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Bending a beam by a generalized ideal elastomeric gel

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A hybrid beam with a gel layer bonded on the top of an elastic non-swellable substrate has been commonly adopted to make various sensors and actuators. Usually, different models need to be developed for the hybrid beam when different gels are used in the system. In this article, based on the generalized ideal elastomeric gel model, we formulate a unified relationship between the swelling of hydrogels and the bending curvature of the elastic beam, which is independent of specific swelling mechanisms of gels. We further illustrate that the equations derived in the article can be used to validate the ideal elastomeric gel model and measure the elasticity of polymer networks of the gels.

1. Introduction

Gel is a network which can swell by absorbing solvent molecules. The amount of swelling can be affected by environmental humidity [1], temperature [2], pH value [3,4], light intensity [5], biological agents [6,7], etc. Thanks to this property, gels are being widely developed for a variety of sensors and actuators [8–10]. Among all the structures, a hybrid system with a gel layer on the top of an elastic non-swellable substrate has often been adopted [4,11]. For example, Bashir *et al.* [12] deposited a pH-sensitive hydrogel on the top of a microcantilever to measure the environmental pH value by detecting the deflection of the microcantilver. Similar structures with different hydrogels have been exploited to measure the concentrations of various ions or even glucose [7]. However, a theoretical formulation and analyses of such a hybrid system is limited, especially when considering the availability of all kinds of responsive gels.

To study the system with a gel layer bonded on an elastic non-swellable substrate, the swelling behaviours of constrained gels in different conditions need to be

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formulated first. In previous studies, nonlinear field theories for various responsive gels have been developed by different researchers including us [2,3]. For example, large deformation and stresses in a temperature-sensitive hydrogel with different constraints have been formulated by us and compared with experiments [2]. We have also studied the deformation of a constrained pH-sensitive hydrogel in solution with different concentrations of ions and pH values. Apparently, the models for different gels are dramatically different. As a consequence, the model of the hybrid beam composed of different types of gels has to be different.

Recently, a highly unified model, the ideal elastomeric gel model, has been proposed by us and verified for several different responsive gels experimentally [13]. Based on two fundamental assumptions, (i) the mixing of polymer, solvent and different solutes is independent of the stretching polymer network, and (ii) individual molecules are incompressible, we obtain the equations of state of different gels in a unified form. The details of the molecular structure of the gel and different environmental conditions are described by a scalar function, the osmosis function, which can be viewed as material properties and measured independently by conducting simple experiments.

In this article, we will generalize the ideal elastomeric gel model and investigate the bending of hybrid beams by a generalized ideal elastomeric gel. Using a generalized ideal elastomeric gel model, we will develop the relationship between the bending curvature of the beam and the swelling of gels. The relationship can be written in a unified form. We will also illustrate that the beam bending experiments can be used to validate the generalized ideal elastomeric gel model. Finally, we adopt the neo-Hookean model and the Gent model for characterizing stretching polymer network to obtain a specific relationship between the swelling of hydrogels and the bending of the hybrid beam.

2. Generalized ideal elastomeric gel

Figure 1 illustrates a block of an elastomeric network. In the reference state, the block is a dry network, of dimensions L_1, L_2 and L_3. In the current state, the block is subject to forces P_1, P_2 and P_3, and is submerged in an aqueous solution of fixed chemical potential of water, μ_s, and of other solutes, μ_i. In the current state, the block absorbs M_s number of water molecules, M_i number of other particles and the dimensions of the block become l_1, l_2 and l_3. The Helmholtz free energy of the block in the current state is denoted as F.

When the block changes the dimensions by small amounts δl_1, δl_2 and δl_3, the applied forces do work $P_1\delta l_1 + P_2\delta l_2 + P_3\delta l_3$. When the number of water molecules in the block increases by δM_s and the number of other molecules in the block increases by δM_i, the chemical potential of water in the surrounding aqueous solution does work $\mu_s\delta M_s$ and $\mu_i\delta M_i$. At a constant temperature, when the hydrogel equilibrates with the applied forces and the surrounding aqueous solution, the change in the Helmholtz free energy of the block equals the sum of the work done by the applied forces and the work done by the chemical potential of water and different solutes:

$$\delta F = P_1\delta l_1 + P_2\delta l_2 + P_3\delta l_3 + \mu_s\delta M_s + \sum_i \mu_i\delta M_i. \tag{2.1}$$

The dimensions of the block, l_1, l_2 and l_3, can vary independently from the number of water molecules in the hydrogel, M_s, and the number of other solutes in the hydrogel, M_i. The condition of equilibrium (2.1) holds for arbitrary and independent small variations of the independent quantities: l_1, l_2, l_3, M_s and M_i. However, these quantities are usually connected by the following considerations [14]. The volume of the block in the current state $l_1l_2l_3$, to a good approximation, equals the sum of the volume of the dry network $L_1L_2L_3$, the volume of the absorbed water $\Omega_s M_s$ and the volume of other solutes $\sum_i \Omega_i M_i$,

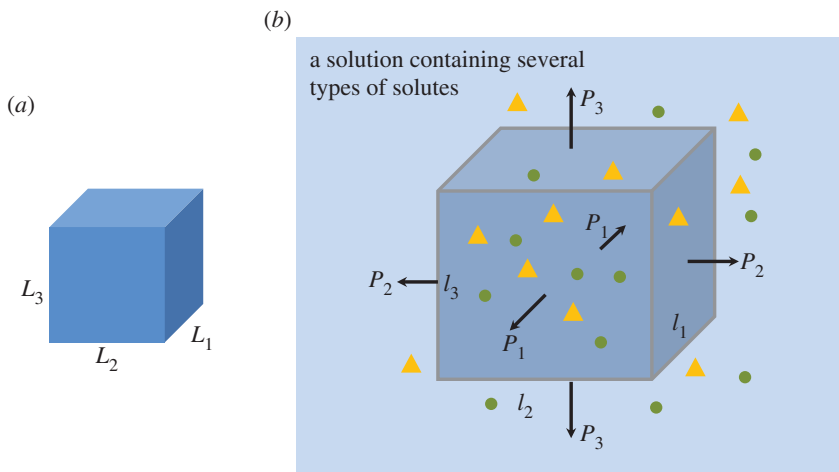$$l_1l_2l_3 = L_1L_2L_3 + \Omega_s M_s + \sum_i \Omega_i M_i, \tag{2.2}$$

Figure 1. (a) In the reference state, a dry network of polymers contains no water and is stress-free. (b) In the current state, the network is in equilibrium with applied forces, and with an aqueous environment of a fixed chemical potential of water and different solutes. (Online version in colour.)

where Ω_s is the volume per water molecule and Ω_i is the volume of other solutes. This approximation is known as molecular incompressibility and commonly adopted in analysing swelling of gels in the literature [14].

The relation (2.2) places a constraint among the variables l_1, l_2, l_3, M_s and M_i. We will regard l_1, l_2, l_3 and M_i as independent variables, in terms of which we express M_s using (2.2). Consequently, the variation of the number of water molecules in the hydrogel relates to the variations of the dimensions and other solutes:

$$\delta M_s = \frac{l_2 l_3}{\Omega_s} \delta l_1 + \frac{l_1 l_3}{\Omega_s} \delta l_2 + \frac{l_1 l_2}{\Omega_s} \delta l_3 - \sum_i \frac{\Omega_i}{\Omega_s} \delta M_i. \tag{2.3}$$

Inserting (2.3) into (2.1), we obtain that

$$\delta F = \left(P_1 + \frac{l_2 l_3 \mu_s}{\Omega_s} \right) \delta l_1 + \left(P_2 + \frac{l_1 l_3 \mu_s}{\Omega_s} \right) \delta l_2 + \left(P_3 + \frac{l_1 l_2 \mu_s}{\Omega_s} \right) \delta l_3 + \sum_i \left(\mu_i - \mu_s \frac{\Omega_i}{\Omega_s} \right) \delta M_i. \tag{2.4}$$

This condition of equilibrium holds for arbitrary and independent variations δl_1, δl_2, δl_3 and δM_i.

Define the concentration of water and other solutes by $c_s = M_s/l_1 l_2 l_3$ and $c_i = M_i/l_1 l_2 l_3$, and stretches by $\lambda_1 = l_1/L_1$, $\lambda_2 = l_2/L_2$ and $\lambda_3 = l_3/L_3$. Dividing both sides of (2.2) by the volume of the swollen elastomer, $l_1 l_2 l_3$, we obtain that

$$\Omega_s c_s = 1 - \frac{1}{J} - \sum_i \Omega_i c_i, \tag{2.5}$$

where $J = \lambda_1 \lambda_2 \lambda_3$. This expression places a constraint among the five variables: λ_1, λ_2, λ_3, c_s and c_i.

Define the nominal density of the Helmholtz free energy by $W = F/(L_1 L_2 L_3)$, true stresses by $\sigma_1 = P_1/(l_2 l_3)$, $\sigma_2 = P_2/(l_1 l_3)$ and $\sigma_3 = P_3/(l_1 l_2)$. Dividing both sides of equation (2.4) by $L_1 L_2 L_3$, we obtain that

$$\delta W = \left(\sigma_1 + \frac{\mu_s}{\Omega_s} \right) \lambda_2 \lambda_3 \delta \lambda_1 + \left(\sigma_2 + \frac{\mu_s}{\Omega_s} \right) \lambda_1 \lambda_3 \delta \lambda_2 + \left(\sigma_3 + \frac{\mu_s}{\Omega_s} \right) \lambda_1 \lambda_2 \delta \lambda_3$$

$$+ \sum_i \left(\mu_i - \mu_s \frac{\Omega_i}{\Omega_s} \right) \lambda_1 \lambda_2 \lambda_3 \delta c_i. \tag{2.6}$$

This condition of equilibrium holds for arbitrary and independent variations $\delta \lambda_1$, $\delta \lambda_2$, $\delta \lambda_3$ and δc_i.

As a material model, the nominal density of the free energy is taken to be a function of independent variables:

$$W = W(\lambda_1, \lambda_2, \lambda_3, c_i). \tag{2.7}$$

Owing to the constraint (2.5), the concentration of water is excluded from the list of the independent variables in (2.7). At a fixed temperature, when the block deforms by small amounts, $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$, and the number of solutes by small amounts, δc_i, the free energy varies by

$$\delta W = \frac{\partial W}{\partial \lambda_1} \delta\lambda_1 + \frac{\partial W}{\partial \lambda_2} \delta\lambda_2 + \frac{\partial W}{\partial \lambda_3} \delta\lambda_3 + \sum_i \frac{\partial W}{\partial c_i} \delta c_i. \tag{2.8}$$

A combination of (2.6) and (2.8) gives that

$$\left[\frac{\partial W}{\partial \lambda_1} - \left(\sigma_1 + \frac{\mu_s}{\Omega_s}\right)\lambda_2\lambda_3\right]\delta\lambda_1 + \left[\frac{\partial W}{\partial \lambda_2} - \left(\sigma_2 + \frac{\mu_s}{\Omega_s}\right)\lambda_1\lambda_3\right]\delta\lambda_2 + \left[\frac{\partial W}{\partial \lambda_3} - \left(\sigma_3 + \frac{\mu_s}{\Omega_s}\right)\lambda_1\lambda_2\right]\delta\lambda_3$$

$$+ \sum_i \left[\frac{\partial W}{\partial c_i} - \left(\mu_i - \mu_s\frac{\Omega_i}{\Omega_s}\right)J\right]\delta c_i = 0. \tag{2.9}$$

When the hydrogel equilibrates with the surrounding aqueous solution and the applied forces, (2.9) holds for arbitrary and independent variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$ and δc_i. Consequently, the coefficient in front of each of the four variations in (2.9) must vanish, leading to four equations:

$$\sigma_1 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\lambda_2\lambda_3\partial \lambda_1} - \frac{\mu_s}{\Omega_s}, \tag{2.10}$$

$$\sigma_2 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1\lambda_3\partial \lambda_2} - \frac{\mu_s}{\Omega_s}, \tag{2.11}$$

$$\sigma_3 = \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1\lambda_2\partial \lambda_3} - \frac{\mu_s}{\Omega_s} \tag{2.12}$$

and

$$\mu_i = -\mu_s J \frac{\Omega_i}{\Omega_s} - \frac{\partial W}{\partial c_i}. \tag{2.13}$$

In many gels, the density of the cross-links is very low. For example, each polymer chain may consist of over a thousand monomers. Consequently, to the first approximation, we may neglect the effect of the cross-links on solution, and simply write the free energy of the gel as the sum

$$W = W_s(\lambda_1, \lambda_2, \lambda_3) + W_{mix}(c_s, c_i), \tag{2.14}$$

where W_s is the free energy due to the stretching of the network, and W_{mix} is the free energy due to the mixing of the polymers and the solvent.

Inserting the above free energy function in (2.10)–(2.13), and using the constraint (2.5), we obtain that

$$\sigma_1 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_2\lambda_3\partial \lambda_1} + \frac{\partial W_{mix}(c_s, c_i)}{J^2\Omega_s\partial c_s} - \frac{\mu_s}{\Omega_s}, \tag{2.15}$$

$$\sigma_2 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_3\lambda_1\partial \lambda_2} + \frac{\partial W_{mix}(c_s, c_i)}{J^2\Omega_s\partial c_s} - \frac{\mu_s}{\Omega_s}, \tag{2.16}$$

$$\sigma_3 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1\lambda_2\partial \lambda_3} + \frac{\partial W_{mix}(c_s, c_i)}{J^2\Omega_s\partial c_s} - \frac{\mu_s}{\Omega_s} \tag{2.17}$$

and

$$\mu_i = -\frac{\Omega_i}{\Omega_s}\mu_s - \frac{\partial W_{mix}(c_s, c_i)}{J\partial c_i}. \tag{2.18}$$

Equations (2.15)–(2.18), along with (2.5), constitute the equations of state of a hydrogel.

By using equation (2.18), the concentration of the solutes c_i can be expressed as a function of the chemical potential μ_i and μ_s, as well as the volume matrix change of the gel, J. As a result, equations (2.15)–(2.17) can be written as

$$\sigma_1 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_2 \lambda_3 \partial \lambda_1} - \Pi, \tag{2.19}$$

$$\sigma_2 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_3 \lambda_1 \partial \lambda_2} - \Pi \tag{2.20}$$

and

$$\sigma_3 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_2 \partial \lambda_3} - \Pi, \tag{2.21}$$

with $\Pi = (\partial W_{mix}(c_s, c_i))/(J^2 \Omega_s \partial c_s) - \mu_s / \Omega_s$.

The equations of state (2.19)–(2.21) describe the mechano-chemical interaction of the hydrogels. In particular, the term Π has the dimension of stress, and is known as osmotic pressure in a solution. Π only depends on the volume of a gel as well as the chemical potential of all the solvent particles, but does not depend on the cross-link density and deformation of the polymer chain. Equations ((2.19)–(2.21)) also work for polymer solutions by letting the elasticity equal zero, and the stress corresponds to osmotic pressure. Therefore, Π can be measured independently using a polymer solution.

3. Beam bending by an ideal elastomeric gel

Figure 2 shows a layer of hydrogel, with initial swelling ratio $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_{pre}$, attached to an elastic and non-swellable substrate. With the swelling of the hydrogel, compressive stresses in lateral directions build up in the layer of hydrogel, which causes the bending of the hybrid beam. Usually, the bending of the substrate is small and the deformation of the hydrogel can be assumed approximately homogeneous. The stress in the vertical direction is zero, i.e.

$$\sigma_3 = 0. \tag{3.1}$$

Inserting (3.1) into (2.21), we have that

$$\Pi = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_2 \partial \lambda_3}. \tag{3.2}$$

Inserting (3.2) into (2.19) and (2.20), we can further obtain that

$$\sigma_1 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_2 \lambda_3 \partial \lambda_1} - \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_2 \partial \lambda_3} \tag{3.3}$$

and

$$\sigma_2 = \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_3 \partial \lambda_2} - \frac{\partial W_s(\lambda_1, \lambda_2, \lambda_3)}{\lambda_1 \lambda_2 \partial \lambda_3}. \tag{3.4}$$

Equations (3.3) and (3.4) indicate that the lateral stresses only depend on the swelling ratio in the vertical direction λ_3, but do not depend on the explicit form of Π. In other words, with fixed lateral stretches for a gel, the same deformation in the vertical direction will give rise to the same amount of lateral stresses, no matter what swelling mechanism is underlined (e.g. variation of humidity, pH, temperature). We would like to point out that if the lateral compressive stress is large enough, creases may develop on the surface of gel. The emergence of creases can change the magnitude of compressive stresses and bending moment, which is beyond the scope the article.

Since the modulus of the gel is usually much smaller than the modulus of the substrate, the bending stiffness of the gel is negligible and the neutral plane for the hybrid beam sits on the middle plane of the elastic non-swellable substrate during the gel swelling. For instance, in one of our recent experiments [15], the modulus of the substrate is 2.5 GPa and the modulus of the hydrogel is only in the order of 1 kPa. In practice, the above assumption should be accurate enough if the modulus of the substrate is at least two orders of magnitude larger than the modulus
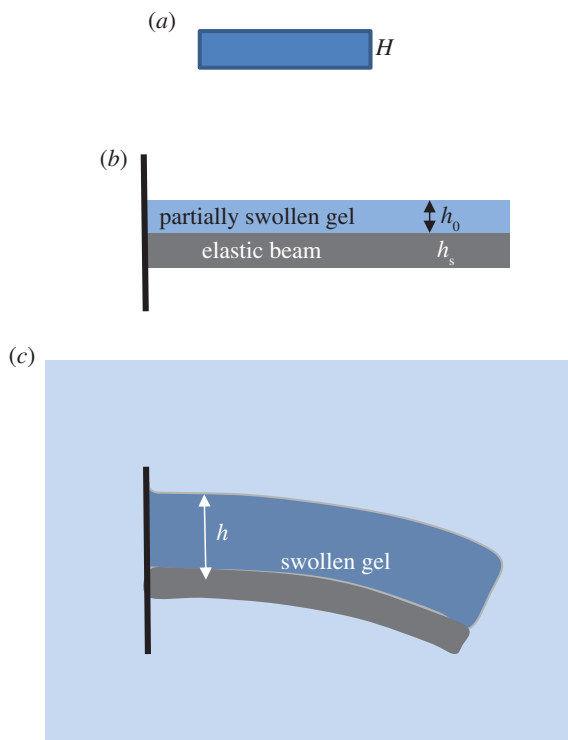
Figure 2. (a) A dry polymer network; (b) a partially swollen network attached to an elastic non-swellable substrate; and (c) the beam is bent by the swelling of gel when it is submerged in a solution. (Online version in colour.)

of the gel. Therefore, the moment per unit length in the width direction applied to the substrate by the hydrogel is given by

$$M = \frac{b_s}{2}\sigma_1\lambda_3 H(\lambda_3 H + h_s), \tag{3.5}$$

where b_s, h_s is the width and thickness of the non-swellable substrate, H is the thickness of the gel layer in dry state.

The curvature can be calculated by using classic beam theory

$$\Delta\kappa = \frac{M}{E_s I_s}, \tag{3.6}$$

where E_s and I_s are Young's modulus and second moment of the transverse plane of the substrate, and $\Delta\kappa$ is the curvature change of the substrate due to the swelling of the hydrogel. As pointed out in Janssen et al. [16], the width of the substrate is usually much larger than its thickness. The elastic substrate should be viewed as a plate instead of a beam. Therefore, to be more precise, E_s in (3.6) should be replaced by $E_s/(1 - \nu_s)$, where ν_s is Poisson's ratio of the substrate. Such replacement is valid for all the following equations in the article.

A combination of (3.5) and (3.6) gives

$$\Delta\kappa = \frac{6\sigma_1}{E_s h_s^3}(\lambda_3^2 H^2 + \lambda_3 H h_s), \tag{3.7}$$

which links the compressive stress in the equilibrium state to the change of the curvature of the beam. In (3.7), we assume the elastic substrate has a rectangular cross section with height h_s. It was pointed out by one of the anonymous reviewers that equation (3.7) can be reduced to Stoney's formula [17] when the thickness of the gel is much smaller than the thickness of the substrate, i.e. $\lambda_3 H \ll h_s$. In consequence, equation (3.7) can be viewed as an extension of Stoney's

formula without assuming the gel thickness to be much smaller than the substrate thickness. A combination of (3.3) and (3.7) is the direct consequence of the generalized ideal elastomeric gel model, so the two equations can be used to validate the model.

4. Elasticity of cross-linked polymer network

To obtain the equations in §3, we made the following assumptions: (i) the system is in the equilibrium state; (ii) the gel is much softer than the elastic substrate; (iii) the deformation and deflections (more precisely, slope) of the elastic substrate is small; (iv) the generalized ideal elastomeric gel model is valid, which is independent of the specific type and swelling mechanisms of gels; and (v) no surface instabilities such as creases happen.

For the free energy of a specific gel, the expression for mixing free energy among polymer, solvent and different solutes is usually very complex and under intensive debates [18–21]. The elasticity of a polymer network, however, can often be well captured by several models, such as the neo-Hookean model [22], the Gent model [23], the Arruda–Boyce model [24], the Mooney–Rivilin model [25,26] and the Ogden model [27]. In this section, we use the neo-Hookean model and the Gent model to get explicit expressions between the swelling of gels and the bending of the beam.

(a) Neo-Hookean model

When the deformation of the polymer network is not very large, its elasticity can usually be modelled by the neo-Hookean model. Specifically, the free energy density of the stretching polymer network is given by

$$W_s = \frac{1}{2} NkT (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\log(J)). \tag{4.1}$$

Plugging (4.1) into (3.2)–(3.4), we have

$$\Pi = NkT \left(\frac{\lambda_3}{\lambda_1 \lambda_2} - \frac{1}{\lambda_1 \lambda_2 \lambda_3} \right), \tag{4.2}$$

$$\sigma_1 = NkT \frac{1}{\lambda_2} \left(\frac{\lambda_1}{\lambda_3} - \frac{\lambda_3}{\lambda_1} \right) \tag{4.3}$$

and
$$\sigma_2 = NkT \frac{1}{\lambda_1} \left(\frac{\lambda_2}{\lambda_3} - \frac{\lambda_3}{\lambda_2} \right). \tag{4.4}$$

For equal-biaxial stretch, we have $\lambda_1 = \lambda_2 = \lambda_{pre}$ and $\sigma_1 = \sigma_2 = \sigma$; (4.2) and (4.3) become

$$\Pi = \frac{NkT}{\lambda_{pre}^2} \left(\lambda_3 - \frac{1}{\lambda_3} \right) \tag{4.5}$$

and

$$\sigma = \frac{NkT}{\lambda_{pre}} \left(\frac{\lambda_{pre}}{\lambda_3} - \frac{\lambda_3}{\lambda_{pre}} \right). \tag{4.6}$$

Plugging (4.6) into (3.7), we obtain that

$$\Delta\kappa = \frac{NkT}{E_s} \frac{6h_0}{h_s^2} \left(1 - \left(\frac{\lambda_3}{\lambda_{pre}} \right)^2 \right) \left(\frac{\lambda_3}{\lambda_{pre}} \frac{h_0}{h_s} + 1 \right), \tag{4.7}$$

where $h_0 = H\lambda_{pre}$. Equations (4.5)–(4.7) are valid for all kinds of hydrogel if the elasticity of the polymer network can be described by the neo-Hookean model. The lateral stress σ as a function of vertical stretch λ_3 is plotted in figure 3a. The change of curvature $\Delta\kappa$ as a function of vertical stretch λ_3 is plotted in figure 3b. A combination of (4.2) and (4.7) can be used to experimentally measure the osmosis function Π.

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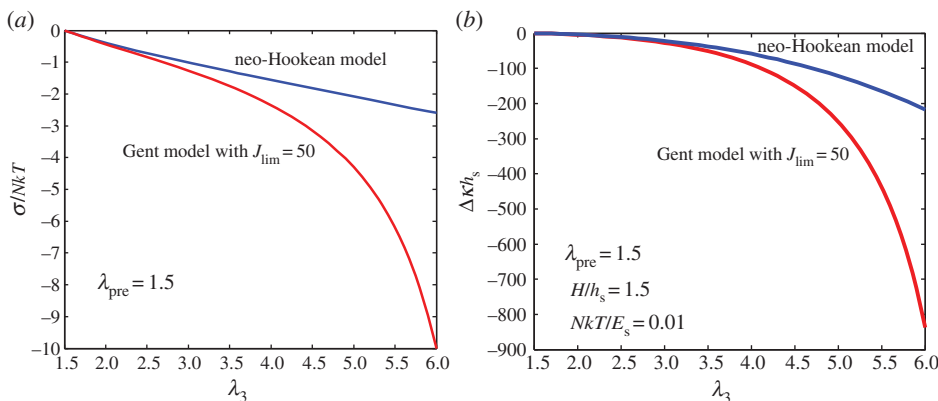rspa.royalsocietypublishing.org   Proc. R. Soc. A **471**: 20140919

Figure 3. (*a*) Lateral compressive stress as a function of the vertical swelling ratio and (*b*) bending curvature as a function of the vertical swelling ratio. (Online version in colour.)

(b) Gent model

When the stretch of the polymer network is large, stiffening effects need to be considered. The Gent model is one of the models taking account of stiffening of polymer chains. The free energy density of the stretching polymer network is given by

$$W_s = \frac{1}{2} NkT \left[J_{lim} \log \left(1 - \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{J_{lim}} \right) + 2 \log J \right]. \tag{4.8}$$

Similarly, for equal-biaxial stretch, the lateral stress and the change of the curvature of the beam are

$$\sigma = \frac{NkT}{\lambda_{pre}} \left(\frac{\lambda_{pre}}{\lambda_3} - \frac{\lambda_3}{\lambda_{pre}} \right) \bigg/ \left(1 - \left(\frac{2\lambda_{pre}^2 + \lambda_3^2 - 3}{J_{lim}} \right) \right) \tag{4.9}$$

and

$$\Delta\kappa = \frac{NkT}{E_s} \frac{6h_0}{h_s^2} \left(1 - \left(\frac{\lambda_3}{\lambda_{pre}} \right)^2 \right) \left(\frac{\lambda_3}{\lambda_{pre}} \frac{h_0}{h_s} + 1 \right) \bigg/ \left(1 - \left(\frac{2\lambda_{pre}^2 + \lambda_3^2 - 3}{J_{lim}} \right) \right). \tag{4.10}$$

The lateral stress σ as a function of vertical stretch λ_3 is also plotted in figure 3*a*. The change of curvature $\Delta\kappa$ as a function of vertical stretch λ_3 is plotted in figure 3*b*. In experiments, figure 3*b* can be used to determine the elasticity of the polymer network of hydrogels.

5. Conclusion

In summary, we investigated the bending of a beam with a gel layer bonded on top of an elastic non-swellable substrate. We adopted the generalized ideal elastomeric gel model to obtain the relationship between the swelling of the gel layer and the bending curvature of the substrate underneath. Since the obtained equations are only based on the ideal elastomeric gel model, they are valid for most gels and independent of specific swelling mechanisms of the gel. Finally, we also plugged specific models of the stretching polymer network to the generalized equations. Unfortunately, we cannot find any quantitative experimental measurement in the literature that can be directly compared with our predictions. We plan to conduct experiments by ourselves and use various gels in the beam bending system to validate the model proposed in the paper. We also expect the model developed in this article to be useful in characterizing different types of hydrogels and the design of gel actuators and sensors.

Data accessibility.   Figure 3 is plotted using Matlab.

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