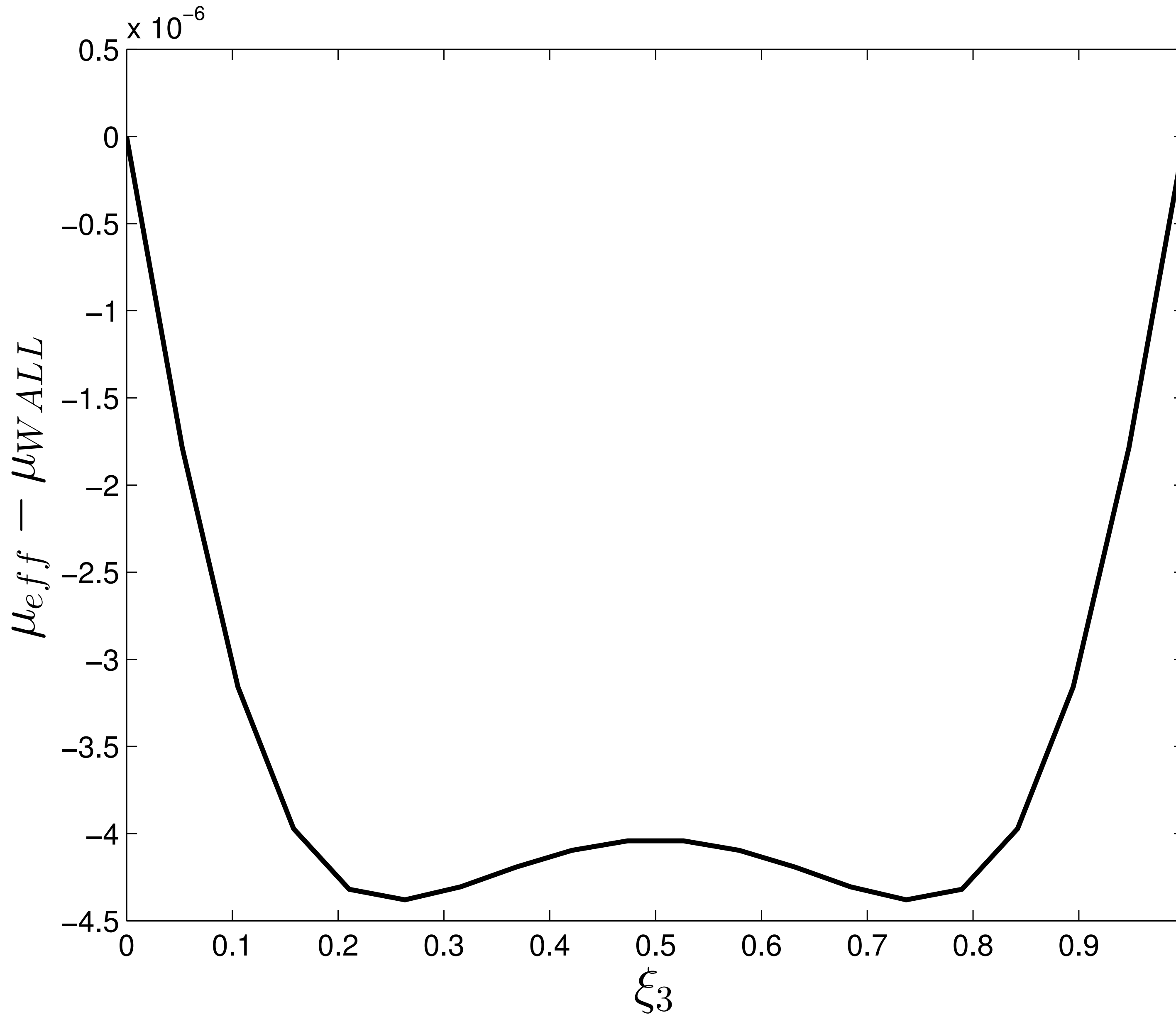

Figure 3.9. Change in effective viscosityfrom the wall as a function of relative function of the aperture ξ_3

Chapter 4

Effective parameters of heat transfer in single fracture metamaterial

4.1 Objective

Cooling material mechanism is important in technology especial in the micro and nano technology. For example keep chips in electronic devices from overheating. This process often becomes tidious and complicated with the miniaturization of the devices and the involvement of the material properties and the biggest impact that the interface physics on the system. As we mentioned it earlier in the introduction, the thermo - fluid study is used to investigate this kind of phenomena. Given the irregularity of the walls of the materials, we introduce a new approach which allows us to effectively capture the physics at the interface. The goal of this study is to find the meaningful effective parameters necessary to characterize a heat convection in a channel with rough boundaries.

4.2 Problem Statement

We consider three - dimensional laminar heat transfer and Stokes flow in infinite domain bounded by two horizontal parallel plates within the region of thermally and hydrodynamically fully- developped. In the limit of imcompressibility of the fluid and in the absence of heat source, the steady state governing equations are written in cartesian cordinate system in tensorial form as follow:

$$(\rho u_i)_{,i} = 0, \tag{4.1}$$

$$-p_{,i} + \tau_{ij,j} = 0, \tag{4.2}$$

$$(\rho c_p u_i T)_{,i} = (kT_{,i})_{,i} + \mu\Phi. \tag{4.3}$$

Where ρ, μ, c_p and k represent the fluid constant density, the viscousity, the isobaric heat capacity and the thermal conductivity of the medium respectively. If the aperture of the channel was constant to b, the solution could be computed in reference to plethora of existing methods; instead, the non homogeneity of the wall surfaces induces an uncertainty in the heat-fluid system making the domain random.

4.3 Approach

The small erradic variability of the surfaces is a driven reason to model these surfaces as random fields $z_u(x,y,\omega)$ and $z_l(x,y,\omega)$ where, ω is a realization or "coordinate" in the sample space Ω. Give that these fields, the direct implication is that the aperture becomes random function defined as $w(x,y,\omega) \equiv z_u(x,y,\omega) - z_l(x,y,\omega)$, which is assumed to be a second-order stationary (statistically homogeneous). We also assume that the random functions $z_u(x,y,\omega)$ and $z_u(x,y,\omega)$ are differentiable almost everywhere (a.e) in Ω with respect to x and y so that no-slip and impermeable flow boundary conditions are defined in each realization ω.

4.3.1 Domain Transformation

As it is posed initially, we are dealing with deterministic differential equations defined in random domain with cannot be solved. To come up with a solvable system, we define a coordinate change that maps the random domain to a deterministic counterpart and to obtain a stochastic differential equations with deterministic domain. Among other transformations, the suitable mapping used here has a non zero determinant and allows us to recover identity in case of roughness free, is assumed to possess one-to-one properties with the original coordinate system (inversibility guaranteed under this circumstance) and continuous up to second derivative (which guarantees the computation of the derivative 1st and second order of the Christoffel symbols second kind). After taking in account the constant properties of the fluid, the resulting governing equations become:

$$(v^i)_{;i} = 0, \tag{4.4}$$

$$-g^{ij} p_{;j} + \tau^{ij}_{;j} = 0, \tag{4.5}$$

$$(v^i T)_{;i} = (\alpha g^{ji} T_{,j})_{;i} + \mu\Phi. \tag{4.6}$$

where $\alpha = k/(\rho c_p)$. In the (**??**) we can further expand the expressions involved in term of simple partial differentiation .

$$(p)_{;k} = (p)_{,k}, \tag{4.7}$$

$$(v^i)_{;k} = (v^i)_{,k} + \Gamma^i_{km} v^m, \tag{4.8}$$

$$(t^i)_{i;k} = t^i_{j;k} - \Gamma^m_{ik} t^i_m + \Gamma^i_{km} t^m_i, \tag{4.9}$$

outlines the definition of the covariant derivative of the scalar, vector and tensor respectively. The $(;k)$ and $(,k)$ denote the covariant derivative (simple partial differentiation) in k direction of the curvilinear component.

4.4 Results and Interpretation

Chapter 5

Conclusion and Future direction

Generally in fluid dynamics, the flow resistance coefficient problems can be grouped into two distinct categories: to problems where the flow rate is fixed (known and constant) and other properties of the fluid are given and to those where only the geometry is fixed (known under certain conditions) and fluid properties are given. It is clear that our problem falls into the second category where the geometry is fixed, but uncertainties of the geometry at each cross-section of the flow channel complicate the understanding of dynamics close to the wall. In this work, we proposed a theoretical novel approach to solving and predicting a laminar three-dimensional channel with highly irregular boundaries. Based on the model, we have shown that roughness has an effect on the laminar flow especially when the relative roughness is higher than 10% in general and more importantly when the correlation length is short which corresponds to a rougher surface.

Despite the presence of correlation in the roughness, there is a change in the effective properties.This is due to the fact that we considered an infinite domain in the horizontal plane. The induced viscosity from the wall due to the roughness affects the flow by decreasing the flow rate. In addition, the correlation between the two surfaces implies that the apparent viscosity increases as the surfaces get rougher. It is mathematically shown that for rough surfaces, the volumetric flow rate decreases faster than in the case of smooth surfaces. As a consequence, the Poiseuille number is affected by the surface texture, as well as the flow resistance coefficient. The friction factor increases as the surfaces get rougher which is a result of the dissipative energy on the wall. Our theoretical model demonstrates that one cannot neglect the effect of the roughness for viscous flow in laminar regime even in microchannels.

We conclude by summarizing the key point of our work. First, we have obtained a closed form expression for the effective roughness and the friction factor in terms of the stochastic functions of the wall roughness which enables one to predict viscosity of fluids that would give the same resulting flow as in a smooth channel. Secondly, we have provided a closed-form relation between the relative roughness height and the coefficient of fluid resistance (CFR) which is very important to improve the existing empirical correlations. Moreover, the theoretical and stochastic approach presented here is practical in the sense that the fluid-boundary interface can be represented using a consistent and rigorous representation owing to the parameters availability which describe the surface texture topology. Finally, our approach offers the ability to predict a bulk flow in a plane channel, given the parameters describing the rough wall.

The advantage of this work is to have the following:

- A closed form expression of the effective roughness and the friction factor in terms of the stochastic functions of the wall roughness, which could enable one to predict viscosity of fluids that would give the same outcome as a smooth channel.
- The ability to link the relative roughness height to the Coefficient of Fluid Resistance (CFR) in closed-form expression is very important to improve the existing empirical correlations.
- This theoretical and stochastic approach is practical in a sense that the fluid/boundary interface will not longer be represented arbitrary but instead can have a consistent and rigorous representation due to the availability of the parameters describing the topology of the surface texture.
- The ability of one to predict a bulk flow in a plane channel if given the parameters

describing the rough wall.

We would like to expand this work to life sciences with an application to hemodynamics. It will deals with microcirculation and hemodynamic to investigate how closely cardiovascular disease (CVD) as is related to the structural damage and malfunctioning of blood vessels specifically tissue-blood flow interaction. The fluid and pressure variations near and far from a rough endothelium surface are key to understaning hemodynamic epigenetic mechanisms that modify gene expression and under what conditions this normal genetic adaptation results in a disease maladaptive remodeling state. The study of a two layer-fluid flow consists of a red blood cell region and a cell-free region ineracting with the endothelium surface.

The Jacobian matrix $\mathbf{J}(\xi)$ of the coordinate transformation $\hat{\mathbf{x}} \mapsto \xi$ is given by

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \zeta_1 & \zeta_2 & w \end{pmatrix}, \qquad \zeta_\mathbf{1} = \frac{\partial \hat{\mathbf{z}}}{\partial \xi_\mathbf{1}}, \quad \zeta_\mathbf{2} = \frac{\partial \hat{\mathbf{z}}}{\partial \xi_\mathbf{2}}, \tag{1}$$

$$\frac{\partial \hat{z}}{\partial \xi_1} = (1-\xi_3)\frac{\partial \hat{z}_l}{\partial \xi_1} + \xi_3 \frac{\partial \hat{z}_u}{\partial \xi_1}, \quad \frac{\partial \hat{z}}{\partial \xi_2} = (1-\xi_3)\frac{\partial \hat{z}_l}{\partial \xi_2} + \xi_3 \frac{\partial \hat{z}_u}{\partial \xi_2} \tag{2}$$

and has determinant $\sqrt{g} \equiv \det \mathbf{J} = \mathbf{w}$. The covariant basis vectors are $\partial \xi_i$ are explicitly given by

$$\varepsilon_1 = (1,0,\zeta_1)^\top, \qquad \varepsilon_2 = (0,1,\zeta_2)^\top, \qquad \varepsilon_3 = (0,0,w)^\top. \tag{3}$$

The covariant metric tensor $g_{ij} = \varepsilon_i \cdot \varepsilon_j$ are written as

$$(g_{ij}) = \begin{pmatrix} 1+\zeta_1^2 & \zeta_1\zeta_2 & w\zeta_1 \\ \zeta_1\zeta_2 & 1+\zeta_2^2 & w\zeta_2 \\ w\zeta_1 & w\zeta_2 & w^2 \end{pmatrix}. \tag{4}$$

The contravariant basis $\varepsilon^i = \partial\xi/\partial\hat{\mathbf{x}}_\mathbf{i}$ are

$$\varepsilon^1 = (1,0,-\zeta_1/w)^\top, \quad \varepsilon^2 = (0,1,-\zeta_2/w)^\top, \quad \varepsilon^3 = (0,0,\frac{1}{w})^\top, \tag{5}$$
$$-\zeta_1/w = \frac{\partial\xi_3}{\partial\hat{x}}, \quad -\zeta_2/w = \frac{\partial\xi_3}{\partial\hat{y}},$$

Using the expression that relates the contravariant and covariant basis in chapter **??**, $\varepsilon^\mathbf{i} = \mathbf{g}^\mathbf{ik}\varepsilon_\mathbf{k}$, the evaluation of the contravariant metric tensor gives

$$(g^{ij}) = (\varepsilon^i \cdot \varepsilon^j) = \begin{pmatrix} 1 & 0 & -\zeta_1/w \\ 0 & 1 & -\zeta_2/w \\ -\zeta_1/w & -\zeta_2/w & (1+\zeta_1^2+\zeta_2^2)/w^2 \end{pmatrix} \tag{6}$$

Christoffel symbols of the second kind, denoted here by $\Gamma^i_{k,m}$ and defined by

$$\Gamma^i_{k,m} = \frac{1}{2}g^{ip}\left(\frac{\partial g_{kp}}{\partial\xi_k} + \frac{\partial g_{mp}}{\partial\xi_m} - \frac{\partial g_{mk}}{\partial\xi_p}\right). \tag{7}$$

The computation of the Christoffel gives

$$\Gamma^1_{k,m} = 0 \quad \text{and} \quad \Gamma^2_{k,m} = 0 \quad \text{for any} \quad k,m = 1,2,3. \tag{8}$$

Next, we compute $\Gamma^3_{k,m}$:

$$\Gamma^3_{k,m} = \frac{1}{2} g^{3p} \left(\frac{\partial g_{kp}}{\partial \xi_k} + \frac{\partial g_{mp}}{\partial \xi_m} - \frac{\partial g_{mk}}{\partial \xi_p} \right). \tag{9}$$

The algebraic refinement leads to the following:

$$\Gamma^3_{k,m} = \frac{1}{w} \left(\frac{\partial^2 \hat{z}_l}{\partial \hat{x}_k \partial \hat{x}_m} + \frac{\partial^2 w}{\partial \hat{x}_k \partial \hat{x}_m} \xi_3 \right), \quad \text{for} \quad k,m = 1,2, \tag{10}$$

and, in particular, for $m = 3$, one has:

$$\Gamma^3_{k,3} = \frac{1}{w} \frac{\partial w}{\partial \hat{x}_k}, \quad \text{for} \quad k = 1,2, \quad \text{and} \quad \Gamma^3_{3,3} = 0, \tag{11}$$

which are in accordance with the results found by [**?**].

E^{ij} is the strain-rate tensor which, in the new coordinates system, is defined by:

$$E^{ij} = \frac{1}{2}(g^{ik} \nabla_k v_j + g^{jk} \nabla_k v_i), \tag{12}$$

where $\nabla_k v$ represents the covariant derivative of the contravriant velocity $\mathbf{v}$ (defined forehand) with respect to ξ_k which writes as

$$\nabla_k v_i = \frac{\partial v_i}{\partial \xi_k} + \Gamma^i_{jk} v_j. \tag{13}$$

Here $v_{i;k} \equiv \nabla_k v_i$ which refers to the covariant derivative of the contravariant velocity. the covariant derivative of E^{ij} with respect to ξ_l gives

$$E^{ij}_{;l} = \frac{1}{2} \left(g^{ik} (v_{j;k})_{;l} + g^{jk} (v_{i;k})_{;l} \right). \tag{14}$$

For $l = j$ and consequently (**??**) simplifies to

$$E^{ij}_{;j} = \frac{1}{2} g^{ik} (v_{j;k})_{;j}. \tag{15}$$

Note that

$$(v_{j;k})_{;l} = (v_{j;l})_{;k} = (\frac{\partial v_j}{\partial \xi_l} + \Gamma^j_{il} v_i)_{;k}. \tag{16}$$

We also know that the viscous stress tensor is related to the strain rate by $\tau^{ij} = 2\mu E^{ij}_{;j}$, where μ is the fluid viscosity. Thus we have

$$g^{ij} \frac{\partial \hat{h}}{\partial \xi_j} = \nabla^2 v_i + 2 g^{jk} \Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + g^{jk} \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r. \tag{17}$$

where general expression of Laplacian in curvilinear is given by

$$\nabla^2 = g^{jk} \frac{\partial^2}{\partial \xi_j \partial \xi_k} - g^{jk} \Gamma^m_{jk} \frac{\partial}{\partial \xi_m}, \tag{18}$$

see [**?**] (pp.110) for complete derivation. We further decompose the right hand side of this general expression of Stokes equations as sum of Laplacian in the orthogonal coordinate system along the principal axes and a force term:

$$g^{ij} \frac{\partial \hat{h}}{\partial \xi_j} = \nabla^2_{\perp} v_i + \left(2 g^{jk} \Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + g^{jk} \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r - g^{jk} \Gamma^m_{jk} \frac{\partial v_i}{\partial \xi_m} \right) + \left(g^{jk} \frac{\partial^2 v_i}{\partial \xi_j \partial \xi_k} \right)_{k \neq j}, \tag{19}$$

where

$$\nabla^2_{\perp} v_i = g^{11} \frac{\partial^2 v_i}{\partial \xi_1^2} + g^{22} \frac{\partial^2 v_i}{\partial \xi_2^2} + g^{33} \frac{\partial^2 v_i}{\partial \xi_3^2}.$$

We introduce the conservative force f_i given by

$$f_i = \left(2 g^{jk} \Gamma^i_{jr} \frac{\partial v_r}{\partial \xi_k} + g^{jk} \frac{\partial \Gamma^i_{kr}}{\partial \xi_j} v_r - g^{jk} \Gamma^m_{jk} \frac{\partial v_i}{\partial \xi_m} \right) + \left(g^{jk} \frac{\partial^2 v_i}{\partial \xi_j \partial \xi_k} \right)_{k \neq j}$$

which can be related to the potential Φ via

$$g^{ij}\frac{\partial \hat{h}}{\partial \xi_j} - \frac{\partial \Phi}{\partial \xi_j} = \nabla_{\perp}^2 v_i. \tag{20}$$

The Stokes equations are then simplified to:

$$\begin{cases} g^{ij}\dfrac{\partial \hat{h}}{\partial \xi_j} - \dfrac{\partial \Phi}{\partial \xi_j} = \nabla_{\perp}^2 v_i & (\text{momentum}), \\ \dfrac{1}{g^{1/2}}\dfrac{\partial (g^{1/2} v_i)}{\partial \xi_i} = 0 & (\text{mass} - \text{conservation}). \end{cases} \tag{21}$$

We recall the dimensionless quantity $\partial H/\partial \xi_j = g^{ij}\partial \hat{h}/\partial \xi_j - \partial \Phi/\partial \xi_j$ the overall resulting pressure head on the flow field. Thus the mass-momentum conservation equations write:

$$\begin{cases} \dfrac{\partial H}{\partial \xi_j} = g^{11}\dfrac{\partial^2 v_j}{\partial \xi_1^2} + g^{22}\dfrac{\partial^2 v_j}{\partial \xi_2^2} + g^{33}\dfrac{\partial^2 v_j}{\partial \xi_3^2} & (\text{momentum}) \\ \dfrac{\partial (w v_j)}{\partial \xi_j} = 0 & (\text{mass} - \text{conservation}), \end{cases} \tag{22}$$

and since we are using dimensionless quantities, the μ appears implicitly in the $\mathbf{v}$. The mapping introduced earlier is a one-to-one mapping and it is worth pointing out that by changing coordinate system through a mapping, flow field remains invariant under no other additional external influence. Thus the velocity field and the pressure force remain invariant under this transformation. Also, to reduce the order to infinitesimal, we now consider the truncation of the terms in (**??**) using the expression $\zeta_1 \& \zeta_2$ in (**??**) in the limit where the deformation on $\hat{z}_u$ and on $\hat{z}_l$ are very small owing the finite linear term of ζ_1, ζ_2. Thus we can write metric with an approximation of $g^{33} \approx 1/w^2$, in components form

$$\begin{cases} \dfrac{\partial H}{\partial \xi_j} = \dfrac{\partial^2 v_j}{\partial \xi_1^2} + \dfrac{\partial^2 v_j}{\partial \xi_2^2} + \dfrac{1}{w^2}\dfrac{\partial^2 v_j}{\partial \xi_3^2} & (\text{momentum}), \\ \dfrac{\partial (w v_j)}{\partial \xi_j} = 0 & (\text{mass} - \text{conservation}). \end{cases} \tag{23}$$

In Compact vector form

$$\begin{cases} (\nabla H)^\top = \nabla_h^2 \mathbf{v} + \dfrac{\mathbf{1}}{\mathbf{w^2}} \dfrac{\partial^2 \mathbf{v}}{\partial \xi_3^2} & \text{(momentum)}, \\ \nabla \cdot (w\mathbf{v}) = \mathbf{0} & \text{(mass} - \text{conservation)}. \end{cases} \tag{24}$$

where ∇_h^2 represents the 2-D Lapacian in the horizontal plane i.e (ξ_1, ξ_2 -plane) rewriting conveniently the latter, we obtain

$$\begin{cases} w^2 \nabla_h^2 \mathbf{v} + \dfrac{\partial^2 \mathbf{v}}{\partial \xi_3^2} = (\mathbf{w^2} \nabla \mathbf{H})^\top, \\ \nabla \cdot (w\mathbf{v}) = \mathbf{0}. \end{cases} \tag{25}$$

This system of equations is stochastic and we therefore need to quantify the solution by computing the first and the second moments.

Ensemble Average

We introduce Reynolds decomposition of all the intrinsic random variables:

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}', \qquad H = \langle H \rangle + H',$$
$$w = \langle w \rangle + w', \qquad \text{where} \qquad \langle w \rangle = 1,$$

where brackets $\langle \cdot \rangle$ represent the ensemble average or mathematical expectation of the intrinsic variables, while the primed quantities indicate the zero-mean fluctuations.

Substituting these decomposed expressions in the momentum and continuity equations while neglecting the higher order moment terms (third, fourth..), and taking the ensemble average, we obtain:

$$\begin{cases} \nabla^2\langle\mathbf{v}\rangle + \boldsymbol{\sigma_w^2}\nabla_{\mathbf{h}}^{\mathbf{2}}\langle\mathbf{v}\rangle + \mathbf{2}\langle\mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle = \nabla\langle\mathbf{H}\rangle + \boldsymbol{\sigma_w^2}\nabla\langle\mathbf{H}\rangle + \mathbf{2}\langle\mathbf{w}'\nabla\mathbf{H}'\rangle, \\ \nabla\cdot\langle\mathbf{v}\rangle + \nabla\cdot\langle w'\mathbf{v}'\rangle = 0. \end{cases} \tag{26}$$

We subtract equations (**??**) from (**??**), and multiply the obtained expressions by $w'(\xi)$ and then take the ensemble average. The expressions obtained are:

$$\begin{cases} 2\sigma_\varepsilon^2\nabla_h^2\langle\mathbf{v}\rangle + \langle\mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle + \dfrac{\partial^{\mathbf{2}}\mathbf{C_{wv}}}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}} = \mathbf{2}\boldsymbol{\sigma_w^2}\nabla\langle\mathbf{H}\rangle + \langle\mathbf{w}'\nabla\mathbf{H}'\rangle, \\ \langle\nabla\cdot\mathbf{v}\rangle + \nabla\cdot\mathbf{C_{wv}} = \mathbf{0}. \end{cases} \tag{27}$$

Similar algebraic technique is perfomed; the result is being multiplied by $w'(\zeta)$ at another location ζ and we then take the expectation to obtain the covariance C_w and cross-covariance $C_{w\mathbf{v}}$:

$$\begin{cases} 2C_w\nabla_h^2\langle\mathbf{v}\rangle + \nabla^{\mathbf{2}}\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle\mathbf{H}\rangle + \nabla\mathbf{C_{wH}}, \\ \nabla\cdot\mathbf{C_{wv}} = 0. \end{cases} \tag{28}$$

Putting together and re-writing these expressions (**??**) – (**??**) as a system of equations gives:

$$\begin{cases} \nabla^2\langle \mathbf{v}\rangle + \sigma_\varepsilon^{\mathbf{2}}\nabla_{\mathbf{h}}^{\mathbf{2}}\langle \mathbf{v}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle = \nabla\langle \mathbf{H}\rangle + \sigma_\varepsilon^{\mathbf{2}}\nabla\langle \mathbf{H}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2\sigma_\varepsilon^2\nabla_h^2\langle \mathbf{v}\rangle + \langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle + \dfrac{\partial^{\mathbf{2}}\mathbf{C_{wv}}}{\partial\xi_{\mathbf{3}}^{\mathbf{2}}} = \mathbf{2}\sigma_{\mathbf{w}}^{\mathbf{2}}\nabla\langle \mathbf{H}\rangle + \langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2C_w\nabla_h^2\langle \mathbf{v}\rangle + \nabla^{\mathbf{2}}\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle \mathbf{H}\rangle + \nabla\mathbf{C_{wH}}, \\ \langle \nabla\cdot\mathbf{v}\rangle + \nabla\cdot\mathbf{C_{wv}} = \mathbf{0}, \\ \nabla\cdot\mathbf{C_{wv}} = 0. \end{cases} \tag{29}$$

We represent each intrinsic variable as an infinite sum as follows:

$$\begin{aligned} \mathbf{v} &= \sum_{k=0}^{\infty}\sigma_w^{2k}\mathbf{v}^{(\mathbf{k})}, & \mathbf{H} &= \sum_{\mathbf{k=0}}^{\infty}\sigma_{\mathbf{w}}^{\mathbf{2k}}\mathbf{H}^{(\mathbf{k})}, \\ \langle \mathbf{v}\rangle &= \sum_{\mathbf{k=0}}^{\infty}\sigma_{\mathbf{w}}^{\mathbf{2k}}\langle \mathbf{v}^{(\mathbf{k})}\rangle, & \langle \mathbf{H}\rangle &= \sum_{\mathbf{k=0}}^{\infty}\sigma_{\mathbf{w}}^{\mathbf{2k}}\langle \mathbf{H}^{(\mathbf{k})}\rangle, \end{aligned} \tag{30}$$

Replacing and collecting the same order terms of σ_ε, one gets for zero order (σ_w^0):

$$\begin{cases} \nabla^2\langle \mathbf{v}^{(\mathbf{0})}\rangle = \nabla\langle \mathbf{H}^{(\mathbf{0})}\rangle, \\ \nabla\cdot\langle \mathbf{v}^{(\mathbf{0})}\rangle = 0. \end{cases} \tag{31}$$

We have for second order i.e (σ_w^2):

$$\begin{cases} \nabla^2\langle \mathbf{v}^{(1)}\rangle + \nabla_{\mathbf{h}}^{\mathbf{2}}\langle \mathbf{v}^{(0)}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle = \nabla\langle \mathbf{H}^{(1)}\rangle + \nabla\langle \mathbf{H}^{(0)}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2\nabla_h^2\langle \mathbf{v}^{(0)}\rangle + \langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle + \dfrac{\partial^{\mathbf{2}}\mathbf{C_{wv}}}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}} = \mathbf{2}\nabla\langle \mathbf{H}^{(0)}\rangle + \langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2C_w\nabla_h^2\langle \mathbf{v}^{(0)}\rangle + \nabla^{\mathbf{2}}\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle \mathbf{H}^{(0)}\rangle + \nabla \mathbf{C_{wH}}, \\ \nabla\cdot\langle \mathbf{v}^{(1)}\rangle = 0. \end{cases} \tag{32}$$

Rearranging (**??**) & (**??**) for convenience,we have

$$\begin{cases} \nabla^2\langle \mathbf{v}^{(0)}\rangle = \nabla\langle \mathbf{H}^{(0)}\rangle, \\ \nabla\cdot\langle \mathbf{v}^{(0)}\rangle = 0, \\ \nabla^2\langle \mathbf{v}^{(1)}\rangle + \nabla_{\mathbf{h}}^{\mathbf{2}}\langle \mathbf{v}^{(0)}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle = \nabla\langle \mathbf{H}^{(1)}\rangle + \nabla\langle \mathbf{H}^{(0)}\rangle + \mathbf{2}\langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2\nabla_h^2\langle \mathbf{v}^{(0)}\rangle + \langle \mathbf{w}'\nabla_{\mathbf{h}}^{\mathbf{2}}\mathbf{v}'\rangle + \dfrac{\partial^{\mathbf{2}}\mathbf{C_{wv}}}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}} = \mathbf{2}\nabla\langle \mathbf{H}^{(0)}\rangle + \langle \mathbf{w}'\nabla \mathbf{H}'\rangle, \\ 2C_w\nabla_h^2\langle \mathbf{v}^{(0)}\rangle + \nabla^{\mathbf{2}}\mathbf{C_{wv}} = \mathbf{2C_w}\nabla\langle \mathbf{H}^{(0)}\rangle + \nabla \mathbf{C_{wH}}, \\ \nabla\cdot\langle \mathbf{v}^{(1)}\rangle = 0. \end{cases} \tag{33}$$

Multiplying the 4th equation by minus two then adding it to the third equation of (**??**) to eliminate some terms in order to solve the system. The resulting system writes:

$$
\begin{cases}
\nabla^2 \langle \mathbf{v^{(0)}} \rangle = \nabla \langle \mathbf{H^{(0)}} \rangle, \\
\nabla \cdot \langle \mathbf{v^{(0)}} \rangle = 0, \\
\nabla^2 \langle \mathbf{v^{(1)}} \rangle - \mathbf{3} \nabla^{\mathbf{2}}_{\mathbf{h}} \langle \mathbf{v^{(0)}} \rangle - \mathbf{2} \dfrac{\partial^{\mathbf{2}} \mathbf{C_{wv}}}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}} = \nabla \langle \mathbf{H^{(1)}} \rangle - \mathbf{3} \nabla \langle \mathbf{H^{(0)}} \rangle, \\
2C_w \nabla^2_h \langle \mathbf{v^{(0)}} \rangle + \nabla^{\mathbf{2}} \mathbf{C_{wv}} = \mathbf{2C_w} \nabla \langle \mathbf{H^{(0)}} \rangle + \nabla \mathbf{C_{wH}}, \\
\nabla \cdot \langle \mathbf{v^{(1)}} \rangle = 0.
\end{cases}
\tag{34}
$$

For a fully developed and unidirectional Poiseuille flow, we have:

$$
\nabla^2_h \langle \mathbf{v^{(0)}} \rangle = \mathbf{0} = \nabla \langle \mathbf{H^{(1)}} \rangle,
$$

$$
\langle \mathbf{v^{(0)}} \rangle \equiv (\langle \mathbf{v_1^{(0)}} \rangle, \mathbf{0}, \mathbf{0}) = \frac{\mathbf{J}}{\mathbf{2}} \xi_{\mathbf{3}} (\mathbf{1} - \xi_{\mathbf{3}}), \qquad -\mathbf{J} \equiv \nabla \langle \mathbf{H} \rangle = \frac{\partial \langle \mathbf{H^{(0)}} \rangle}{\partial \xi_{\mathbf{1}}}.
$$

Substituting the latter in the system from note-31, we have :

$$
\begin{cases}
\nabla^2 \langle \mathbf{v^{(0)}} \rangle = -\mathbf{J}, \\
\nabla^2 \langle \mathbf{v^{(1)}} \rangle = \mathbf{3J} + \mathbf{2} \dfrac{\partial^{\mathbf{2}} \mathbf{C_{wv}}}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}}, \\
\nabla^2 C_{w\mathbf{v}} = -2JC_w + \nabla C_{wH}, \\
\nabla \cdot \mathbf{C_{wv}} = 0.
\end{cases}
\tag{35}
$$

To obtain the expression for $\nabla^2 \langle \mathbf{v^{(1)}} \rangle$ in the second equation of (**??**), it is necessary to solve the third equation in terms of J and C_w. In particular, for the special case where $C_{wH} = 0$, we have:

$$\begin{cases} \nabla^2 C_{w\mathbf{v}} = -2C_w J \quad |\xi_1| < \infty, \quad |\xi_2| < \infty, \quad \text{and} \quad 0 \le \xi_3 \le 1, \\ C_{w\mathbf{v}}(\xi_1, \xi_2, 0) = 0, \\ C_{w\mathbf{v}}(\xi_1, \xi_2, 1) = 0. \end{cases} \tag{36}$$

Let G be the Green's function associated with (**??**) which is defined as follows:

$$\begin{cases} \nabla^2 G = \delta(\xi - \zeta) \quad |\xi_{\mathbf{1}}| < \infty, \quad |\xi_{\mathbf{2}}| < \infty, \quad \text{and} \quad \mathbf{0} \le \xi_{\mathbf{3}} \le \mathbf{1}, \\ G(\xi_1, \xi_2, 0, \zeta_1, \zeta_2, \zeta_3) = 0, \quad |\xi_1| < \infty, \quad |\xi_2| < \infty, \\ G(\xi_1, \xi_2, 1, \zeta_1, \zeta_2, \zeta_3) = 0, \quad |\xi_1| < \infty, \quad |\xi_2| < \infty. \end{cases} \tag{37}$$

The Green's function is given by

$$\begin{aligned} G(\xi, \zeta) =& \frac{1}{4\pi} \sum_{-\infty}^{\infty} \frac{1}{\sqrt{(\xi_1 - \zeta_1)^2 + (\xi_2 - \zeta_2)^2 + (\xi_3 - \zeta_3 - 2n)^2}} \\ & - \frac{1}{\sqrt{(\xi_1 - \zeta_1)^2 + (\xi_2 - \zeta_2)^2 + (\xi_3 + \zeta_3 - 2n)^2}}. \end{aligned}$$

[**?**]. Let $\rho_w(\|\zeta - \eta\|)$ be the correlation function of $C_w(\zeta, \eta)$:

$$C_w(\zeta, \eta) = \sigma_{\mathbf{w}}^{\mathbf{2}} \rho_{\mathbf{w}}(\|\zeta - \eta\|). \tag{38}$$

From (**??**)–(**??**),

$$\nabla^2 \langle \mathbf{v}^{(\mathbf{1})} \rangle = \mathbf{3J} - \mathbf{4J} \int_{\blacksquare} \rho_{\mathbf{w}}(\|\zeta - \eta\|) \frac{\partial^{\mathbf{2}} \mathbf{G}(\xi, \eta)}{\partial \xi_{\mathbf{3}}^{\mathbf{2}}} \mathbf{d}\eta. \tag{39}$$

In particular, up to the first order we have:

$$\nabla^2 \langle \mathbf{v} \rangle = \nabla^{\mathbf{2}} \left(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{1}} \sigma_{\mathbf{w}}^{\mathbf{2k}} \langle \mathbf{v}^{(\mathbf{k})} \rangle \right) = \nabla^{\mathbf{2}} \langle \mathbf{v}^{(\mathbf{0})} \rangle + \sigma_{\mathbf{w}}^{\mathbf{2}} \nabla^{\mathbf{2}} \langle \mathbf{v}^{(\mathbf{1})} \rangle. \tag{40}$$

It follows from (**??**) that

$$\nabla^2\langle\mathbf{v}\rangle = -\mathbf{J} + \mathbf{3}\sigma_\mathbf{w}^\mathbf{2}\mathbf{J} - \mathbf{4}\sigma_\mathbf{w}^\mathbf{2}\mathbf{J}\int_{\blacksquare}\rho_\mathbf{w}(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta, \tag{41}$$

where $\sigma_w^2\left(3J - 4J\int_\Omega \rho_w(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta\right)$ is the correction term.

By analogy to the classical Stokes flow, one can express the dimensionless effective viscosity as

$$\mu_{\text{eff}} = \frac{1}{1-\sigma_w^2\left(3 - 4\int_\Omega \rho_w(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta\right)}, \tag{42}$$

where

$$\int_\Omega \rho_w(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta = \int_\mathbf{0}^\mathbf{1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_\mathbf{w}(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta_\mathbf{1}\mathbf{d}\eta_\mathbf{2}\mathbf{d}\eta_\mathbf{3}. \tag{43}$$

and

$$\begin{aligned}\frac{\partial^2 G}{\partial\xi_3^2} = -\frac{1}{4\pi}\sum_{-\infty}^{\infty} & \frac{1}{\left((\xi_3+\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{3}{2}}} \\ & -\frac{1}{\left((\xi_3-\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{3}{2}}} \\ & -\frac{3\,(2\xi_3+2\eta_3-4n)^2}{4\left((\xi_3+\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{5}{2}}} \\ & +\frac{3\,(2\xi_3-2\eta_3-4n)^2}{4\left((\xi_3-\eta_3-2n)^2+(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2\right)^{\frac{5}{2}}}.\end{aligned} \tag{44}$$

We obtain the general form of the effective viscosity:

$$\mu_{\text{eff}} = \frac{1}{1-\sigma_w^2\left(3 - 4\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_w(\|\zeta-\eta\|)\frac{\partial^\mathbf{2}\mathbf{G}(\xi,\eta)}{\partial\xi_\mathbf{3}^\mathbf{2}}\mathbf{d}\eta_\mathbf{1}\mathbf{d}\eta_\mathbf{2}\mathbf{d}\eta_\mathbf{3}\right)}. \tag{45}$$

We also check if the variation of the μ_{eff} is sensitive to the variation in ξ_3 direction by computing the average effective viscosity which is given by:

$$\langle \mu_{\text{eff}} \rangle = \int_0^1 \left(\frac{1}{1-\sigma_w^2\left(3-4\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_w(\|\zeta-\eta\|)\frac{\partial^2\mathbf{G}(\xi,\eta)}{\partial\xi_3^2}\mathbf{d\eta_1 d\eta_2 d\eta_3}\right)} \right) d\xi_3. \tag{46}$$

Consequently the ratio between the effective and the original flowrates can be written as

$$\frac{Q_{eff}}{Q} = 1-\sigma_w^2\left(3-4\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_w(\|\zeta-\eta\|)\frac{\partial^2\mathbf{G}(\xi,\eta)}{\partial\xi_3^2}\mathbf{d\eta_1 d\eta_2 d\eta_3}\right). \tag{47}$$

The resistance in the flow system is defined as the ratio of the pressure drop over the volumetric flow rate:

$$R_F = \frac{\Delta p}{Q}. \tag{48}$$

We divide both sides by the specific weight γ and the characteristic length L_{ξ_1} (stream-wise direction) of the channel to obtain :

$$R_F/(\gamma L_{\xi_1}) = \frac{\Delta p}{\gamma L_{\xi_1} Q}. \tag{49}$$

We define the ratio R_F/L_{ξ_1} to be the coefficient of the fluid resistance (CFR). and we rewrite the expression as follows:

$$CFR = \frac{\frac{\Delta p}{\gamma L_{\xi_1}}}{Q} = \frac{12\mu_{eff}}{\langle d\rangle^3}. \tag{50}$$

Further algebraic refinement leads to a simplified expression of CFR with the dimensionless average channel height $\langle d\rangle$ being unit:

$$CFR = 12\mu_{eff}. \tag{51}$$

The effective viscosity's expression obtained (**??**) shows that there exists a positive real number

$K_{\sigma_w} \geq 1$ such that:

$$\mu_{eff} = K_{\sigma_w} \mu \tag{52}$$

Note here that K_{σ_w} is the ratio of μ_{eff} over μ. To carry our analysis further, we multiply both sides of the expression of (**??**) by $1/\bar{\mathbf{V}}$ and using the expression (**??**), we obtain:

$$CFR/\bar{\mathbf{V}} = \frac{24 K_{\sigma_w} \mu}{2\bar{\mathbf{V}}}, \tag{53}$$

where $\bar{\mathbf{V}}$ is the average velocity defined as $2/3\langle \mathbf{v_1} \rangle_{\mathbf{max}}$.

Recall $CFR/\bar{\mathbf{V}}$ the rough channel friction factor f_w. We can then express this parameter in term of σ_w and the Reynold's number Re after some algebraic refinements:

$$f_w = \frac{24 K_{\sigma_w}}{Re} \tag{54}$$

Appendix B

$$(p)_{;i} = \frac{\partial p}{\partial x^i}, \quad \text{for a given scalar p,} \tag{55}$$

$$(u^i)_{;j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jm} u^m, \quad \text{for a given vector } u^i, \tag{56}$$

$$(t^i_j)_{;k} = (t^i_j)_{;k} - \Gamma^m_{jk} t^i_m + \Gamma^i_{km} t^m_i \quad \text{for a given tensor.} \tag{57}$$

The energy equation can be written as :

$$(v_i T)_{;i} = (\alpha^i_j T_{,j})_{;i} + \mu \Phi \tag{58}$$