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Gravity-induced wrinkling of thin films on soft substrates

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Abstract – In this letter, we investigate the wrinkling instability of a stiff thin film bonded on a soft substrate, induced by the gravity of densely packed pillars adhered on the surface of the film. By using linear perturbation analysis, we show that the gravity of the pillars can induce wrinkling instability of the system when the gravitational force of the pillars is large enough. Our calculation results give the instability criterion and illustrate how the wavelength of the wrinkles varies with several parameters of the system. The results of this article may be useful in the applications of similar pillar structures.

When a stiff thin film bonded on a soft substrate is under compression, different wrinkling patterns have been observed in various experiments [1,2]. Wrinkling instability in such a system can be understood by energetic considerations. When the compressive strain is large, compared with a flat surface, a wrinkled surface has lower elastic energy [3]. The wavelength of the wrinkles can be calculated by minimizing the total elastic energy of the thin film and soft substrate underneath. Recently, controllable wrinkles have been explored to change the adhesion and friction properties of surfaces [4–6].

Inspired by some recent experimental observations, in this letter, we propose a different way of inducing wrinkling instability in the system of a stiff thin film bonded on a soft substrate. In the experiments [7–10], it has been observed that periodically packed pillars adhered on a compliant substrate tend to tilt and bundle together. Corresponding analysis shows that the gravity of the pillars is very crucial in determining the patterns of the structure [11,12]. In this letter, we study the wrinkling instability of a thin film on a soft substrate driven by the gravity of pillars clamped on the surface, as shown in fig. 1. When the gravity of the pillars is large enough, the increase of the elastic energy of the system due to wrinkling can be totally compensated by the decrease of the gravitational potential energy of the pillars. By using linear perturbation analysis, we show how the critical gravitational force depends on the elastic properties and the geometry of the system.

As sketched in fig. 1, identical pillars are homogenously clamped on the surface of a thin elastic film, which is then bonded on a soft substrate. The total free energy of the system, W, is the summation of the elastic energy of the soft elastic substrate, the bending energy of the thin film and the gravitational potential energy of the periodic pillars. Denoting volume elements and surface elements by $\mathrm{d}V$ and $\mathrm{d}S$ respectively, we have

$$W = \int_V U_s \mathrm{d}V + \int_S U_f \mathrm{d}S + \int_S U_g \mathrm{d}S, \tag{1}$$

where U_s is the elastic energy density of the soft substrate, U_f is the bending energy of the thin film per area and U_g is the gravitational potential energy of the pillars per area.

We assume the displacement u_i and the strain ε_{ij} in the soft substrate is infinitesimal. The strain energy density of the substrate is

$$U_s = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}, \tag{2}$$

where σ_{ij} is Cauchy stress tensor and the infinitesimal strain tensor is defined as $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$. For simplicity, the soft substrate is assumed to be incompressible and the stress tensor σ_{ij} can be written as

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{3}$$

where p is the hydrostatic pressure in the soft substrate, δ_{ij} is the Kronecker delta and μ is the shear modulus.

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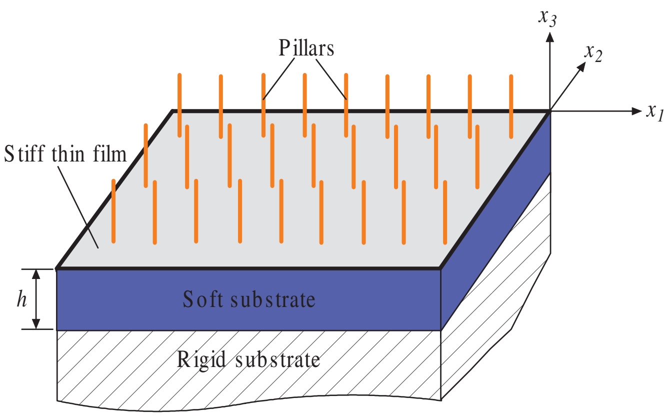

Fig. 1: (Color online) Schematic of periodic pillars adhered on a stiff thin film bonded to a soft substrate with thickness h.

The bending energy density U_f of the thin film can be written as [13]

$$U_f = \frac{1}{2} D \left(\nabla^2 u_3^f\right)^2 - (1-\nu)\left[\frac{\partial^2 u_3^f}{\partial x_1^2}\frac{\partial^2 u_3^f}{\partial x_2^2} - \left(\frac{\partial^2 u_3^f}{\partial x_1 \partial x_2}\right)^2\right], \tag{4}$$

where $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is a 2D Laplace operator, D is the bending stiffness of the film, ν is the Poisson ratio and u_3^f is the deflection of the film.

The pillars are densely packed on the top of the thin film, so we adopt a continuous description in the following formulations. The change of the gravitational potential energy of the pillars, due to the deformation of the system, can be approximated as

$$U_g = n\rho g V_0 H, \tag{5}$$

where n is number of pillars per surface area, ρ is the mass density of the pillars, g is the gravitational acceleration, V_0 is the volume of a single pillar, and H is the position change of the gravity center of pillars in the vertical direction.

We further assume that the pillars are always perpendicular to the surface of the thin film. Through simple geometrical analysis, H can be linked to the deflection of the thin film by

$$H = u_3^f + l\left\{\left[1 + \left(\frac{\partial u_3^f}{\partial x_1}\right)^2 + \left(\frac{\partial u_3^f}{\partial x_2}\right)^2\right]^{-1/2} - 1\right\}, \tag{6}$$

where l is the half-length of the pillar.

Since the displacement is small, eq. (6) can be approximated by Taylor's expansion to the second order as

$$H = u_3^f - \frac{l}{2}\left[\left(\frac{\partial u_3^f}{\partial x_1}\right)^2 + \left(\frac{\partial u_3^f}{\partial x_2}\right)^2\right]. \tag{7}$$

Plugging (7) into (5), we obtain

$$U_g = G\left[\frac{u_3^f}{l} - \frac{1}{2}\left(\frac{\partial u_3^f}{\partial x_1}\right)^2 - \frac{1}{2}\left(\frac{\partial u_3^f}{\partial x_2}\right)^2\right], \tag{8}$$

where $G = n\rho g V_0 l$ is a loading parameter.

The vertical displacement is continuous on the interface between the soft substrate and the thin film. Therefore, in this problem, u_i are the only independent variables. Minimizing the free energy in eq. (1) with respect to u_i gives the equilibrium conditions

$$\mu\left(\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}\right) - \frac{\partial p}{\partial x_i} = 0,\ (i = 1,2,3),\ \text{in } V, \tag{9}$$

and the boundary conditions

$$\sigma_{3i} = -\left(\frac{G}{l} + G\nabla^2 u_3 + D\nabla^2\nabla^2 u_3\right)\delta_{3i},\ \text{at } x_3 = 0. \tag{10}$$

The bottom of the soft substrate is fixed, so

$$u_i = 0,\ \text{at } x_3 = -h. \tag{11}$$

The above boundary value problem has a simple homogeneous solution: $u_i^{(0)} = 0$ and $p^{(0)} = G/l$. To study the stability of the problem, we search for another solution: $u_i = u_i^{(0)} + u_i^{(1)}$ and $p = p^{(0)} + p^{(1)}$, in which $u_i^{(1)}$ and $p^{(1)}$ are small perturbations of $u_i^{(0)}$ and $p^{(0)}$, respectively. Due to the incompressibility, $u_i^{(1)}$ can be represented by $u_1^{(1)} = -\partial\phi_2/\partial x_3$, $u_2^{(1)} = \partial\phi_1/\partial x_3$ and $u_3^{(1)} = \partial\phi_2/\partial x_1 - \partial\phi_1/\partial x_2$, with ϕ_1 and ϕ_2 being stream functions. We assume that the solutions are of the periodic forms as

$$\begin{aligned} \phi_1 &= \Phi_1(x_3)\, e^{i(k_1x_1 + k_2x_2)}, \\ \phi_2 &= \Phi_2(x_3)\, e^{i(k_1x_1 + k_2x_2)}, \\ p^{(1)} &= P(x_3)\, e^{i(k_1x_1 + k_2x_2)}, \end{aligned} \tag{12}$$

where $i = \sqrt{-1}$, k_1 and k_2 are wave numbers per unit length in the x_1 and x_2 directions, respectively.

A combination of eqs. (12) and eqs. (9) gives,

$$\begin{aligned} &\mu\left(\frac{\mathrm{d}^3\Phi_1}{\mathrm{d}x_3^3} - k^2\frac{\mathrm{d}\Phi_1}{\mathrm{d}x_3}\right) - ik_2P = 0, \\ &\mu\left(\frac{\mathrm{d}^3\Phi_2}{\mathrm{d}x_3^3} - k^2\frac{\mathrm{d}\Phi_2}{\mathrm{d}x_3}\right) + ik_1P = 0, \\ &\mu\left(\frac{\mathrm{d}^2\Phi}{\mathrm{d}x_3^2} - k^2\Phi\right) + i\frac{\mathrm{d}P}{\mathrm{d}x_3} = 0, \end{aligned} \tag{13}$$

where $\Phi = k_1\Phi_2 - k_2\Phi_1$ and $k = \sqrt{k_1^2 + k_2^2}$.

The general solutions of the above ordinary differential equations are

$$\begin{aligned} \Phi_1 &= -c_1e^{-kx_3} + c_2e^{kx_3} + ik_2\left(c_5e^{-kx_3} - c_6e^{kx_3}\right), \\ \Phi_2 &= -c_3e^{-kx_3} + c_4e^{kx_3} - ik_1\left(c_5e^{-kx_3} - c_6e^{kx_3}\right), \\ P &= 0, \end{aligned} \tag{14}$$

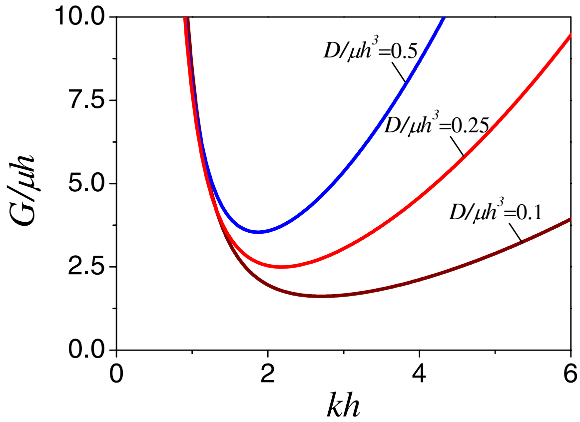

Fig. 2: (Color online) Variations of the dimensionless loading parameter $G/\mu h$ with the wave number kh of the wrinkles, for several different bending stiffnesses of the thin film.

where c_1, c_2, $\ldots$, and c_6 need to be determined through boundary conditions.

Inserting the general solution (14) into boundary conditions (10) and (11), we obtain six homogenous linear algebraic equations with the six constants c_1, $c_2, \ldots$, and c_6 to be determined. To ensure the existence of nontrivial solutions, the determinant of the coefficient matrix must vanish, yielding

$$\frac{G}{\mu h} = \frac{D}{\mu h^3}(kh)^2 + \frac{2\left[1+2(kh)^2+\cosh(2kh)\right]}{kh\left[2kh-\sinh(2kh)\right]}, \qquad (15)$$

where $D/\mu h^3$ is a normalized bending stiffness of the thin film.

Equation (15) gives the criterion for the stability of the system: the system wrinkles if positive root of k_1 and k_2 exists for a fixed value of G. In general, G attains the minimal G_c at $k_1 = k_{1c}$ and $k_2 = k_{2c}$, where the critical wave numbers k_{1c} and k_{2c} are determined by $\partial G/\partial k_1 = 0$ and $\partial G/\partial k_2 = 0$. The magnitude of G_c gives the critical loading condition for wrinkling, while the wave vector $\mathbf{k} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2$ characterizes the wrinkling mode. Obviously, the above relation is symmetric about k_1 and k_2, and so is the corresponding instability mode (k_1, k_2). As a consequence, the wrinkling pattern of the film will be statistically isotropic or disordered, which is similar to the simulation results obtained in [14].

With eq. (15), in fig. 2, we plot the dimensionless loading parameter $G/\mu h$ as a function of the normalized wave number kh, for several different normalized bending stiffnesses of the thin film $D/\mu h^3$. If the relevant material parameters are taken as $D = 2.5 \times 10^{-4}\,\mathrm{N \cdot m}$, $\mu = 1\,\mathrm{MPa}$ and $h = 1\,\mathrm{mm}$, we obtain the normalized bending stiffness of the thin film, $D/\mu h^3 = 0.25$. In fig. 2, we find that, when $D/\mu h^3 = 0.25$, $G/\mu h$ has the minimal $G_c/\mu h = 2.50$ at the critical wave number $k_c h = 2.22$. The plotting results indicate that the pillars can wrinkle the surface of the system when the loading parameter G is beyond a threshold G_c.

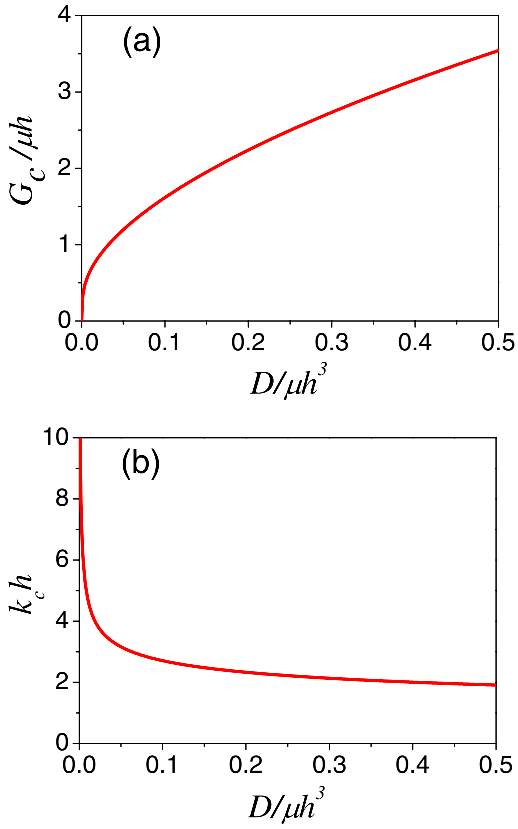

Fig. 3: (Color online) Variations of (a) the critical load $G_c/\mu h$ for wrinkling the system and (b) the critical wave number of the wrinkles $k_c h$ with the normalized bending stiffness of the thin film $D/\mu h^3$.

In fig. 3(a), we plot how the critical loading parameter $G_c/\mu h$ varies with the bending stiffness of the thin film $D/\mu h^3$. The value of $G_c/\mu h$ increases with increasing $D/\mu h^3$. The result implies that a stiff thin film tends to prevent the wrinkling of the system, which is consistent with our intuition. In fig. 3(b), we plot the critical wave number of the wrinkles $k_c h$ as a function of $D/\mu h^3$. The critical wave number can be tuned in a wide range by changing the bending stiffness of the thin film. From figs. 3(a) and (b), we find that when $D/\mu h^3$ approaches zero, $G_c/\mu h$ also approaches zero and $k_c h$ approaches infinity. In this limit, our continuous description of pillars may not be valid. This is because when $k_c h$ is large, namely the wavelength of the wrinkles is small, the distance between each two pillars can be comparable or even bigger than the wavelength of the wrinkles. Therefore, the results in figs. 3(a) and (b) for small $D/\mu h^3$ may not be accurate or even correct.

In conclusion, we have studied the wrinkling instability of the system with a thin film bonded on a soft substrate, induced by the gravity of pillars adhered on the surface. The calculation results show that the pillars can wrinkle the surface of the system when the gravitational force is larger than a critical value, and the wrinkling patterns can be regulated by changing the properties of the thin film. According to our knowledge, there has not been any experimental observation on the gravity-induced wrinkling instabilities. We hope our predictions can be verified by future experiments, and the results obtained in this article may find applications in generating wrinkling patterns in pillar structures.

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