Corrigendum:

A.K. Didwania, F. Cantelaube, and J.D. Goddard,
Static multiplicity of stress states in granular heaps.

Equations (2.8), (2.9), (2.12) and (2.15) contain typographical errors having no consequence for any other equation or result in the above paper. The amended forms read:

\[
\frac{1}{r^n} \frac{\partial}{\partial r} \left( r^n \sigma_{rr} \right) + \frac{1}{r \sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left( \sin^{n-1} \theta \sigma_{r\theta} \right) - \frac{\sigma_{\theta \theta} + (n-1)\sigma_{\phi \phi}}{r} = \cos \theta, \quad (2.8)
\]

\[
\frac{1}{r^{n+1}} \frac{\partial}{\partial r} \left( r^{n+1} \sigma_{r\theta} \right) + \frac{1}{r \sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left( \sin^{n-1} \theta \sigma_{\theta \theta} \right) - \frac{(n-1) \cot \theta \sigma_{\phi \phi}}{r} = -\sin \theta, \quad (2.9)
\]

\[
\psi = \Psi - \theta = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{r\theta}}{\sigma_{rr} - \sigma_{\theta \theta}} \right), \quad (2.12)
\]

\[
\begin{align*}
\chi_p' &= \frac{\chi(\theta) \sin 2\psi - \sin(2\psi + \theta)}{\cos 2\psi - \sin \varphi}, \\
\chi_e' &= \frac{\chi(\theta) \sin \varphi \cos \theta - \cos(2\psi + \theta) + \kappa(\theta) \cos(2\psi + \theta)}{\sin \theta \cos 2\psi - \sin \varphi}, \\
\Psi_p' &= \frac{\sin \varphi \cos(2\psi + \theta) + \chi(\theta) \cos \varphi - \cos \theta}{2\chi(\theta) \sin \varphi \cos 2\psi}, \\
\Psi_e' &= \kappa(\theta) \left( \sin \varphi \sin(2\psi + \theta) - \sin \theta \right) + \chi(\theta) \cos^2 \varphi \sin \theta, \\
\text{with} \quad \kappa(\theta) &= \frac{\sigma_{\phi \phi}}{r},
\end{align*}
\]

(2.15)
Missing Reference:

Static multiplicity of stress states in granular heaps

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Abstract

Arching, Granular heaps, Static indeterminacy, Elastoplastic states

As a summary and extension of previous work on arching in granular heaps (Cantelaube et al. 1997, 1998), an analysis is given of statically admissible stress distributions in infinite planar wedges and axisymmetric cones composed of an isotropic linear-elastic material subject to a non-cohesive Mohr-Coulomb yield criterion. The treatment is based on a combination of analytical solutions for elastic and simple plastic states, together with numerical integration of the (Sokolovskii-Kötter) ordinary differential equations appropriate to more complex plastic states.

For wedges, we obtain a one-parameter family of continuous elastoplastic solutions, with only three isolated symmetric solutions for symmetric wedges, one of which has a central pressure dip or "arch". For axisymmetric cones subject to a well-known closure for plastic hoop stress, only one continuous elastoplastic state is found, and it exhibits an arch.

In addition to the above continuous solutions, a class of discontinuous plastic-limit states is considered, which exhibit a central pressure dip associated with the discontinuous transition from active to passive states proposed by Savage (1998). The only solutions of this type found for symmetric wedges and cones involve central pressure dips. A brief discussion is given of the relation of this work to an extensive recent literature on the central pressure dip observed in certain experiments on granular heaps.

1 INTRODUCTION

"Previous researches on this subject are based...on some mathematical artifice or assumption...[which,] although leading to true solutions of many special problems, are both limited in the application of their results, and unsatisfactory in a scientific point of view."

(Rankine 1857).

"[The] transfer of pressure from a yielding mass of soil onto adjoining stationary parts is commonly called the arching effect, and the soil is said to arch over the yielding part of the support."
(Terzaghi 1943, Chapt. V).

A subject of long-standing interest in soil mechanics, arching has been revived in the recent literature as a possible explanation of the experimentally observed “dip” in pressure beneath certain granular heaps. The countervailing opinions and theoretical methods in that literature are well represented in the recent reviews of Savage (1997 & 1998) and of Cates et al. (1998). Those articles provide surveys of the relevant literature and the various philosophical viewpoints on the stress distribution in large heaps of non-cohesive, highly-rigid granules (e.g. dry sand) deposited on nearly-rigid flat surfaces. Since the paper by Cates et al. (1998) reflects the outlook and substance of their numerous co-authored publications, we direct the reader to it as the primary reference to that body of literature. Certain differences should be apparent below in our interpretation of other literature on the subject.

Here, we merely recall that one key issue is whether the stress distribution in a large heap is more crucially dependent on the mechanics of deposition than on the nature and subsequent deflections of the bottom surface, an issue which is perhaps ultimately best resolved by experiment. A second and, from our vantage point, more important issue is whether new models, beyond those embodied by standard elastoplastic continuum theories, are necessary to address the problem theoretically.

On the first issue Savage (1997 & 1998) favors the view, supported by certain experimental evidence and reflected in the epigrammatic definition by Terzaghi (1943), that arching may be traced to bottom deflection, especially with 2-d (two-dimensional) heaps, or that it may be attributed to certain plastic discontinuities to be discussed below. On the other hand, while conceding the importance of bottom deflection in wedges, Cates et al. (1998) cite experiments on 3-d conical piles to suggest that the mode of deposition is a decisive factor.

As to theoretical methods, we share the view of Savage (1998) that standard continuum mechanics and plasticity theories are generally appropriate for the problem at hand. This does not rule out the need for dynamic models, or for new insights into the static models of mechanical states that may arise from deposition or creation in situ of a granular material, a fundamental problem in geophysics and geotechnical engineering.

Setting aside the complex problem of how various stress states might arise from various deposition processes, the present work explores the nature and multiplicity of statically admissible states in granular wedges and cones, according to a standard elastoplastic model. We recall that this problem already has been addressed, for the case of wedges, by the work of Cantelaube et al. (1997 & 1998), whose results have been questioned by Savage (1998) and perhaps overlooked by Cates et al. (1998). With the intent being partly to allay such questions and oversights, we shall summarize some of these results below, presenting them from a clearer theoretical perspective, along with new results.
for continuous-elastoplastic and discontinuous-plastic states in both wedges and cones.

2 Basic model and theoretical issues

As in our previous work (Cantelaube and Goddard 1997, Cantelaube et al. 1998), denoted respectively, as CG and CDG in the following, we adopt the well-known model of a linear-elastic isotropic medium subject to the Mohr-Coulomb yield (plastic limit-state) criterion. As in CG & CDG, we deal primarily with the stress field $\mathbf{T}(\mathbf{x})$ without immediate appeal to associated deformations or displacements. Thus, in addition to the static-equilibrium (Cauchy) equations:

$$\nabla \cdot \mathbf{T} + \rho \mathbf{g} = 0$$  \hspace{1cm} (1)

the stress $\mathbf{T} = \mathbf{T}^T$ satisfies:

$$F := (\sigma_{II} - \sigma_I)^2 - (\sigma_{II} + \sigma_I)^2 \sin^2 \varphi \leq 0$$  \hspace{1cm} (2)

where $\sigma_I$ and $\sigma_{II}$ denote extremal principal stresses and $\varphi$ the internal friction angle. We further denote regions where $F < 0$ as elastic and those on the yield locus $F = 0$ as plastic (CG,CDG) or plastic limit states.

In continuous elastic regions, we impose stress compatibility (Love 1944, Lure 1964, Jaunzemis 1967):

$$\nabla \times (\nabla \times \mathbf{T})^T + \frac{3\nu}{\nu + 1} \nabla \times (\nabla \times \mathbf{p})^T = 0$$  \hspace{1cm} (3)

where $\nu$ denotes Poisson’s ratio, $\mathbf{T}$ (Cauchy) tensile stress, and $\mathbf{p} = -\text{tr}(\mathbf{T})/3$ pressure.

In the following, we shall have occasion to consider the coexistence of different elastic states, separated by a singular surface at which compatibility fails, essentially because of discontinuities in $\nabla \mathbf{T}$. As in CDG, we regard this surface as a plastic limit state to which the yield condition (2) applies.

We consider infinitely deep heaps in the form of 2-d triangular wedges or 3-d axisymmetric cones, which leads to plane-stress and axisymmetric-stress problems, respectively. For plane stress, (3) reduces to the single cartesian form

$$\sigma_{xx,yy} + \sigma_{yy,xx} - 2\sigma_{xy,xy} = 0$$  \hspace{1cm} (4)

which, in elastic regions, provides closure of the two equations obtained from (1) (cf. CG, CDG). These three equations admit general solutions for stress fields $\sigma_{xx}$, $\sigma_{yy}$ & $\sigma_{xy}$, in which the elastic moduli do not appear explicitly. Consequently, in traction boundary-value problems the internal stresses are generally independent of elastic moduli, which is also expected for elastoplastic wedges (CD), where (2) provides a well-known closure for (1) in plastic regions (Sokolovskii 1965). On the other hand, for 3-d problems the elastic moduli appear solely as Poisson’s ratio, so that solutions to traction boundary-value problems should also be independent of stiffness (Young’s modulus) $E$. 

3
Owing to the lack of dependence on $E$, the imposition of elastic compatibility eliminates rigid-body indeterminacies in the (rigid-plastic) limit $E \to \infty$ (CG,CDG), but there still remain problems of static indeterminacy. An immediately evident indeterminacy arises in 3-d plasticity problems, where the number of stresses exceeds the number of equations arising from the (1) and the (2). A well-known aspect of static plasticity (Nedderman 1992), this indeterminacy as well as its weaker 2-d counterpart are often resolved by kinematic assumptions relating to incipient flow together with plastic "flow rules". This approach, well-represented by the review of Savage (1998), appears quite distinct from the procedures advocated by Cates et al. (1998). We now illustrate the basic nature of this indeterminacy by means of a simpler problem.

In the discussion to follow, and in contrast to the convention in CG & CDG, we adopt the soil-mechanics convention for stress, denoting compressive stresses, i.e. the components of $-\mathbf{T}$, by the symbol $\sigma$.

### 2.1 Rankine indeterminacy and plastic discontinuity

With apologies for the textbook example (Terzaghi 1943, Nedderman 1992, Parry 1995), we recall that static indeterminacy or multiplicity is already evident in the renowned problem of the "earth pressure" (Rankine 1857). For the special case of a horizontal layer, free of horizontal shear stress $\sigma_{xy}$, the principal stress $\sigma_{xx}(\equiv \rho gx)$ at any depth $x$ can lie on, or anywhere between the so-called
passive and active plastic limit-state values. For a homogeneous Mohr-Coulomb material, the latter are, respectively,

$$\frac{\sigma_{yy}}{\sigma_{xx}} = C^{\pm 1}, \text{ where } C = \frac{1 + \sin \varphi}{1 - \sin \varphi},$$

(5)
is the standard earth-pressure coefficient and $\varphi$ the friction angle of (2). While this multiplicity can be resolved, trivially by specification of lateral stress, or, more conventionally by assumption of incipient plastic deformation, neither resolution is particularly appropriate to states arising from an unspecified deposition process. We note that similar indeterminacies arise in Rankine's (1857) infinite slope (cf. van R. Marais 1969) and have counterparts in the granular wedges and cones discussed below. The further important points that:

1. static states need not be everywhere plastic (limit states),
2. not all plastic stresses are necessarily continuous,
3. compatibility resolves rigid-body indeterminacy, and
4. loss of elastic compatibility generally occurs at the yield locus,
are illustrated by the variant on the Rankine problem depicted in Fig. 1(a). There, one has an indefinite number of discrete horizontal strata, with some in plastic states $P, P', \ldots$, either active or passive, and others in intermediate states $E, E', \ldots$.

Without further specification of lateral tractions or displacements (imposed, say, by imaginary segmented or flexible boundaries at $|y| = \infty$), the stress state is indeterminate, a point relevant not only to the soil mechanics behind retaining walls (Terzaghi 1943, the Preface) but also to arching induced by flexible boundaries (ibid., the prefatory quote above).

If the intermediate states $E, E', \ldots$, are assumed elastic, then compatibility restricts the possible stress distributions within these regions. In particular, (4) leads to the stress trajectories illustrated in Fig.1(b) and represented by

$$\sigma_{xx} = \rho gx + c \text{ and } \sigma_{yy} = ax + b$$

(6)
in which the constants $a, b, c$ appropriate to a given elastic layer are determined by a yield locus such as (2). It is also worth noting, for later reference, that plastic discontinuities, in $\sigma_{yy}$, which may be imagined to arise from $a \to \infty$ in an elastic layer of zero thickness, represent discontinuous jumps between active and passive states (cf. Savage 1998). Also, we note that rigid plasticity, freed from elastic compatibility, allows for arbitrary, statically indeterminate stress-paths in Fig.1(c) where, incidentally, $\sigma_{yy}$ need not be continuous as depicted.

Finally, we observe that one may have weak elastic discontinuities between contiguous elastic strata, corresponding to different constants $a$ in (6), as long as these are separated by plastic layers of vanishing thickness. This failure of compatibility corresponds to pointwise "reflection" of elastic paths in Fig.1(a) off the yield loci, and it illustrates the elastic-matching principle mentioned above and to be applied below.
2.2 Reduced equations

With the constitutive model adopted here, we seek stress fields $T(x)$ linear in radial distance $r = |x|$ from the apex of wedge or cone. This scaling is evident from the earliest treatments of linear-elastic wedges and cones (Terzaghi 1943, Love 1944, Samsioe 1955, Lure 1964) and from Sokolovskii’s (1965) non-cohesive plastic wedges. Although solutions involving other powers of $r$ are feasible (Sokolovskii 1965, Lure 1964, pp. 330 ff., England 1971), these involve unwanted singularities at $r = 0$ or $r = \infty$ (CG) or otherwise fail to represent self-weight in uniform gravity for the elastoplastic models considered here. Other constitutive models, involving non-linear elasticity, e.g. due to non-linear granular contact mechanics, or cohesive plasticity generally do not admit the same $r$-scaling (invoked as "RSF" by Cates et al. 1998).

We employ both cartesian coordinates $(x, y, z)$ and the associated cylindrical or spherical polar coordinates $(r, \theta, z)$ and $(r, \theta, \phi)$, for the planar and axisymmetric problems, respectively, with $\theta$ representing polar angle measured from the direction of gravity ($x$ or $z$). We further assume the extremal principal stresses (7) to lie in $r\theta$ planes (i.e. the plane of the wedge or any meridional plane of the cone), with

$$\sigma_{I,II} = \frac{1}{2} \left\{ \sigma_{rr} + \sigma_{\theta\theta} \pm \sqrt{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2} \right\}$$  \hspace{1cm} (7)

The only non-vanishing out-of-plane stress, $\sigma_{zz}$ for wedges and $\sigma_{\phi\phi}$ for axisymmetric cones, represents the principal stress lying between those in (7) and is generally indeterminate. With the yield criterion (2), the indeterminacy of $\sigma_{zz}$ is inconsequential for the idealized 2-d wedges considered here, although in 3-d prisms, such as Samsioe’s (1955) granular ”dams”, it must vary over the cross-section and vanish, along with all other stresses, at the free surface. By contrast, the ”hoop-stress” $\sigma_{\phi\phi}$ appears in the axisymmetric stress equilibrium (1), representing a special case of the 3-d indeterminacy mentioned earlier. A long-standing subject in the plasticity literature (Haar & Von Kármán 1909, Drescher 1991, Nedderman 1992) and evident in the analysis by Cates et al. (1998), this indeterminacy is also dealt with below.

Following our previous work (CD,CDG), we adopt nondimensional forms with $x$ scaled by an arbitrary depth $H$ and $T$ by $\rho g H$, so that (1) reduces in polar coordinates to

\[
\frac{1}{r^n} \frac{\partial}{\partial r} (r^n \sigma_{rr}) + \frac{1}{r \sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left( \sin^{n-1} \theta \sigma_{r\theta} \right) - \frac{\sigma_{\theta\theta} + (n-1)\sigma_{\phi\phi}}{r} = -\cos \theta \hspace{1cm} (8)
\]

\[
\frac{1}{r^{n+1}} \frac{\partial}{\partial r} (r^{n+1} \sigma_{r\theta}) + \frac{1}{r \sin^{n-1} \theta} \frac{\partial}{\partial \theta} \left( \sin^{n-1} \theta \sigma_{\theta\theta} \right) - \frac{(n-1) \cot \theta \sigma_{\phi\phi}}{r} = \sin \theta \hspace{1cm} (9)
\]

where $n = 1$ for wedges and $n = 2$ for axisymmetric cones with $\sigma_{r\phi}, \sigma_{\theta\phi} = 0$.

In plastic zones, the stresses $\sigma_{rr}, \sigma_{\theta\theta}$ and $\sigma_{r\theta}$ can be written as

$$\begin{align*}
\sigma_{rr} &= \sigma (1 + \varphi \cos 2\psi) \\
\sigma_{\theta\theta} &= \sigma (1 - \varphi \cos 2\psi) \\
\sigma_{r\theta} &= \sigma \sin \varphi \sin 2\psi
\end{align*}$$  \hspace{1cm} (10)
in terms of the two variables $\sigma(r, \theta)$ and $\psi(r, \theta)$:

$$\sigma = r \chi(\theta) = \frac{\sigma_{II}}{2} = \frac{\sigma_{rr} + \sigma_{\theta \theta}}{2}$$

$$\psi = \Psi - \theta = \frac{1}{2} \tan^{-1} \left( \frac{2 \sigma_{r \theta}}{\sigma_{rr} - \sigma_{r \theta}} \right)$$

where the mean stress $\sigma$ is subject to the above $r$-scaling, and $\Psi$ denotes the angle of the major principal stress relative to a fixed cartesian axis, taken here as the direction of gravity.

When substituted into (8)-(9), (12) yield generalized form of the ordinary differential equations given in Sokolovskii (1965), which we shall refer to as the SK equations, after Sokolovskii and Kötter (Suklje 1969, Terzaghi 1943, pp.61):

$$\frac{d \chi}{d \theta} = \chi'(\chi, \psi, \theta) + \chi''(\chi, \psi, \theta)$$

$$\frac{d \Psi}{d \theta} = \frac{d \psi}{d \theta} + 1 = \Psi'(\chi, \psi, \theta) + \Psi''(\chi, \psi, \theta)$$

whose rhs involves the following functions of $\chi, \psi, \theta$

$$\chi'_p = -\frac{\sin(2\psi + \theta) + \chi(\theta)\sin 2\psi}{\cos 2\psi - \sin \varphi},$$

$$\chi'_e = -\frac{\chi(\theta)\cos(2\psi + \theta) + \sin \varphi \cos \theta}{\sin \theta(\cos 2\psi - \sin \varphi)} + \frac{\sigma_{\phi \phi} \cos(2\psi + \theta)}{\sin \theta(\cos 2\psi - \sin \varphi)},$$

$$\Psi'_p = -\frac{\sin \varphi \cos(2\psi + \theta) + \chi(\theta)\cos^2 \varphi - \cos \theta}{2\chi(\theta)\sin \varphi(\sin \varphi - \cos 2\psi)},$$

$$\Psi'_e = -\frac{\sigma_{\phi \phi}(\sin \theta - \sin \varphi \sin(2\psi + \theta))}{2\chi(\theta)\sin \theta \sin \varphi(\sin \varphi - \cos 2\psi)}$$

where the subscript $p$ refers to terms appearing in the planar SK equations, applicable here to the wedge, and subscript $e$ denotes the extra terms derived by us to describe axisymmetric stress states. Our limited experience with the numerical treatment of (15) suggests that an extension of Heurtaux's (1959) (cf. Sokolovskii 1965) mathematical analysis of singularities and stability of the planar SK equations would be most useful.

In the following, we obtain various admissible solutions for infinite wedges and axisymmetric cones with traction free surfaces. Although the main focus is on elastoplastic states, considered in the next two sections, we present afterwards some new numerical solutions to the SK equations involving discontinuous plastic solutions.
3 Elastoplastic wedges

We consider a wedge illustrated in Fig. 2, defined by polar angle $\theta$ in $[\beta', \beta]$. Following CG and CDG, we let $t := \tan \theta$ and

$$t'_\beta := \tan \beta' \leq \hat{t}'_\beta \leq t_0 := \tan \beta_0 \leq \hat{t}_\beta := \tan \hat{\beta} \leq t_{\beta} := \tan \beta. \quad (16)$$

anticipating plastic zones $P : \hat{t}_\beta \leq t \leq t_{\beta}$, and $P' : t'_\beta \leq t \leq \hat{t}'_\beta$, adjacent to the free surfaces. We further postulate an elastic core consisting of two possibly distinct elastic zones $E : t_0 \leq t \leq t_{\beta}$, and $E' : \hat{t}'_\beta \leq t \leq t_0$ separated by a singular surface $t = t_0$ on which the yield condition (2) is satisfied.

3.1 Elastic states

The only compatible set of elastic stresses linear in $r$ are readily found as:

$$\sigma_{ij} = (a_{ij} + b_{ij}t)r \cos \theta, \text{ for } i, j = 1, 2 \quad (17)$$

where $a$, $b$ are constants, symmetric in the indices $i, j$ and satisfying

$$a_{12} = -b_{22} \text{ and } b_{12} = -(a_{11} + 1) \quad (18)$$

on the cartesian coordinate system $x_1 = x, x_2 = y$. With (17) denoting solutions for region $E$, a similar set holds in $E'$, involving a generally distinct set of constants $a', b'$. The relations (17)-(18) correspond to the cartesian forms given in CG & CDG and previously by Samsioe (1955).

3.2 Plastic states

The planar form of the SK equations (13)-(14) apply to the plastic zones $P$ and $P'$ and define trajectories on the Mohr-Coulomb yield surface, represented by the cone in Fig. 3. There, distance from the symmetry axis represents
(maximum) shear stress, distance along the axis represents (mean) pressure $\sigma$, and the apex represents a free surface where all stresses vanish.

In the vicinity of a free surface, a suitable solution for stress is

$$\sigma_{ij} = r \cos \theta (1 - t \beta) c_{ij}, \text{ for } i, j = 1, 2$$  \hspace{1cm} (19)

on the same cartesian system employed in (17), where the $c_{ij}$ are constants. These are given in CDG (and denoted by $a_{ij}$) along with the corresponding vertical/horizontal stress ratio:

$$R \equiv \frac{c_{11}}{c_{22}} = \frac{(1 + \sin^2 \varphi) \sin \beta \pm 2 \sqrt{\sin(\varphi - \frac{\pi}{2} + \beta) \sin(\varphi + \frac{\pi}{2} - \beta)}}{\sin \beta \cos^2 \varphi}$$  \hspace{1cm} (20)

The $\pm$ signs distinguish active and passive states, respectively, which become co-incident for $|\beta| = (\pi/2 - \varphi)$, representing the angle of repose. The corresponding form of the functions in (12) are

$$\chi = \frac{t \beta (1 + R) \sin(\theta - \beta)}{2(t \beta R - 1) \cos \beta},$$

$$\Psi = \psi + \theta = \frac{1}{2} \tan^{-1} \left\{ \frac{2}{(R - 1) t \beta} \right\} \text{ (const.)}$$  \hspace{1cm} (21)

The solutions represented by (19-21) are those designated as "simple stress states" and traced back to Rankine (1857) by Sokolovskii (1965, p.222). The
corresponding stress trajectories are represented by generators of the cone in Fig. 3, with principal-stress axes fixed in space. The coincidental agreement with the FPA (fixed-principal-axis) "manifesto" of Cates et al. (1998) does not in our view elevate the latter to the status of a general principle. At any rate, Sololovskii (1965, pp.224ff.) indicates that for wedges such simple plastic solutions apply only in regions near free surfaces that are inclined less steeply than the angle of repose, corresponding to $|\beta| < \pi/2 - \phi$ in (20). Hence, for a wedge everywhere in a plastic-limit state (the "IFE" condition of Cates et al. 1998), such simple solutions must be joined in the interior to more complicated solutions to the SK equations. Thus, continuous plastic states of this kind are seen to consist of pairs of generators emanating from the apex of the cone in Fig. 3 and connected by a curved trajectory lying on the cone. As implied by Sokolovsky (1965) and shown by the computations of van R. Marais (1969) for the symmetric wedge at angle of repose, the simple solutions apply in a vanishing small neighborhood of the apex, where they represent tangents to curves defined by (12).

None of the above, purely plastic solutions exhibits a central minimum in vertical pressure. However, as suggested by the work of Samsoie (1955) and emphasized by Savage (1998) there exist other plastic-limit solutions to (12) which involve discontinuities or plastic "shocks" in $\sigma_{rr}$. Allowed by the hyperbolicity of (1) and (2), and well-established in the soil-mechanics literature (Sokolovsky 1965, Savage et al. 1969, Parry 1995), these discontinuities and the associated switching between active and passive states are viewed by Savage (1998) as a plausible representation of the pressure dip in granular heaps. In Fig 3, such discontinuities involve jumps from one point on the yield cone to another, with different values of $\Psi$ and $\sigma$. As apparent from the Rankine problem above and the computations presented below, such jumps may be realized by abrupt changes over elastic zones of vanishing thickness.

Anticipated to some extent by the analysis of Samsoie (1955) and explored previously in CG and CDG, there exists another class of solutions involving the elastoplastic transitions illustrated in Fig. 3, which we now consider.

### 3.3 Continuous elastoplastic states

Based on the Rankine problem, one is led to suspect that a large, if not arbitrary number of elastoplastic transitions may be admissible. While this is an interesting theoretical question in its own right, we shall restrict our investigation to the class of solutions illustrated by the stress trajectory in Fig. 3, involving at most two distinct elastic sectors $E, E'$ bounded by superficial plastic sectors $P, P'$. To de-limit the parameter space further, we restrict attention to the asymmetric wedges or "berms" treated previously in CDG, with left face at the angle of repose and right face inclined no more steeply, such that:

$$\beta' = \phi - \frac{\pi}{2} < 0 \text{ and } -\beta' \leq \beta \leq \frac{\pi}{2}$$

\(^1\)whose solutions are not the same as those given in CG, contrary to the surmise of Cates et al. 1998
At the lower limit on $\beta$ the radical in (20) vanishes, while the upper limit generally admits two values of $R$. Finally, we insist on continuity of all stresses at the interface between various zones, requiring that the elastic solutions touch the yield surface and ruling out plastic discontinuities of the type discussed above.

As pointed out in CDG and in the above introductory remarks, the existence of contiguous elastic solutions of the form (17) with different values of the constants $a_{ij}, b_{ij}$ involve loss of compatibility$^2$ and requires imposition of (2) at the boundary between $E$ and $E'$ as well as matching of all stresses.

We require elastic states to be at the plastic limit at the boundary $t = \hat{t}_\beta$ between zones $P$ and $E$ states, achieved by the aforementioned stress matching:

$$c_{ij}(1 - \hat{t}_\beta/t_\beta) = (a_{ij} + b_{ij}\hat{t}_\beta) \text{ for } i, j = 1, 2 \quad (23)$$

with similar matching between $P'$ and $E'$. Furthermore, we rule out discontinuities in stress, although generally not in stress gradients, by matching all stresses at the boundary $t = t_0$ between $E'$ and $E$:

$$(a_{ij} + b_{ij}t_0) = (a'_{ij} + b'_{ij}t_0) \text{ for } i, j = 1, 2 \quad (24)$$

Finally, we impose the yield condition (2) on the elastic stresses at $t_0$, as represented by either side of (24). Hence, the elastic trajectories in Figure 3 are "reflected" off the yield surface, as in the Rankine problem considered above.

Inspection reveals that the above problem for the wedge represents a non-linear algebraic system in the twenty-one quantities $a_{ij}, a'_{ij}, \ldots, \hat{t}_\beta, \hat{t}'_\beta, t_0$, (i.e. eighteen $a$’s, $b$’s and $c$’s plus three $\hat{t}$’s), subject to twenty independent equations arising from the eight equilibrium conditions in zones $P', E', \ldots$, together with (nine) stress matching conditions and (three) yield conditions. Thus, the problem is statically indeterminate with one parametric degree of freedom. While the solution becomes determinate for the special case of symmetric distributions, treated in CG, CDG and in the following subsection, the problem otherwise admits continuous families of solutions, found in CDG and considered further below.

### 3.4 Symmetric wedges

For geometrically symmetric wedges having both faces at the angle of repose, we first consider the symmetric states treated in CG and CDG, with

$$\beta = -\beta' = \frac{\pi}{2} - \varphi, \quad t'_\beta = -t_\beta, \quad \hat{t}'_\beta = -\hat{t}_\beta, \quad t_0 = 0 \quad (25)$$

which reduces (23) and (24) to ten equations in ten unknowns and leads, after some algebra, to exactly three values of $\hat{t}_\beta$:

$$\hat{t}_{\beta \pm} = \left\{ \frac{\sqrt{1 + (1 \pm \cos \beta)^2} \pm 1}{(1 \pm \cos \beta)^2} \right\} \sin \beta \quad (26)$$

$^2$mis-stated as a discontinuity in stress rather than in its derivative, in an otherwise inconsequential passage of CDG
\[ i_{\beta 0} = \frac{\sin \beta}{\sqrt{1 + \cos^2 \beta}} \]  

(27)

corresponding to three distinct stress states, with vertical pressure distributions illustrated schematically in Fig. 4.

Figure 4: Schematic symmetric pressure profiles in a symmetric wedge

We denote by \( \beta_- \), \( \beta_0 \) and \( \beta_+ \) the respective values of \( \beta = \arctan i_{\beta} \) corresponding to an arch (pressure dip), a plateau or a peak in \( \sigma_{xx}(y) \). Of these, only the plateau is endowed with fully compatible elastic core, with the other two possessing discontinuous stress gradients at \( t = 0 \) and representing the extremal states identified originally in CG, where a plot of (26)-(27) is given. The discontinuities provide a measure of arching defined by:

\[ \sigma_{xx,y}\mid_{y=0^+} = -\sigma_{xx,y}\mid_{y=0^-} = b_{11} = \frac{1}{i_{\beta \pm}} \left\{ 2 + \frac{(i_{\beta \pm}^2 - i_{\beta}^2)}{i_{\beta} i_{\beta \pm}} \cos \beta \right\} \]  

(28)

for the values of \( i_{\beta} \) in (26) representing peaks and arches, whereas \( b_{11} \equiv 0 \) for the plateau represented by (27).

As found in CDG, there also exist continuous families of asymmetric solutions even for symmetric wedges, which emerge as limits of asymmetric wedges discussed below, corresponding to the ordinates \( \beta = -\beta' \) in Figs. 5 & 6.

Before proceeding further, we address Savage’s (1998) concern that linear-elastic displacements derived from the stresses in \( \mathbf{E} \) and \( \mathbf{E}' \) generally exhibit material overlap or separation along the wedge centerline \( y = 0 \). Based perhaps on a reading of CG without the subsequent clarifications of CDG, this concern seems to discount the widely accepted connection between incompatibility and inelastic deformation (Jaunzemis 1967, Mura 1982). In any event, because of the dimensional scaling discussed above and in CG, the displacements in question have characteristic magnitude \( \rho g H^2 / E \), which tends to be small for a typical geomaterial at moderate depths \( H \) (e.g. fractions of a millimeter at depths of a meter). In our view, the corresponding rigidity is an essential feature of arching, particularly the sensitivity to boundary displacements emphasized by Savage (1998).
3.5 Asymmetric wedges

As anticipated above, continuous families of solutions are found for asymmetric wedges with $\beta > \beta'$ in (22). To characterize these, we adopt the *arching parameter* of CDG:

$$K := \frac{b_{11} - b'_{11}}{[1 + b^2_{11} + b'^2_{11}]^{1/2}} \quad \text{where} \quad b_{11} \equiv \sigma_{xx,y}$$

(29)

as a bounded measure of convexity resembling curvature for differentiable $\sigma_{xx}(y)$.

After a partial analytical reduction (to ten equations in eleven unknowns), the one-parameter solution space of the above set of non-linear equations was treated numerically by a parametric continuation algorithm, for $\beta' = -65^\circ$, at various $\beta \geq -\beta'$, for the special case $\varphi = 25^\circ$. In particular, the non-linear equations were solved numerically by Broyden’s algorithm (Press et al. 1992, pp. 373) for various $t_0$. Then, this set of solutions for various $t_0$ and a particular value of $\beta$ is used as an initial guess to search for a set with a new, nearby value of $\beta$.

The solutions thus determined are presented in Fig. 5 as a kind of bifurcation diagram of $K$ vs. $\beta$, with positive values of $K$ corresponding to arches, negative values to peaks and zeroes to plateaus. Fig. 6 presents an enlargement of the near symmetric wedge region $\beta \approx -\beta'$ including both arch ($K > 0$) and peak ($K < 0$) solutions. The shaded regions in Figs. 5-6 correspond to the continuous set of solutions we were able to identify by means of the above algorithm. Although we find certain isolated points, disconnected from the continuous sets and denoted by open circles in Fig. 5, we cannot rigorously rule out other isolated solutions, either points or regions (*isola*), or certain void regions within the shaded ones. Beyond the plateaus found for the symmetric wedge, we discovered no others. Thus, as a largely theoretical matter, a more exhaustive investigation may be called for.

Figs. 7 & 8 illustrate solutions for $\beta = 80^\circ$ with arch and peak, respectively, giving the vertical pressure scaled by $\rho g H$ at arbitrary depth $H$. The upper half of the figures depicts the wedge shape and boundaries between $P', E', E, P$ (from left to right, respectively, as in Fig. 2). To avoid clutter, the corresponding vertical-pressure distribution is displayed in the lower half as Figs. 7, 8, 9 & 10 (i.e. as a tensile stress).

As indicated in Fig. 6, there exists a continuous family of asymmetric arches even in symmetric wedges. Fig. 9 shows the vertical pressure distribution for one of these, with the angular location of the elastoplastic zone boundaries cited in the caption. Figs. 9 and 10 show the counterparts for peaks.

It can be noted that some solutions, such as those in Figs. 9 & 10 involve virtual discontinuities across thin elastic zones, of the type anticipated above in the Rankine problem. In such cases, one has in effect only one elastic zone, with the other serving to represent a plastic discontinuity. A more complete

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3not to be confused with the earth-pressure coefficient (5), often denoted by the symbol $K$
Figure 5: Arching parameter $K$ vs. right-hand slope angle $\beta$ for a wedge with left-hand slope at $\pi/2 - \varphi = 65^\circ$.

Figure 6: Near symmetric wedge region of Figure 5: arches on the left hand plot and peaks on the right.
Figure 7: Boundaries of elastoplastic zones (upper plot) and vertical pressure (lower plot) vs. $\tan \theta$, for $\beta' = -65^\circ$, $\beta = 80^\circ$ : $\beta = 40^\circ$, $\beta_0 = 24^\circ$, $\beta' = 19^\circ$, $K = 1.14$

Figure 8: Elastoplastic zones and vertical pressure, as in Fig. 6, for $\beta' = -65^\circ$, $\beta = 80^\circ$ : $\beta = 31^\circ$, $\beta_0 = 12.3^\circ$, $\beta' = -49^\circ$, $K = -0.310$
Figure 9: Zones and pressure, as in Fig. 6, for an asymmetric arch in a symmetric wedge, with $\beta = \pi/2 - \varphi = 65^\circ$: $\hat{\beta} = -30.00^\circ$, $\beta_0 = 23.98^\circ$, $\beta' = 25^\circ$, $K = 1.008$

theoretical investigation of the governing algebraic equations might serve to isolate such states.

4 Elastoplastic cones

We restrict attention to axisymmetric stresses in axisymmetric cones with surface $\theta = \beta = \pi/2 - \varphi$. Guided by the above analysis of wedges, we consider a class of statically admissible, continuous solutions involving a plastic surface layer $P: \hat{\beta} \leq \theta \leq \beta$ with, provisionally, a conical elastic core $E: 0 \leq \theta \leq \hat{\beta}$.

4.1 Plastic states

An admissible closure for the statically indeterminate hoop stress $\sigma_{\phi\phi}$ is

$$\sigma_{\phi\phi} = \lambda \sigma_{II} + (1 - \lambda) \sigma_I = \quad \text{with} \quad 0 \leq \lambda \leq 1$$

which is satisfied, exactly or fortuituously, for the problem at hand by the forms:

$$\sigma_{\phi\phi} = \sigma_I \quad \text{(Haar-Von Kármán), or}$$

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} \quad \text{(Levy)}$$

where $\sigma_I$ is the minimum principal stress in (7). The relation (31) is generally attributed to Haar and von Kármán (1909), whereas the terminology for (32) is based on its origins in Levy’s flow rule, e.g. applied to flows in axisymmetric cylindrical bunkers (Nedderman 1992). The two closures are very similar and
their relationship to each other is discussed in Nedderman (1992 pp. 280).
Because of its close analogy with isotropic linear elasticity, the Levy rule is identical with that given below for regular elastic states.

4.2 Elastic states

Candidate elastic solutions for region $E$ are given by the axisymmetric $r$-linear forms in Lure (1964) as

$$
\begin{align*}
\sigma_{rr} &= -8(1 + \nu)A + \frac{(3 - \nu)}{5(1 - \nu)}r \cos \theta - 4B(1 + \nu) \ln \frac{1 + \mu}{1 - \mu}r \cos \theta \\
&\quad + 8B(1 + \nu)r, \\
\sigma_{r\theta} &= [-2(1 + \nu)A - \frac{(1 - 2\nu)}{6(1 - \nu)}r \sin \theta - B(1 + \nu) \ln \frac{1 + \mu}{1 - \mu}r \sin \theta \\
&\quad - 2B(\frac{1 + \nu}{1 - \mu^2})r \sin \theta, \\
\sigma_{\theta\theta} &= [-16(1 + \nu)A + \frac{(1 + 3\nu)}{5(1 - \nu)}r \cos \theta - 8B(1 + \nu) \ln \frac{1 + \mu}{1 - \mu}r \cos \theta \\
&\quad + 4B(\frac{3 - 2\nu}{1 - \mu^2})r \cos \theta + 4B(7 + 2\nu)r, \\
\sigma_{\phi\phi} &= [-16(1 + \nu)A + \frac{(1 + 3\nu)}{5(1 - \nu)}r \cos \theta - 8B(1 + \nu) \ln \frac{1 + \mu}{1 - \mu}r \cos \theta \\
&\quad - 4B(\frac{3 - 2\nu}{1 - \mu^2})r \cos \theta + 4B(1 + 6\nu)r
\end{align*}
$$

where $\mu := \cos \theta$, $\nu$ denotes Poisson’s ratio and $A, B$ are arbitrary constants.
Figure 11: Stress vs. $\tan \theta$ in a cone: $\sigma_{rr}$ (solid curve), $\sigma_{\theta\theta}$ (dashed curve), $\sigma_{r\theta}$ (dotted curve)

Since certain terms multiplying $B$ in (33) involve axial singularities at $\theta = 0$, we must reject them or else admit yet another elastoplastic region surrounded by $E$. We opt for the simpler regular-elastic states with $B = 0$ and, coincidentally, with $\sigma_{\phi\phi} \equiv \sigma_{\theta\theta}$ in (33), which allows continuous matching with plastic states $P$ subject to (32). By means of a standard Runge-Kutta stiff integrator (MatLab$^TM$, "ODE23S"), we integrate (13)-(14) from the cone surface $\theta = \beta = (\pi/2) - \varphi$ inwards subject to $\chi(\beta) = 0$. The requirement that $d\psi/d\theta$ remain bounded at the surface yields $(d\Psi/d\theta)(\beta) = 0$, implying by (14) that $\psi(\beta) = -\beta/2$. The transition from plastic to elastic states occurs at an angle $\hat{\theta}$ to be determined, at which all plastic stresses must match the elastic stresses in (33) with $B \equiv 0$. Fig. 11 presents the resulting stress profiles for Poisson's ratio $\nu = 1/4$ with the closure (32). Fig. 12 presents the corresponding vertical-pressure profile. All stresses are seen to be continuous.

The stress profiles obtained by means of the closure (31) are very similar to the ones shown for (32), although $\sigma_{\phi\phi}$ for (31) is not strictly continuous at the plastic-elastic transition. Moreover, in calculations not shown here, we find that stress profiles like those in Figs. 11-12 are fairly insensitive to variations in $\nu$ and $\varphi$, over ranges $0.15 \leq \nu \leq 0.4$ and $25^\circ \leq \varphi \leq 40^\circ$.

5 Discontinuous plastic states

As discussed above, there exist purely plastic states involving discontinuities in $\sigma_{rr}$ and, hence, in the vertical pressure. For the sake of completeness and comparison, we present new numerical solutions of this type to the SK equations, for the special case of a symmetric wedge, touched on by Samsioe (1955) and discussed at length by Savage (1998), and, then, for an axisymmetric cone.
5.1 Symmetric wedges

We seek solutions to the planar SK equations for symmetric wedges with surfaces at the angle of repose, and with discontinuity in $\sigma_{rr}$ allowed along a ray $\theta = \hat{\theta}$. Assuming simple plastic solutions of the type (21), we apply continuity conditions on the expressions given for $\sigma_{r\theta}, \sigma_{\theta\theta}$ in (10) at $\theta = \hat{\theta}$. We note that $\chi(\beta) = 0, \psi(\beta) = -\beta/2$ and $\psi(0) = \pi/2$, representing the above-mentioned switch from active state at the surface to passive state at the center. The stress continuity conditions are solved numerically to give $\hat{\theta} = 22.35\ldots^\circ$.

Stresses are then obtained by integrating (13) inwards from $\theta = \beta$, allowing for discontinuity in $\chi$ at $\hat{\theta}$, an approach that seems simpler than the one described by (Savage 1998). In any event, we have not been able to ascertain the exact relation between his discontinuous solutions and that found here, for
which the vertical pressure is shown in Fig. 13.

5.2 Axisymmetric cones

As with wedges, we consider cones with surface at the angle of repose and seek plastic solutions with stress discontinuity along a ray $\theta = \hat{\theta}$. However, unlike the symmetric wedge, we can find no simple solutions with constant $\Psi$ in either active or passive regions. A shooting technique was employed to integrate (13)-(14), with $\chi(0)$ adjusted until continuity of $\sigma_{rr}$ and $\sigma_{\theta\theta}$ was achieved along a ray having an allowed discontinuity in $\sigma_{rr}$. Only one discontinuous solution was found for each of the closures (31) and (32). The vertical pressure shown in Fig. 14 arises from (32) and is found to be very close to that obtained from the alternative form (31), essentially because $\sigma_{\theta\theta} \approx \sigma_I$.

![Figure 14: Vertical pressure for discontinuous plastic state in a cone.](image)

6 Conclusions

Our key findings are adequately summarized in the above abstract. Our results for axisymmetric cones support the conclusions of CDG for wedges, namely, that conventional elastoplasticity produces stress distributions qualitatively similar to those associated with "arching" in granular heaps.

As for the relevance to stress fields in real granular heaps, it is plausible that the static solutions found above correspond to states that are physically accessible from simpler, uniform states. Moreover, it seems evident that certain states may be generated from others by relatively simple far-field "bottom" displacements under a large granular mass (Trollope and Burman 1980, Savage 1998) or by "avalanching" or "slumping" induced by these or other mechanical disturbances. Of course, the elementary models employed above do not distinguish between different friction angles $\varphi$ arising from such processes, i.e. from history-dependent granular plasticity (cf. Herrmann 1998, who indicates there
may be many different surface angles depending on the mode of deposition). Similarly, the above models do not allow for depositional anisotropy or "fabric", the effects of which may be reflected in the experiments on mono-sized sphere assemblies reported in CG.\(^4\)

The notion that the process of deposition might be represented by a combination of elastic and inelastic deformations superimposed on a uniform reference state (e.g. a state of zero stress) is anticipated in the continuum theory of dislocations. In this respect, the association by Jaunzemis (1967, p. 338) of elastic incompatibility with a hypothetical "manufacturing" process seems particularly germane. While lending a certain physical significance to various static states, that association cannot, of course, inform us as to which states are favored by dynamics and stability any more, it should be added, than can the various prescriptions in the contemporary literature.

While the foregoing considerations suggest that plausible but more complex elastoplastic models might serve to capture the effects of deposition, it should be amply clear that their static states would generally exhibit the \textit{Rankine indeterminacy} of the simpler models considered in the present work. We interpret this to mean experimentally that, whatever the static state resulting from a given deposition process, there exist nearby static states generally accessible through small bottom deflections or other mechanical disturbance.

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\(^4\)We hasten to add, however, that these experiments involve a relatively small number of particles and exhibit a large scatter in data.


