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The *M* waves emitted by an expanding phase boundary create a "lacuna"

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The *M* waves introduced by Burridge and Willis (1969) are emitted by the surface of a self-similarly expanding elliptical crack, and they give Rayleigh waves at the corresponding crack speed. In the analysis for the self-similarly expanding spherical inclusion with phase change (dynamic Eshelby problem) the *M* waves are related to the waves obtained on the basis of the dynamic Green's function containing the contribution from the latest wavelets emitted by the expanding boundary of phase discontinuity, and they satisfy the Hadamard jump conditions for compatibility and linear momentum across the moving phase boundary of discontinuity. In the interior of the expanding inclusion they create a "lacuna" with zero particle velocity by canceling the effect of the *P* and *S*. It is shown that the "lacuna" and Eshelby properties are also valid for a Newtonian fluid undergoing phase change in a self-similarly expanding ellipsoidal region of a fluid with different viscosity.

I. Introduction

The seminal paper by Burridge and Willis [1] introduced the *M* slowness surface for the emission of the *M* waves by the surface of a self-similarly elliptically expanding crack [2]. Subsequently, following the same method, Ni and Markenscoff [3] solved the self-similarly expanding ellipsoidal inclusion with uniform transformation strain, which constitutes the dynamic generalization of the Eshelby inclusion problem [4],[5]. It emits, *P, S, and M* waves with corresponding slowness surfaces, with the *M* waves emitted by the surface of the expanding ellipsoid [1],[3] . In this paper the connection is made of the *M* waves to the waves emitted by the material due to the Green's function during the course of the expansion of the inclusion up to the current time [6], so that their physical meaning becomes evident. We show the important physical properties of the *M* waves to satisfy the Hadamard jump conditions of conservation of linear momentum and compatibility across the moving surface of discontinuity. Moreover, the *M* waves cancel the particle velocity due to the *P* and *S* in the interior domain of a self-similarly expanding ellipsoidal inclusion with uniform transformation strain [7] creating a "lacuna" [1] (a "traveling zone of absolute quiet" [8]) with zero particle velocity (and hence zero kinetic energy), and they give the static Eshelby inclusion field in a particular limit [3]. In the Eshelby inclusion [4] seminal paper the applicability of the property for the solid ellipsoid to a Newtonian fluid was included, and here we show that the dynamic generalization of the Eshelby problem is also valid for a Newtonian fluid undergoing phase change, generating a "lacuna" as well. Mathematically " lacuna" is a topological property of a system of hyperbolic differential equations [9], noting also that Petrovsky lacunae are occurring inside the cusps in anisotropic wave propagation [8],[10].

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II. The self-similarly expanding ellipsoidal inclusion with transformation strain and the waves emitted by the slowness surfaces *P*, *S*, and *M*.

We present first the fundamental derivation of the slowness surface for the *M* waves emitted by an expanding ellipsoidal inclusion containing a region of phase change with uniform eigenstrain. We consider the dynamic generalization of the Eshelby inclusion problem [4], which is a self-similarly expanding ellipsoidal inclusion with uniform transformation strain [3] expanding from zero dimension with constant axes speeds,

$$\varepsilon^*_{lm}(\vec{x},t) = \varepsilon^*_{lm} H(t-(s_r^2 x_r^2)^{1/2}) \tag{1}$$

where x_i is the position vector, t the time, $1/s_1, 1/s_2, 1/s_3$, are the subsonic *constant* axes speeds (the inverse of the slowness s_i) of the expanding ellipsoid in the argument of the Heaviside step function $H(.)$.The governing equations express the conservation of linear momentum in elastodynamics , where C_{ijkl} are the elastic stiffness tensor and ρ is the density are [3] ,[4],

$$\rho \frac{\partial^2 u_j}{\partial t^2} - C_{jklm} \frac{\partial^2 u_l}{\partial x_k \partial x_m} = -C_{jklm} \varepsilon^*_{lm} \frac{\partial}{\partial x_k} H(t-(s_r^2 x_r^2)^{1/2}) \tag{2}$$

with initial conditions

$$\vec{u}(\vec{x},0) = 0 \ \text{ and } \ \frac{\partial \vec{u}}{\partial t}(\vec{x},0) = 0 \tag{3a}$$

and applied zero radiation conditions at infinity

$$\lim_{r\to\infty} \vec{u}(r,t) = 0 \ \text{ and } \ \lim_{r\to\infty} \frac{\partial \vec{u}}{\partial t}(r,t) = 0 \tag{3b}$$

In terms of the Navier operator of elastodynamics

$$L_{jl}(\frac{\partial}{\partial t}, \nabla) = \rho \frac{\partial^2}{\partial t^2} \delta_{jl} - C_{jklm} \frac{\partial^2}{\partial x_k \partial x_m} \tag{4}$$

with δ_{ij} denoting the Kroenecker delta, the system (2) is written as

$$L_{jl}(\frac{\partial}{\partial t}, \nabla) u_l = -K_j(\nabla) H(t^2 - s_r^2 x_r^2) \quad \text{with} \quad K_j(\nabla) = C_{jklm} \varepsilon^*_{lm} \frac{\partial}{\partial x_k} \tag{5}$$

The system (2) was solved by the Radon transform (e.g., Willis, 1973 [11], Wang and Achenbach [12]) in Ni and Markenscoff [7]. Here we demonstrate the method of the *M* operator [1] (with *M* following alphabetically the Navier *L* operator), introducing

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$$M\{\frac{\partial}{\partial t},\nabla\}=\frac{\partial^2}{\partial t^2}-\frac{1}{s_r^2}\frac{\partial^2}{\partial x_r^2} \tag{6}$$

which satisfies the identity

$$M^2\{\frac{\partial}{\partial t},\nabla\}H(t^2-s_r^2x_r^2)=\frac{8\pi}{s_1s_2s_3}\delta(t)\delta(x) \tag{7}$$

so that, as in [1], with the application of Duhamel's principle, the governing system of equations (2) becomes a *homogeneous system of equations*

$$M^2\{\frac{\partial}{\partial t},\nabla\}L\{\frac{\partial}{\partial t},\nabla\}u=0 \qquad t>0 \tag{8}$$

with the *inhomogeneous initial conditions*

$$\partial^k u_i(0)/\partial t^k=0 \text{ for } k=1,2,3,4 \text{ and } \partial^5 u_i(0)/\partial t^5=-8\pi/(\rho s_1s_2s_3)K_i\delta(x) \tag{9}$$

The solution of the system (8) under the initial conditions (9) consists of contributions from the poles of the differential operators L and M, which give waves emanated from the *three slowness surfaces*, S^P, S^S, S^M (the first two corresponding to the poles of L, and the third one to those of M), which, for an isotropic material with $C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$, where λ and μ denote the Lame' constants, are:

$$S^P:\ |\xi|^2=\frac{\rho}{\lambda+2\mu}=\frac{1}{a^2},\ S^S:\ |\xi|^2=\frac{\rho}{\mu}=\frac{1}{b^2},\ S^M:\ \frac{\xi_1^2}{s_1^2}+\frac{\xi_2^2}{s_3^2}+\frac{\xi_3^2}{s_3^2}=1 \tag{10}$$

with a and b denoting the pressure and shear wave speeds.

The solution of (8) with (9) is obtained following the procedure in [1] in terms of integrals over the slowness surfaces [13], rather than over the unit sphere as in [7]. The solution for the displacement is obtained in [13], as

$$\begin{aligned}u_l(\mathbf{x},t)&=\frac{1}{\pi s_1s_2s_3}\sum\int_{S^L}\frac{N_{lk}(1,\xi)K_k(\xi)}{\partial D/\partial\gamma(1,\xi)M^2(1,\xi)}(t+\xi\cdot\mathbf{x})\mathrm{sgn}(t+\xi\cdot\mathbf{x})\frac{dS}{\left|\nabla\gamma^L(\xi)\right|}\\&-\frac{1}{4\pi s_1s_2s_3}\int_{S^M}\left\{\rho L_{lk}^{-2}(1,\xi)K_k(\xi)\left[(t+\xi\cdot x)\mathrm{sgn}(t+\xi\cdot x)-(t-\xi\cdot x)\mathrm{sgn}(t-\xi\cdot x)\right]\right.\\&\left.+L_{lk}^{-1}(1,\xi)K_k(\xi)(\xi\cdot x)H(t-|\xi\cdot x|)\right\}\frac{dS}{\left|\nabla\gamma^M(\xi)\right|}\end{aligned} \tag{11}$$

The above expression is valid for general anisotropy. For isotropy we have,

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$$K_k(\vec{\xi}) = C_{jklm}\varepsilon^*_{lm}\xi_j = \lambda\varepsilon^*_{mm}\delta_{jk}\xi_j + \mu(\varepsilon^*_{jk} + \varepsilon^*_{kj})\xi_j$$

$N_{ij}(\gamma,\xi_i)$ and $D(\gamma,\xi_i)$ are defined by $L^{-1}{}_{ij}(\gamma,\xi_i) = \dfrac{N_{ij}(\gamma,\xi_i)}{D}$

with $N_{lj}(\gamma,\xi) = \rho[(\gamma^2 - a^2|\xi|^2)\delta_{jl} + (a^2 - b^2)\xi_j\xi_l]$ and $D(\gamma,\xi) = \rho^2(\gamma^2 - b^2|\xi|^2)(\gamma^2 - a^2|\xi|^2)$

$$\gamma^{\mathrm{M}}(\vec{\xi}) = \pm(\xi_k^2/s_k^2)^{1/2}, \quad \left|\nabla\gamma(\vec{\xi})\right| = (\xi_k^2/s_k^4)^{1/2} \tag{12}$$

For a self-similarly expanding spherical inclusion with uniform general transformation strain the expressions were evaluated by Ni and Markenscoff in [14], and, by a different method, the solution for dilatational eigenstrain was obtained in [15].

Here, we will provide the application and illustration of these above opaque expressions in the simple example for the self-similarly expanding spherical inclusion with dilatational eigenstrain, and show the physical structure and the important properties of the *M* waves.

III. The *M* waves emitted by the expanding spherical inclusion and their properties

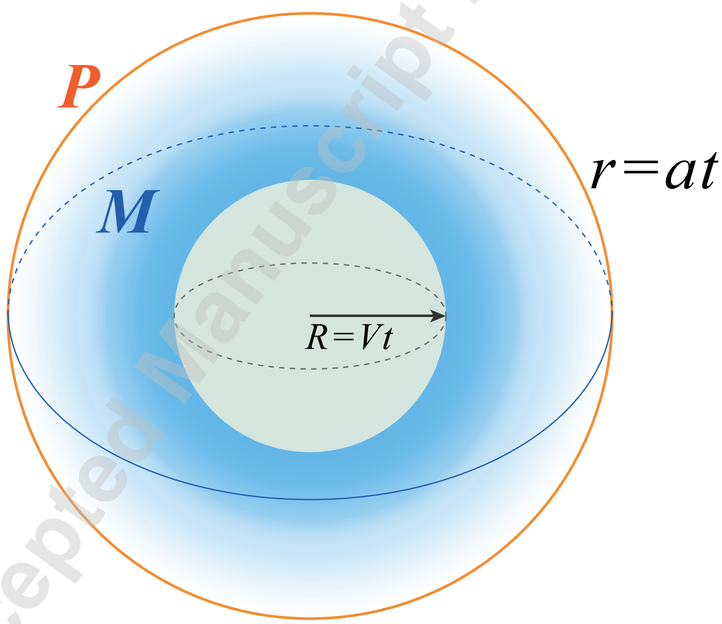

Figure 1: A spherical inclusion with uniform eigenstrain ε^*_{ij} expanding self-similarly as $R=Vt$, with $(V<a)$, emits *M* waves from the moving boundary of phase change. The interior of a self-similarly expanding anisotropic ellipsoidal inclusion with phase change is a "lacuna" with zero particle velocity ("they are traveling zones of absolute quiet"), a property also valid for a Newtonian fluid with different viscosity [19].

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In this section the structure of the M waves is presented showing the connection to the formulation of an expanding inclusion in terms of the Green's function that emits P and S waves. Markenscoff and Ni [6] analyzed a spherical inclusion with dilatational transformation strain expanding spherically in general motion $R = R_0 + l(t)$ by applying the expression given by Willis [16] in terms of the dynamic Green's function (where, in eqtn (2), the RHS is a delta function for a point load), and integrating it over an expanding region as

$$u_i(\boldsymbol{x},t) = \int_{-\infty}^{\infty} dt' \int_{V(t)} C_{jklm}\varepsilon^*_{lm}(\boldsymbol{x}';t')\frac{\partial G_{ij}(\boldsymbol{x}-\boldsymbol{x}',t-t')}{\partial x_k}\, dV = \int_{-\infty}^{\infty} dt' \int_{S(t)} C_{jklm}\varepsilon^*_{lm}(\boldsymbol{x}';t')n_k(\boldsymbol{x})G_{ij}(\boldsymbol{x}-\boldsymbol{x}',t-t')\, dS. \quad (13)$$

Naturally, it should be expected that, when applied to self-similar expansion, the solution in [6] based on integrating the dynamic Green's function over time on an expanding phase boundary should coincide with the one based on the self-similar approach, that produces waves emitted by the *P, S* and *M* slowness surfaces [3],[7],[13],[14]. The nature of the *M* waves was seen from the analysis in [1], as a slowness surface, where the governing equation for the expanding inclusion is made homogeneous by the application of the *M* operator as in Burridge and Willis [1], and an additional slowness surface to the *P* and *S* ones is generated [1]. The *P*, *S* and *M* *together* satisfy zero initial and radiation conditions for the self-similar problem, as required by the physical problem solved. An important property of the self-similarly expanding ellipsoidal inclusion is that the system of governing eqtns (1) is stretch-invariant and becomes elliptic in a range [7]. Based on analytic properties, from the zero initial conditions (3), through the Cauchy-Kowalewskaya theorem, follows that the particle velocity is zero in the whole region of analyticity which is the internal domain [17], a fact confirmed by the explicit calculation where it was shown that the particle velocity due to the *P, S* and *M* slowness surfaces cancel each other [7]. This consideration came as a reply to a question asked by J.R. Rice at the Broberg Symposium in Sweden (2015) on what follows from dimensional analysis: from dimensional analysis and analytic properties alone, from the Cauchy-Kowalewskaya theorem follows the lacuna property (also obtainable from [9]), and, from it, since there is no inertia term in the interior domain, Eshelby's proof is extended, by showing that the elliptic equations reduce to a well posed problem (with D. Gintides, in preparation). Thus, the Eshelby constant strain/stress property is valid in the interior domain of the self-similarly expanding ellipsoidal inclusion, in agreement with the explicit derivation in [7], with the above arguments valid both in isotropy and in anisotropy.

The governing equation (2) for a spherically inclusion with dilational eigenstrain expanding self-similarly according to $R=Vt$, (subsonically with $V<a$, $a^2 = (\lambda + 2\mu)/\rho$), takes the form in spherical coordinates

$$(\lambda+2\mu)\partial/\partial r\{1/r^2(r^2u_r)\} - \rho\partial^2 u_r/\partial t^2 = (3\lambda+2\mu)\varepsilon^*\partial/\partial r\{H(Vt-r)\} \quad (14)$$

The M waves that arise in the self-similar expansion can be shown to be produced from the contributions (in the term u_r^{III} in [6] in eqtns (2.20)-(2.27)) of the wavelets emitted by the expanding surface of the phase boundary at the latest time τ_2 (and traveling at the P wave speed a) (Figure 2). The wavelets emitted from the earlier position $R(\tau_1)$ of the expanding inclusion have the time to reach the field point (r,t) if τ_1 satisfies $r + R(\tau_1) = a(t-\tau_1)$, so that for $R(t') = Vt'$, $\tau_1 = (at-r)/(V+a)$, and these are P waves . The wavelets emitted by the latest

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position $R = V\tau_2$ that have the time to reach the field point (r,t) at time that satisfies $|r - R(\tau_2)| = a(t-\tau_2)$, which, due to the absolute value, has different solution for points interior or exterior to the inclusion at $R = V\tau_2$, i.e.,

$$\tau_2 = (at-r)/(a-V) \text{ for } r \geq V\tau_2 , \quad \text{and} \quad \tau_2 = (at+r)/(a+V) \text{ for } r \leq V\tau_2 , \qquad (15)$$

and these wavelets produce the discontinuity in the derivatives of the M waves across the moving boundary, which are obtained below.

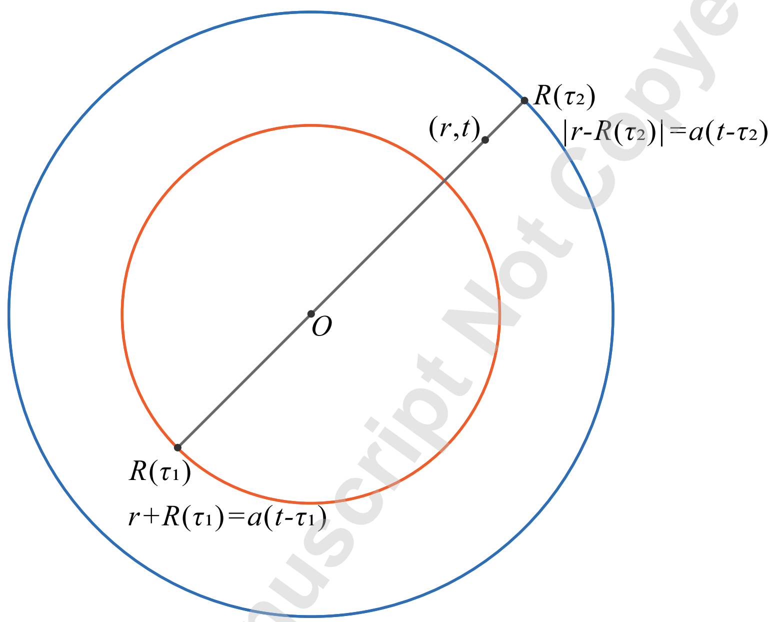

Figure 2. A spherical inclusion expands self-similarly starting from zero dimension. The contributions at the field point (**r**,t) due to wavelets emitted by the expanding inclusion according to $R(t') = Vt'$ consists of the integral of the contributions from the emitted wavelets at the earliest time τ_1 (the inclusion being at $\mathbf{R}(\tau_1)$) to the latest time τ_2 (the inclusion being at $\mathbf{R}(\tau_2)$), so that they had the time to reach the field point $(\boldsymbol{r},t)$, satisfying respectively the above equations. The M waves contain the contributions from the latest time, which (due to the absolute value) create jumps in the particle velocity and the traction across the moving boundary satisfying the Hadamard jump conditions.

The displacement field $u_r(r,t)$ of the self-similarly expanding spherical inclusion $(R(t)=Vt)$ with dilatational transformation strain $\varepsilon^*_{ij} = \varepsilon^*\delta_{ij}$ was evaluated in [14] based on the P and M slowness surfaces. We present here the solution of eqtn (2) (with (3)) as superposition of the P and M contributions, $u_r = u_r^{(P)} + u_r^{(M)}$, with the P contributions being

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$$u_r^{(P)} = 2K\varepsilon^* V^3 r / \{a\rho(a^2 - V^2)^2\} H(at - r) - K\varepsilon^* V^3 r\left(a^2(t/r)^3 - 3(t/r)\right) / \rho(a^2 - V^2)^2 H(r - at)$$
(16)

and the M contributions

$$u_r^{(M)} = K\varepsilon^*(a^2 - 3V^2) r / \{\rho(a^2 - V^2)^2\} H(Vt - r) + K\varepsilon^* V^3 r\left(a^2(t/r)^3 - 3(t/r)\right) / \{\rho(a^2 - V^2)^2\} H(r - Vt)$$

(17)

For the particle velocity the above expressions give

$$\partial u_r^{(M)} / \partial t = 3K\varepsilon^* V^3 (a^2 t^2 / r^2 - 1) / \{\rho(a^2 - V^2)^2\} H(r - Vt) \qquad (18)$$

$$\partial u_r^{(P)} / \partial t = -3K\varepsilon^* V^3 (a^2 t^2 / r^2 - 1) / \{\rho(a^2 - V^2)^2\} H(r - at) \qquad (19)$$

From (16) and (17) follows that $u_r = u_r^{(P)} + u_r^{(M)} = 0$ for $r \ge at$ and we verify that the intitial and radiation conditions are satisfied by the sum of the two contributions. With the sum of the two contributions the particle velocity is zero in front of the surface *r=at*, which is the moving pressure wave-front with the *M* waves behind it.

It can be seen from eqtns (16) and (17) that it is the *M* waves that contribute to the RHS inhomogeneous term in equation (14), satisying the Navier equations, while the derivatives of the *P* components being continuous across the moving boundary produce a zero RHS in eqtn (14).

We easily verify the properties of the dynamic Eshelby problem

(a) constant strain $\partial u_r / \partial r = K\varepsilon^*(a + 2V) / \{\rho a(a + V)^2\}$ for $r < Vt$ (20)

(b) $\partial u_r / \partial t = \partial u_r^{(P)} / \partial t + \partial u_r^{(M)} / \partial t = 0$ for $r < Vt$, (21)

This verifies that the interior domain is a "lacuna", also shown by the calculation in [14].

The displacement $u_r^{(M)}$ is verified to be continuous across the inclusion boundary $r = Vt$ while its derivatives are discontinuous and contribute to the jump conditions.

We can immediately show the properties of the *M* waves:

(i) *The M waves satisfy the Hadamard jump conditions on the moving interface r=Vt*

(a) $[[\partial u_r / \partial t]] = -V[[\partial u_r / \partial r]]$ (22)

which expresses ***the compatibility of the deformation*** across the moving boundary of discontinuity; from (16)-(19) we obtain

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$$[[\partial u_r / \partial t]] = [[\partial u_r^{(M)} / \partial t]] = 3K\varepsilon^* V / \{\rho(a^2 - V^2)\} \tag{23}$$

$$[[\partial u_r / \partial r]] = [[\partial u_r^{(M)} / \partial r]] = -3K / \{\rho(a^2 - V^2)\} \tag{24}$$

(b), the second Hadamard jump condition

$$[[\sigma_{rr}]] = -\rho V [[\partial u_r / \partial t]] \tag{25}$$

expresses ***the conservation of linear momentum*** across the moving interface with velocity V; with the above we obtain the jump in the normal traction at $r=Vt$ as

$$[[\sigma_{rr}]] = (\lambda + 2\mu)[[\partial u_r^{(M)} / \partial r]] - [[3K\varepsilon^*]] = -3K\varepsilon^* a^2 / (a^2 - V^2) + 3K\varepsilon^* =$$
$$-3K\varepsilon^* V^2 / (a^2 - V^2) = -\rho V [[\partial u_r^{(M)} / \partial t]] \tag{26}$$

As the displacement $u_r^{(P)}$ given by (16) is continuous across the interface $r = Vt$, only the M waves contribute to the above jump conditions.

(ii) *The M waves give the static Eshelby ellipsoidal inclusion solution in a limit*

It was shown in [3] that the static Eshelby ellipsoidal inclusion solution is obtained from the M waves contribution in the limit as time tends to infinity and the expansion speed tends to zero, while their product tends to a constant, which would be a static inclusion, of radius R_0. We verify this limit here, and obtain the static field

$$\varepsilon_{rr} = \lim_{V\to 0, t\to\infty, Vt\to R_0} \partial u_r^{(M)} / \partial r = \lim_{V\to 0, t\to\infty, Vt\to R_0} (K\varepsilon^* V^3 r\{a^2(t/r)^3 - 3(t/r)\}) / \rho(a^2 - V^2)^2 = K\varepsilon^* R_0^3 / r^3 - 1/3\varepsilon^* K R_0^3 / r^3 =$$
$$= -2/3(3\lambda + 2\mu)/(\lambda + 2\mu)\varepsilon^* R_0^3 / r^3 = -2/3(1+\nu)/(1-\nu)\varepsilon^* R_0^3 / r^3$$

(27)

which is in agreement with the classical solution, e.g., Mura [18, eqtn (18.18)].

It is noted that the M waves manifest themselves in the interior deformation field of a phase change model for the source of a deep earthquake as a self-similarly expanding flattened ellipsoidal inclusion in the penny-shape asymptotic limit under full isotropy [13]. The lacuna property results in the energy due to "volume collapse" escaping out as seismic energy without kinetic energy losses in the interior of the expanding inclusion of phase change.

III. The self-similarly expanding ellipsoid with phase change in Newtonian fluids produces a "lacuna"

Bilby, Eshelby and Kundu [19] implemented the Eshelby ellipsoidal inclusion solution/ methodology with the Eshelby Tensor to the slow motion of a viscous Newtonian incompressible immiscible fluid in a matrix with different viscosity (with corresponding variables of strain to rate of strain in the fluid analogy [20]), as it was initially outlined in the classical Eshelby inclusion

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paper [4]. For a Newtonian isotropic fluid with pressure p, the Navier-Stokes equations are (e.g. [21])

$$\rho dv_i / dt = \rho f_i - \partial p / \partial x_i + (\lambda + \mu)\partial(\vec{\nabla}.\vec{v}) / \partial x_i + \mu \nabla^2 v_i \tag{28}$$

$$\text{with } f_i = -C_{iklm}\varepsilon^*_{lm} \frac{\partial}{\partial x_k} H(t - (s_r^2 x_r^2)^{1/2})$$

with external body force f_i due to the rate of eigenstrain ε^*_{ij} as in (1), and where v_i is the particle velocity, λ, μ the Lame moduli for a Newtonian fluid (linear Stokesian fluid [21]), p the pressure, f_i the body force and ρ the density. We note that in the fluid analogy [21] *the strain in the solid in eqtns (1) and (2) is a deformation rate in the fluid in eqtn (28)*; in the sequel for brevity we will be using the words strains and eigenstrains rather than rates of them. We assume zero initial conditions, as in equations (3) above.

The system (2) was shown to be stretch invariant under stretching of the variables x_i, t, u_i and to yield an elliptic system in the new variables in a corresponding region [7], [15]. The system (28) will be stretch invariant under stretching of the variables $\hat{x}_i = \alpha x_i, \hat{t} = \alpha t, \hat{v}_i = \alpha v_i, \hat{p} = \alpha p$. As a result of the stretch invariance, the system (28) will be elliptic in the new variables, $\vec{z} = \vec{x} / t$, $\vec{\phi}(\vec{z}, t) = \vec{u}(\vec{x}, t) / t$, and for zero initial conditions , according to the Cauchy-Kowalewskaya theorem, the particle acceleration dv_i / dt will be zero in the entire region of analyticity [17], which is the interior domain of the expanding ellipsoid, now with a Newtonian fluid. This is [1] mathematically a weak lacuna [9] (lacunae "are traveling zones of absolute quiet" [8]). In the generalization to self-similar expansion, in the interior domain we will have a constant deformation rate (Eshelby property) obtained through the Dynamic Eshelby Tensor [3]. The constant interior strain allows to solve the phase change problem of a Newtonian viscous fluid undergoing phase change as an equivalent eigenstrain, as in Bilby *et al* [19], but now within a self-similarly expanding ellipsoidal region of different viscosity.

IV. **Conclusions**

We have considered the self-similar expansion of an ellipsoidal region of phase change and demonstrated the particular physical properties of the *M* waves emitted by the moving surface of discontinuity, illustrating them in the example of the self-similarly expanding spherical inclusion with dilatation eigenstrain. It should be noted that the self-similar solution grasps the early or very late response of the system [22]. We have shown that it is the *M* waves that satisfy the fundamental physical Hadamard jump conditions (of conservation of linear momentum and compatibility) across the moving surface of discontinuity, and that they also produce the static Eshelby inclusion solution in a limit. The *M* waves cancel the ones due to the *P* and *S* slowness surfaces [12] and create a "weak lacuna" [1],[9], a topological property of hyperbolic system of pde's ("traveling zone of absolute quiet" [8]), in the interior of the self-similarly expanding ellipsoid with phase change, following from dimensional analysis and analytical properties alone. The "lacuna" property was shown here to be valid also for an expanding ellipsoidal inclusion containing a Newtonian fluid of different viscosity, extending the application of the dynamic Eshelby ellipsoidal inclusion to Newtonian fluids as in [19]. The phenomenon of the self-similarly expanding ellipsoidal region of phase change can have wide applications in dynamic

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phase transformations under pre-stress or shock-loading, such as amorphization (with ellipsoidal amorphized regions observed in [23]), failure waves, etc., phenomena currently investigated in materials under extreme conditions.

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