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Regularization by compressibility of the $\mu(I)$ model of dense granular flow

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The following article deals with the role of compressibility in regularizing the well-known $\mu(I)$ model, i.e., eliminating the short-wavelength (Hadamard) instability revealed by Barker *et al.* ["Well-posed and ill-posed behaviour of the $\mu(I)$-rheology for granular flow," J. Fluid Mech. **779**, 794–818 (2015)]. In particular, we discuss the compressible-flow models proposed in the recent papers by Heyman *et al.* ["Compressibility regularizes the $\mu(I)$-rheology for dense granular flows," J. Fluid Mech. **830**, 553–568 (2017)] and Barker *et al.* ["Well-posed continuum equations for granular flow with compressibility and $\mu(I)$-rheology," Proc. R. Soc. A **473**(2201), 20160846 (2017)]. In addition to a critique of certain aspects of their proposed constitutive models, we show that the main effect of their regularizations is to add viscous effects to the shear response in a way that appears unfortunately to eliminate quasi-static yield stress. Another goal of the present work is to show how the development and analysis of visco-plastic constitutive relations are facilitated by dissipation potentials and the dissipative analog of elastic potentials. We illustrate their utility in Sec. IV of this article, where it is shown that a constant non-zero yield stress leads to loss of convexity that can only be restored by substituting viscous effects or else by adding spatial-gradient effects proposed previously by the present authors [Goddard, J. and Lee, J., "On the stability of the $\mu(I)$ rheology for granular flow," J. Fluid Mech. **833**, 302–331 (2017)]. *Published by AIP Publishing.* https://doi.org/10.1063/1.5040776

I. INTRODUCTION

The present paper is largely motivated by two recent publications[3,10] denoted, respectively, by Ref. 2 and Ref. 1 in the following (reflecting the order in which they are analyzed below). Both articles have proposed constitutive equations with compressibility effects to regularize the incompressible $\mu(I)$ model against the Hadamard (short-wavelength) instability. As shown below, this is tantamount to adding a more pronounced dependence of shear stress on the rate of deformation, reflecting a material time scale and the associated viscous effect. This to be contrasted with the length-scale or gradient effect in models that depend on higher spatial gradients, such as that proposed by the present authors,[9] referred to as Ref. 3 in the following. In addition to a critique of the models in Refs. 1 and 2, one goal of the present discussion is to highlight the utility of dissipation potentials in the formulation of visco-plastic constitutive equations. The connection between the convexity of potentials and material stability, as reflected by the resulting ellipticity of the quasi-static field equations, is well established in the solid-elasticity literature. In the same literature, one finds useful variational principles which are illustrated by other studies on the mechanics of visco-plastic fluids.[9,11,12,14]

As a brief recapitulation of certain notation employed in previous studies, we shall, for the sake of comparison with Refs. 1 and 2, make implicit use of their norm $\|\mathbf{A}\| = |\mathbf{A}|/\sqrt{2}$ for second-rank tensors $\mathbf{A} = [A_{ij}]$, instead of the Euclidean (or Frobenius) norm $|\mathbf{A}| := \sqrt{\mathbf{A}^T \cdot \mathbf{A}}$ employed in Ref. 3. As standard in various continuum mechanics literature, we denote the deviator or traceless part of the second rank tensors $\mathbf{A}$ by primes, with $\mathbf{A}' = \mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}/3$, where $\mathbf{I} = [\delta_{ij}]$ denotes the unit tensor and $\mathrm{tr}(\mathbf{A}) = \mathbf{I} \cdot \mathbf{A}$ is the trace, and we denote the transformation by a second-rank tensor $\mathbf{A}$ of a vector $\mathbf{x} = [x_i]$ by the symbol $\mathbf{Ax} = [A_{ij}x_j]$ suppressing a dot for the indicial contraction which we employ in certain expressions. Here, as below, we frequently use square brackets [ ] to indicate the components of tensors on a Cartesian basis, and we employ the Cartesian summation convention for contraction on repeated indices. Finally, we take the three isotropic invariants of a symmetric second-rank tensor $\mathbf{A}$ to be

$$A_1 = \mathrm{tr}(\mathbf{A}),\ A_2 = \mathrm{tr}(\mathbf{A}'^2)/2 = \|\mathbf{A}'\|^2,$$
$$A_3 = \det(\mathbf{A}') = \mathrm{tr}(\mathbf{A}'^3)/3,$$

with

$$\partial_{\mathbf{A}'} A_k = (\mathbf{A}')^{k-1},\ k \geq 2, \tag{1}$$

which are easily related to standard invariants. We shall have occasion then to deal with the special cases, $p = -\sigma_1/3 = -\mathrm{tr}(\boldsymbol{\sigma})/3$, and $\mathbf{A} = \mathbf{D}$ in (1), respectively, for Cauchy stress $\boldsymbol{\sigma}$, and deformation rate $\mathbf{D} = \mathrm{sym}(\nabla\mathbf{v})$, with the "shear rate" defined as $\mathbf{D}'$.

We consider here a class of visco-plastic models that represent strongly dissipative or "hyperdissipative" materials,

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i.e., those with the rate of dissipation given by positive-definite stress power,

$$\mathfrak{D} = \boldsymbol{\sigma} \cdot \mathbf{D} > 0, \text{ for } |\mathbf{D}||\boldsymbol{\sigma}| > 0, \tag{2}$$

and which are therefore endowed with a frame-indifferent, non-negative, and convex dissipation potential $\psi(\mathbf{D}) = \psi(D_1, D_2, D_3)$ such that[6,8]

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p\mathbf{I} = \partial_{\mathbf{D}}\psi = (\partial_{D_1}\psi)\mathbf{I} + (\partial_{D_2}\psi)\mathbf{D}' + (\partial_{D_3}\psi)\mathbf{D}'^2$$

with

$$p = -(\partial_{D_1}\psi + \frac{2}{3}D_2\partial_{D_3}\psi)$$

and

$$\boldsymbol{\sigma}' = -\frac{2}{3}D_2\partial_{D_3}\psi\mathbf{I} + (\partial_{D_2}\psi)\mathbf{D}' + (\partial_{D_3}\psi)\mathbf{D}'^2, \tag{3}$$

where $\boldsymbol{\sigma}'$ represents the shear stress and p represents the pressure, and we employ the chain rule

$$\partial_{\mathbf{D}} = \mathbf{I}\partial_{D_1} + \partial_{\mathbf{D}'} = \mathbf{I}\partial_{D1} + \mathbf{D}'\partial_{D_2} + \mathbf{D}'^2\partial_{D_3}. \tag{4}$$

The rate of dissipation is thus given by

$$\mathfrak{D} = \boldsymbol{\sigma} \cdot \mathbf{D} = \boldsymbol{\sigma}' \cdot \mathbf{D}' - pD_1 = \mathbf{D}' \cdot \partial_{\mathbf{D}'}\psi + D_1\partial_{D_1}\psi$$
$$= \sum_{k=1}^{3} kD_k\partial_{D_k}\psi, \tag{5}$$

which is non-negative owing to the non-negativity and convexity of ψ. The model (3) is a dissipative form of Reiner's "dilatant" fluid (later the "Reiner-Rivlin" fluid) proposed some years ago as a model for granular media[7] without regard for dissipation potentials. This enlarges the class of non-dimensional models beyond that declared to be unique in Ref. 1 [in the text preceding their Eq. (3.8)].

In the present paper, we consider the special case $\partial_{D_3}\psi = 0$ which represents the restricted form common to much of the current modeling, which might be called "planar," since the $D_3 \equiv 0$ implies that one principal shear rate vanishes identically. This gives a restricted form of the Reiner-Rivlin fluid that is appropriate to the study of the planar flows that are the object of the following discussion. Included of course is the further restricted class of incompressible visco-plastic fluids with $\partial_{D_1}\psi \equiv 0$, and $\mathbf{D}' \equiv \mathbf{D}$, such as the $\mu(I)$ model,[9]

$$\psi = \psi_0(D_2) = \frac{2p}{\theta}\left[\mu_\infty I + (\mu_0 - \mu_\infty)I_* \ln\left(\frac{I + I_*}{I_*}\right)\right]$$

with

$$\boldsymbol{\sigma} = (\partial_{\mathbf{D}}\psi)_p = \psi'(I)(\partial_{D_2}I)\mathbf{D} = p\mu(I)\frac{\mathbf{D}}{\sqrt{D_2}}$$

and

$$\mu(I) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{I/I_* + 1},$$

where

$$I = \theta\sqrt{D_2} \quad \text{and} \quad \theta = 2d\sqrt{\rho_s/p}. \tag{6}$$

Here, $\psi' = (\partial_I\psi)_p$, while θ is an inertial time constant, μ_0 and μ_∞, with $\mu_0 \le \mu_\infty$, are the limiting (Coulomb) coefficients of rate-independent friction, and I_* is a non-dimensional parameter mediating the transition between the quasi-static $I = 0$ limit and rapid-flow ("Bagnold") limit $I = \infty$.

Recent papers show that the model (6) is ill-posed, exhibiting Hadamard (short-wavelength) instability.[2,3,9,10] We believe that this instability arises from the limiting regimes of constant μ at $I = 0$ and $I = \infty$ where the potential ψ_0 in (6) is marginally convex, as pointed out in Ref. 3 and shown more clearly below. As also indicated in Ref. 3, the mathematical term "ill-posed" should not be construed as "unphysical," since short-wavelength instability signals more often than not the emergence of spatial discontinuities in numerous physical settings. However, whatever one's motivation for modifying this instability, it seems evident to us that this can only be achieved either by length-scale ("gradient") effects, as proposed in Ref. 3, and/or by time-scale ("viscous") effects, inherent to the compressible-fluid models of Refs. 1 and 2. The partial regularization proposed in another work by certain of the same co-authors[1] appears to rely on the latter. While proper dissipative behavior is guaranteed by the model of Ref. 3, it is not immediately evident that this is the case for the models proposed in Refs. 1 and 2, a matter to be addressed in the following.

To illustrate the utility of the dissipation potential, we note that the stress derived solely from it is subject to Edelen's non-linear Onsager symmetry[6,8] which amounts to the equality of various cross derivatives. For the planar-shearing form $\psi = \psi(D_1, D_2)$, this restriction is expressed by the Maxwell-type relation

$$\partial_{D_1}\partial_{\mathbf{D}'}\psi = \partial_{D_1}\boldsymbol{\sigma}' = -\partial_{\mathbf{D}'}p = \partial_{\mathbf{D}'}\partial_{D_1}\psi,$$

where

$$\partial_{\mathbf{D}'} = \mathbf{D}'\partial_{D_2}$$

and hence

$$\partial_{D_1}(\boldsymbol{\sigma}' \cdot \mathbf{D}') = -2D_2\partial_{D_2}p, \tag{7}$$

whenever $\boldsymbol{\sigma}'$ and p are given as functions of $\mathbf{D}'$ and D_1, as is the case with Ref. 1, considered next. We note that for compressible flow in general, the potential ψ depends on a variable particle fraction ϕ, whose evolution is governed by the mass balance for dry granular media with rigid particles, namely,

$$\dot{\phi} = \partial_t\phi + \mathbf{v} \cdot \nabla\phi = -\phi\nabla \cdot \mathbf{v} = -\phi D_1. \tag{8}$$

References 1 and 2 involve either a "dilatancy relation" $D_1 = \nu(D_2, p, \phi)$ or else an "equation of state" $p = p(D_1, D_2, \phi)$ which is tantamount to the inverse. In the following, we shall often suppress the notation for dependence on ϕ whose evolution is governed by (8), recalling that dissipation potentials may depend on any number of such evolutionary internal variables.

We emphasize at the outset that we did not deem it incumbent on us to follow closely the numerical analyses of Refs. 1 and 2, as certain details are not immediately evident and, more importantly, because it would constitute a distraction from fundamental questions regarding the formulation and interpretation of the constitutive modeling.

II. MODEL OF REF. 1

The model of Ref. 1 is given essentially by

$$\boldsymbol{\sigma}' = \frac{p_{\mathrm{eq}}\mu_a(\phi)}{\sqrt{D_2}}\mathbf{D}' \text{ and } p = \left(1 - \mu_b(\phi)\frac{D_1}{\sqrt{D_2}}\right)p_{\mathrm{eq}},$$

where

$$p_{\text{eq}} = \frac{\rho D_2 d^2}{(\phi - \phi_{\max})^2}. \tag{9}$$

Here, we denote by μ_a the quantity denoted by the somewhat overworked symbol μ in Ref. 1. A functional form is proposed for $\mu_a(\phi)$ in Ref. 1, whereas the function $\mu_b(\phi)$ is provisionally left arbitrary. We note that $\boldsymbol{\sigma}'$ exhibits the Bagnold scaling with apparent viscosity proportional to $\sqrt{D_2}$.

We also note in passing a certain puzzling relation in Ref. 1, where the inertial number I is defined in the first paragraph of p. 557 by the relation given above in (6) with p_{eq} replacing p. However, according to the definition of p_{eq} in (9), and with their norm $\|\mathbf{S}\| = \sqrt{D_2}$, one obtains a nugatory relation $I \equiv (\phi_{\max} - \phi)$ so that the mass balance in terms of I in their Eq. (13) reduces trivially to that given previously in their Eq. (3.1). Without understanding the significance and consequences for their subsequent analysis, we proceed to the main issue of concern in the present work.

Thus, the relevant derivatives in (7) are found from (9) to be

$$\partial_{D_1}\boldsymbol{\sigma}' = \mathbf{0} \text{ and } \partial_{\mathbf{D}'}p = \mathbf{D}'\partial_{D_2}p = \frac{\rho d^2}{(\phi_{\max} - \phi)^2}\left[1 - \frac{\mu_b D_1}{\sqrt{D_2}}\right]\mathbf{D}', \tag{10}$$

which clearly are unequal, indicating the absence of Onsager symmetry. [Note that this symmetry can be restored by including a factor, as a definite function of $D_1/\sqrt{D_2}$, in the definition (9) of $\boldsymbol{\sigma}'$]. Moreover, the nominal dissipation rate is given in terms of a quadratic form in $R = D_1/\sqrt{D_2}$ as

$$\mathfrak{D} = \boldsymbol{\sigma}' \cdot \mathbf{D}' - pD_1 = p_{\text{eq}}\sqrt{D_2}(2\mu_a - R + \mu_b R^2). \tag{11}$$

This has real positive roots $R = R_\pm = (1 \pm \sqrt{1 - 8\mu_a\mu_b})/(2\mu_b)$ for $8\mu_a\mu_b < 1$ such that $\mathfrak{D}$ is non-positive for $R_+ \geq R \geq R_-$. Hence, to avoid negative dissipation, one should take $8\mu_a\mu_b > 1$, whereas the relation $\mu_b = (1 - 7\mu_a/6)$ proposed in Ref. 1 as the demarcation line between stable and unstable equations appears to violate this condition over a considerable range of the μ_a values displayed on the abscissa of the plot in their Fig. 2(a). Without further detailed analysis, we cannot ascertain whether or how their stability analysis may have been influenced by a possible negative dissipation.

Whenever $\mathfrak{D}$ is non-negative, we may make use of Edelen's formula[8] to write the dissipation potential in terms of dissipation rate $\mathfrak{D}(\mathbf{D})$ as

$$\psi(\mathbf{D}) = \int_0^1 \mathfrak{D}(\lambda\mathbf{D})\frac{d\lambda}{\lambda} = \frac{3}{2}\mathfrak{D}(\mathbf{D}) \tag{12}$$

since $\mathfrak{D}(\mathbf{D})$ is homogeneous of degree 3/2 in $\mathbf{D}$. Then, according to another formula of Edelen,[8] the constitutive equations of Ref. 1 can be written as

$$\boldsymbol{\sigma}' = \partial_{\mathbf{D}'}\psi + \boldsymbol{\sigma}'_0 \text{ and } p = -\partial_{D_1}\psi + p_0,$$

where

$$\boldsymbol{\sigma}'_0 \cdot \mathbf{D}' - p_0 D_1 \equiv 0, \text{ for } \forall\mathbf{D}. \tag{13}$$

That is, $\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}'_0 - p_0\mathbf{I}$ is "powerless" or gyroscopic, and one may make use of the preceding expressions for ψ, $\boldsymbol{\sigma}'$, and p to write down explicit expressions for $\boldsymbol{\sigma}_0$ and p_0. That these do not vanish identically again signals the failure of nonlinear Onsager symmetry and the principle of minimum dissipation potential for quasi-static flows,[3] which we hasten to add does not *per se* invalidate their constitutive equation.

As a more important matter, it is evident that the Bagnold scaling of $\boldsymbol{\sigma}'$ in (9) represents "Bagnold shear-thickening" without yield stress, which is not expected to admit the Hadamard instability. It also seems apparent that the resulting stabilization could have been achieved by any number of viscosity models, without appeal to compressibility.

III. MODEL OF REF. 2

The constitutive relations of Ref. 2 are given in what may be regarded as an implicit form in stress, about which Rajagopal and Srinivasa[17] have written extensively with a view to pressure sensitive viscosity or plasticity, particularly in the nearly incompressible regime. Thus, the shear stress is allowed to depend on the pressure, no longer a work-free reaction against the incompressibility constraint,

$$\begin{aligned}\boldsymbol{\sigma}' &= p\left[\alpha(I) - \frac{p}{C(\phi)}\right]\frac{\mathbf{D}'}{\sqrt{D_2}} \text{ and } D_1 = \nu(D_2, p)\\ &= 4\sqrt{D_2}\left[\alpha(I) - \mu(I) - \frac{p}{C(\phi)}\right].\end{aligned} \tag{14}$$

The second relation represents the sort of dilatancy relation referred to above, while $\mu(I)$ is the function defining $\mu(I)$-rheology, and $\alpha(I)$ is given in Ref. 2 by

$$\alpha(I) = \frac{4}{5}\mu(I) + \frac{12}{25}I^{-2/5}\int_0^I s^{-3/5}\mu(s)ds. \tag{15}$$

In the regimes $I \ll 1$ and $I \gg 1$ where μ becomes constant, it follows that $\alpha \to 2\mu$.

At this juncture, we should express our concern that their constitutive parameter $C(\phi)$ derives its stress units from an assumed proportionality in Eq. (2.30) of Ref. 2 to the acceleration of gravity, an *extrinsic* quantity which, by d'Alembert's principle, is basically equivalent to accelerations of the material itself. Since this includes linear accelerations and rigid-body rotations, the model cannot be considered frame-indifferent and, however seriously one views this, it should be clearly pointed out. By contrast, the critical-state theory of soil mechanics, to which the authors appeal, involves an *intrinsic* parameter which serves to cap off Drucker-Prager cones, and C may serve as a similar function in the present model (*vide infra*). Setting aside such concerns, we proceed with the analysis of the authors' constitutive theory as it stands.

A bit of thought shows that the principal dependent variables in the above model are D_2 and p, the latter dependence inherited from the $\mu(I)$ model. Hence, it is expeditious to define a new potential $\varphi(\mathbf{D}', p)$ given by the Legendre transformation

$$\varphi(\mathbf{D}', p) = \varphi(D_2, p) = \psi(D_2, D_1) + pD_1, \tag{16}$$

where we suppress the notation for dependence on the particle volume fraction ϕ for the reasons stated above following (8). We note that (16) represents a precise analog of the standard thermodynamic transformation from Helmholtz free energy ψ to Gibbs free energy φ if D_2 is interpreted as temperature and D_1 as volume. At any rate, it is easy to show that

$$\boldsymbol{\sigma}' = \partial_{\mathbf{D}'}\varphi(\mathbf{D}', p) = \partial_{\mathbf{D}'}\psi(\mathbf{D}', D_1)$$

or

$$\partial_{D_2}\varphi(D_2, p) = \partial_{D_2}\psi(D_2, D_1)$$

and

$$\partial_p\varphi(D_2, p) = D_1 = \nu(D_2, p), \tag{17}$$

from which one obtains another Maxwell-type relation

$$\partial_p\boldsymbol{\sigma}' = \partial_{\mathbf{D}'}\nu = \mathbf{D}'\partial_{D_2}\nu \text{ or } \partial_p(\boldsymbol{\sigma}'\cdot\mathbf{D}') = 2D_2\partial_{D_2}\nu, \tag{18}$$

where $\nu(\mathrm{D}_2, p)$ is the dilatancy function appearing in the second equation of (14). The left-hand and right-hand sides of the second equality are given, respectively, by (14) as

$$\partial_p\boldsymbol{\sigma}' = \left[\alpha - \frac{1}{2}I\alpha' - 2\frac{p}{C}\right]\frac{\mathbf{D}'}{\sqrt{D_2}}$$

and

$$\partial_{\mathbf{D}'}\nu = 2\left[\alpha - \mu + I(\alpha' - \mu') - \frac{p}{C}\right]\frac{\mathbf{D}'}{\sqrt{D_2}}, \tag{19}$$

where primes denote derivatives with respect to I. The requirement of equality between these expressions yields the ordinary differential equation given in Ref. 2 with solution given by (15). Hence, we conclude that the model of Ref. 2 is strongly dissipative, that is, in principle derivable from a dissipation potential, provided the nominal dissipation rate

$$\mathfrak{D} = 2p\sqrt{D_2}(2\mu - \alpha + \frac{p}{C}) \tag{20}$$

is non-negative, which seems to be the case for $p/C > \alpha - 2\mu$, with $p > 0$ and $C > 0$, for any function $\alpha(I) \le 2\mu(I)$, including the authors' assumed form (15).

For the sake of completeness, we note that one can determine the function $\varphi(D_2, p)$ by making use of the relation $\boldsymbol{\sigma}' = \partial_{\mathbf{D}'}\varphi = (\partial_{D_2}\varphi)\mathbf{D}'$ and the first equation of (14) to give

$$\varphi(D_2, p) = 2p\sqrt{D_2}\left[\frac{1}{3}(5\alpha - 4\mu) - p/C\right]. \tag{21}$$

From (16) and (21), one can derive an expression for ψ as a function of D_2 and p that has no immediate analytical value since the second member of (14) is generally not analytically invertible to give p and ψ as a function of the kinematic quantities D_1 and D_2. The exceptional case of constant μ is considered below.

For the above reasons, one cannot generally obtain an explicit analytic expression for $\psi^*(\boldsymbol{\sigma})$, the complementary or dual force potential to $\psi(\boldsymbol{\sigma})$, such that $\mathbf{D} = \partial_{\boldsymbol{\sigma}}\psi^*$. In that connection, we note that, contrary to the terminology of Ref. 2, the relation $\nu = \partial_p\varphi$ for $\nu(D_2, p)$ in (14) represents but one component of a "flow rule," one which is distinct from that based on a plastic potential $\psi^*(\boldsymbol{\sigma})$ in the standard plasticity literature[15,16] (Sec. IV C 1).

Given the likely role of Coulomb yielding in the Hadamard instability of the $\mu(I)$ model, it is worthwhile investigating the case where there exists a regime in which $\mu = \mu_c$, a constant, to terms $o(1)$ in I, from which it is easy to show that (14) gives the stress in that regime in terms of kinematic quantities as

$$\begin{aligned} p = C\left(\mu_c - \frac{D_1}{4\sqrt{D_2}}\right) \text{ and } \boldsymbol{\sigma}' &= (2\mu_c - p/C)p\frac{\mathbf{D}'}{\sqrt{D_2}} \\ &= C\left(\mu_c^2 - \frac{D_1^2}{16D_2}\right)\frac{\mathbf{D}'}{\sqrt{D_2}}. \end{aligned} \tag{22}$$

According to the second equality, the effective friction coefficient is $\mu = (2\mu_c - p/C)$ which reduces to μ_c at the critical state $D_1 \equiv 0$. In the kinematic form (22), the restriction to positive p/C requires that $D_1/4\sqrt{D_2} \le \mu_c$, while co-directionality of $\boldsymbol{\sigma}'$ and $\mathbf{D}'$ requires that $|D_1|/4\sqrt{D_2} \le \mu_c$. For non-isochoric flow with $D_1 \neq 0$, it is evident that the magnitude of the shear stress increases monotonically with D_2 from zero at $D_2 = D_1^2/16\mu_c^2$, while $\mu = (2\mu_c - p/C)$ decrease monotonically from $p = 0$ with increasing p. Hence, Drucker-Prager yielding at $D_2 = 0$ is replaced by a non-linear viscous effect. In the region $0 \le D_2 \le D_1^2/16\mu_c^2$, which corresponds to $p > 2\mu_c C$, the stress $\boldsymbol{\sigma}'$ is directed oppositely to the shear rate according to (22), an unquestionably unstable situation. The restriction $p \le 2\mu_c C$ may be regarded as a flat truncation of the Drucker-Prager cone. Beyond this truncation, a new defining equation for stress seems to be required. If shear stress and pressure are to be continuous, one possibility is to take $\boldsymbol{\sigma}' \equiv 0$ and $p \equiv 2\mu_c C$ in the interval $0 \le D_2 \le D_1^2/16\mu_c^2$, avoiding an unstable decrease in stress with increasing strain rate.

The above considerations raise the question as to whether a simpler modification of the $\mu(I)$ model without compressibility might eliminate the Hadamard instability and, unlike the models of Refs. 1 and 2, preserve a non-zero yield stress at $D_2 = 0$. We shall show next that the answer is in the negative for models described by a dissipation potential.

IV. DISSIPATION POTENTIAL AND HADAMARD INSTABILITY

As pointed out in Ref. 3, the classic work of Browder[4] implies that the Hadamard instability can be attributed to the loss of generalized ellipticity in the quasi-static field equations. Hence, despite the recent heroic efforts to investigate the dynamics analytically and numerically, it suffices to consider the stability of homogeneous and unbounded creeping flows such that both the effects of boundary conditions on finite domains and of continuum-level inertia are negligible.

Given the compelling analogy between "hyperdissipativity" and "hyperelasticity," both involving stress derived from a potential, one may also appeal to classic studies of nonlinear elasticity,[5,13] where ellipticity of the quasi-static field equations turns on the convexity of the potential, as defined by its Hessian. This fourth-rank tensor determines the so-called acoustic tensor of elasticity which is directly related to the second-rank tensor governing linear stability. In the case of the dissipation potential, the Hessian is given by $\boldsymbol{\mathcal{C}} = [\mathcal{C}_{ijkl}] = \partial^2_{\mathbf{D}}\psi = [\partial_{D_{ij}}\partial_{D_{kl}}\psi]$, and we note that the value of this tensor $\boldsymbol{\mathcal{C}}^{(0)}$ evaluated at the homogeneous base state arises from the linearization of $\nabla \cdot \boldsymbol{\sigma}$ about this state, which happens to determine the linear stability of the dynamical equations of motion. We recall that one obtains a similar result from a constitutive model involving higher velocity gradients, as shown in Ref. 3. Of course, when the stress $\boldsymbol{\sigma}$ is not given explicitly by a dissipation potential, linear stability is determined by a more general fourth-rank tensor $(\partial_{\mathbf{D}}\boldsymbol{\sigma})^{(0)}$ that generally involves gyroscopic stresses.

In the following, we consider a more general dissipation potential than that given by the $\mu(I)$-model, which presumably includes models such as that proposed recently by Barker and Gray.[1] Moreover, it allows for a general treatment of the role of yield stress on the Hadamard instability.

Thus, given a potential $\psi(D_1, D_2)$, the components of the Hessian are derived in the Appendix and we write it here in the direct tensor notation as

$$\mathcal{C} = (\partial^2_{D_1}\psi)\mathbf{I}\otimes\mathbf{I} + (\partial_{D_2}\partial_{D_1}\psi)(\mathbf{D}'\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{D}') + (\partial_{D_2}\psi)(\tilde{\mathcal{I}} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}) + (\partial^2_{D_2}\psi)\mathbf{D}'\otimes\mathbf{D}', \tag{23}$$

where $\tilde{\mathcal{I}} = [\tilde{\mathcal{I}}_{ijkl}] = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]/2$ is the standard symmetrized form of the fourth-rank identity $\mathcal{I} = [\mathcal{I}_{ijkl}] = [\delta_{ik}\delta_{jl}]$. Then, with proper scaling of time t, this leads to the differential operator $\mathbf{A}(\nabla)$ in the partial differential equation of linear stability $\partial_t\boldsymbol{v} = \mathbf{A}(\nabla)\boldsymbol{v}$ for perturbation $\boldsymbol{v}(\mathbf{x}, t)$ on the base-state velocity field $\mathbf{v}^{(0)}(\mathbf{x})$ with a homogeneous deformation rate $\mathbf{D}'^{(0)} = \mathrm{sym}(\nabla\mathbf{v}^{(0)})$, a constant. In particular, $\mathbf{A}(\nabla)$ is given by its *principal symbol*[4,18]

$$\mathbf{A}(\boldsymbol{\xi}) = [A_{jk}] = \boldsymbol{\xi}\cdot\mathcal{C}^{(0)}\cdot\boldsymbol{\xi} := [\xi_i \mathcal{C}^{(0)}_{ijkl}\xi_l]. \tag{24}$$

Here as below, the superscript (0) indicates that all partial derivatives and tensors in (23) and (A1) are to be evaluated at the base state.

Convexity of ψ requires that $\mathbf{A}(\boldsymbol{\xi})$ be a positive-definite form (quadratic, here) in the real vector $\boldsymbol{\xi}$ or, equivalently, that $\mathbf{A}(\imath\mathbf{k})$ be negative-definite in real vector $\mathbf{k}$. In the first instance, $\boldsymbol{\xi}$ is the symbolic representation of ∇ which is tantamount to the Fourier space representation $\nabla \to \imath\mathbf{k}$, where $\mathbf{k}$ is the spatial wave-vector. This in turn requires that $\boldsymbol{v}\cdot\mathbf{A}\boldsymbol{v} = [A_{jk}v_j v_k]$ be positive-definite in $\boldsymbol{v}$, a quadratic form in both $\boldsymbol{v}$ and $\boldsymbol{\xi}$. Clearly this is equivalent to the condition $\boldsymbol{v}\cdot\mathbf{A}(\imath\mathbf{k})\boldsymbol{v} < 0$ which translates to the decay of energy in Fourier mode $\boldsymbol{v}(\mathbf{k})$ identified in Ref. 3.

In the case of pressure-sensitive incompressible materials, including those studied by Rajagopal[17] as well as the $\mu(I)$ model of interest here, all terms involving ∂_{D_1} vanish in (23) and (A1), and the above stability analysis requires modification. In particular, we must now allow ψ to depend on both D_2 and the pressure p regarded as a reaction (Lagrange multiplier) against the compressibility constraint, as is the case in past analyses of the $\mu(I)$ model.[2,9] For simplicity, and in view of its relevance to the present discussion, we assume that a function $\psi = 2p\Psi(I)/\theta$ captures the dependence on both D_2 and p, with $\Psi'(I) = \mu(I)$, as in the special case (6).

It is easy to show that, given the relations $\mathbf{D}' \equiv \mathbf{D}$ and $\boldsymbol{\sigma}' = \partial_{\mathbf{D}}\psi$, the relevant momentum balance can now be reduced to the symbolic form with $\nabla \to \boldsymbol{\xi}$,

$$\partial_t\boldsymbol{v} = \tilde{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{v}, \text{ where } \tilde{\mathbf{A}} = \mathbf{P}\mathbf{A}, \text{ with } \mathbf{P}(\boldsymbol{\xi}) = \mathbf{I} - \frac{\mathbf{a}\otimes\boldsymbol{\xi}}{\mathbf{a}\cdot\boldsymbol{\xi}}$$

and

$$\mathbf{a} = (\partial_p\partial_{\mathbf{D}}\psi - \mathbf{I})^{(0)}\boldsymbol{\xi} = \left[\frac{1}{2}(\Psi' - \frac{1}{2}I\Psi'')\mathbf{E} - \mathbf{I}\right]^{(0)}\boldsymbol{\xi},$$

with

$$\mathbf{E} = \mathbf{D}/\sqrt{D_2}. \tag{25}$$

Here, primes on Ψ represent derivatives with respect to I, while $\mathbf{P}$ is an oblique projection onto the space of solenoidal vectors $\mathbf{v}$ such that $\boldsymbol{\xi}\cdot\mathbf{v} = 0$, which serves to eliminate p from the momentum balance (Ref. 3).

According to the above, the tensor $\mathbf{A}$ defined by (24) can now be written as

$$\mathbf{A}(\boldsymbol{\xi}) = \left(\frac{p\theta}{2I}[\Psi'(\frac{1}{3}\boldsymbol{\xi}\otimes\boldsymbol{\xi} + \xi^2\mathbf{I}) + (I\Psi'' - \Psi')(\mathbf{E}\boldsymbol{\xi})\otimes(\mathbf{E}\boldsymbol{\xi})]\right)^{(0)},$$

with

$$\xi^2 = \xi_i\xi_i, \tag{26}$$

which, given Ψ(I), allows for the calculation of the stability operator $\tilde{A}$. It is evident that P is independent of $\xi = |\boldsymbol{\xi}|$ so that the instability depends only on the direction of $\boldsymbol{\xi}$ or k (cf. Ref. 3). Since $\tilde{\mathbf{A}}$ involves the projection $\mathbf{P}$, its determinant vanishes giving one null eigenvalue and a second eigenvalue $\tilde{\lambda} = \tilde{A}_{11} + \tilde{A}_{22} = \mathrm{tr}(\tilde{\mathbf{A}})$. From the expressions for $\mathbf{A}$ and $\mathbf{P}$ given in the Appendix, we find

$$\tilde{\lambda} = \xi^2\left(\frac{p\theta}{2I}\right)^{(0)}\left(\frac{4(\Psi' - I\Psi'')\cos^2 2\vartheta + \Psi'(I\Psi'' - 2\Psi')\cos 2\vartheta + 4I\Psi''}{(I\Psi'' - 2\Psi')\cos 2\vartheta + 4}\right)^{(0)}. \tag{27}$$

We recall that the terms $\mu_0 = \Psi'(0) > 0$ and $\Psi''(0)I > 0$ represent, respectively, a friction coefficient and a non-dimensional viscosity, and we consider first the limiting case of a non-vanishing friction coefficient $\mu_0 = \Psi'(0) \gg I\Psi''(0)$, assumed to dominate in the limit $I \to 0$. Ignoring the positive pre-factor $p\theta/2I$ in (27), which can be incorporated into the time t in the first equation of (25), or totally ignored in the quasi-static limit, we find that the inequality $\tilde{\lambda} > 0$ (which corresponds to a negative eigenvalue in the usual dynamic stability analysis) can be readily reduced to the form

$$(\cos 2\vartheta - \frac{1}{2}\mu_0)\cos 2\vartheta > 0, \text{ for } 0 \leq \mu_0 \leq 1. \tag{28}$$

Figure 1 shows the resulting stability diagram, with $\mathbf{U}$ and $\mathbf{S}$ denoting the unstable and stable regions, respectively. The unstable regions located at $\vartheta \approx \pm 45°$ relative to the principal

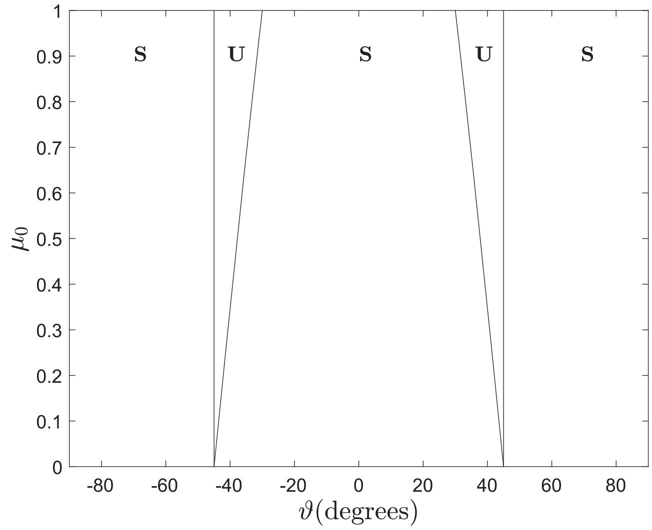

FIG. 1. Stability diagram for a purely frictional regime at $I = 0$ according to (28).

axis of $\mathbf{D}$ are close to those found in previous studies,[2,9] where the effects of viscosity were accounted for. The importance of viscous effects is illustrated by considering the limiting viscous behavior $\mu_0 = 0$, $\Psi''(0) > 0$, for which (27) yields $\tilde{\lambda} > 0$ provided that

$$\frac{I\Psi''(0)\sin^2 2\vartheta}{1 + \frac{1}{4}I\Psi''(0)\cos 2\vartheta} > 0, \tag{29}$$

which is satisfied for $0 < |\vartheta| < \pi/2$ and $I\Psi''(0) < 4$. As indicated by typical values of I in previous studies[2,9] of the $\mu(I)$ model, the latter inequality is unlikely to be violated near $I = 0$.

Without a more detailed investigation, it is plausible that many of the above results will apply to the limiting state of constant μ for large I of the type exhibited by the $\mu(I)$ model. We note that the recently proposed modification of the $\mu(I)$ by Barker and Gray[1] stabilizes at $I = 0$ by taking $\mu(0) = 0$.

We therefore conclude that any strongly dissipative model possessing sufficiently regular viscosity without yield stress will not exhibit the Hadamard instability arising from purely frictional behavior, *irrespective of compressibility effects*. The same may be true for any properly dissipative model, whether or not it exhibits Onsager symmetry and, in the absence of a proof, we believe it worthy of further investigation.

V. CONCLUSIONS

The main conclusion of the present article is that the regularizations of the $\mu(I)$ model obtained by Ref. 10 (Ref. 1) and Ref. 3 (Ref. 2) depend mainly on the replacement of a constant (Coulomb) friction coefficient by an effective viscosity, which appears to eliminate yield stress. The second conclusion is that one can probably achieve regularization without introducing compressibility, whatever the other merits of the latter. As separate issues, we have raised questions as to proper dissipation in the model of Ref. 1 and as to frame-indifference in the model of Ref. 2.

Finally, an effort has been made to illustrate the merits of models based on dissipation potentials and nonlinear Onsager symmetry for the analysis and modeling of visco-plasticity. In that connection, the model of Ref. 2 is found to be Onsager-symmetric, while that of Ref. 1 is not and thus involves physically admissible stresses that do not work in any deformation, even in the regime where it is properly dissipative.

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APPENDIX: TENSOR COMPONENTS

For the planar models discussed here, with $\psi = \psi(D_1, D_2)$, the components of the Hessian $\mathcal{C} = \partial^2_{\mathbf{D}}\psi$ are calculated as follows:

$$\begin{aligned}\mathcal{C}_{ijkl} &= \partial_{D_{ij}}\partial_{D_{kl}}\psi = \partial_{D_{ij}}[\delta_{kl}\partial_{D_1}\psi + D'_{kl}\partial_{D_2}\psi] \\ &= \partial_{D_{ij}}[\delta_{kl}\partial_{D_1}\psi + (D_{kl} - \frac{1}{3}D_1\delta_{kl})\partial_{D_2}\psi] \\ &= \delta_{ij}\delta_{kl}\partial^2_{D_1}\psi + (D'_{ij}\delta_{kl} + \delta_{ij}D'_{kl})\partial_{D_2}\partial_{D_1}\psi \\ &\quad + (\tilde{\mathcal{I}}_{ijkl} - \frac{1}{3}\delta_{ij}\delta_{kl})\partial_{D_2}\psi + D'_{ij}D'_{kl}\partial^2_{D_2}\psi.\end{aligned} \tag{A1}$$

Restricting ourselves to planar flows and adopting as orthogonal coordinates the principal axes of $\mathbf{E}$, we may employ the matrix representations

$$\boldsymbol{\xi} = \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix}1 & 0 \\ 0 & -1\end{bmatrix} \text{with } \mathbf{a} = \begin{bmatrix}\frac{1}{2}(\Psi' - I\Psi''/2 - 2)\xi_1 \\ -\frac{1}{2}(\Psi' - I\Psi''/2 + 2)\xi_2\end{bmatrix}^{(0)}, \tag{A2}$$

which further gives

$$\mathbf{A} = \left(\frac{p\theta}{2I}\right)^{(0)} \begin{bmatrix}(\xi^2 - \frac{2}{3}\xi_1^2)\Psi' + I\Psi''\xi_1^2 & \left(\frac{4}{3}\Psi' - I\Psi''\right)\xi_1\xi_2 \\ \left(\frac{4}{3}\Psi' - I\Psi''\right)\xi_1\xi_2 & (\xi^2 - \frac{2}{3}\xi_2^2)\Psi' + I\Psi''\xi_2^2\end{bmatrix}^{(0)} \tag{A3}$$

and

$$\mathbf{P} = \begin{bmatrix}1 - b_1\xi_1 & -b_1\xi_2 \\ -b_2\xi_1 & 1 - b_2\xi_2\end{bmatrix}, \quad \text{where } \mathbf{b} = \mathbf{a}/(\mathbf{a}\cdot\boldsymbol{\xi}) \text{ and } \mathbf{b}\cdot\boldsymbol{\xi} = 1. \tag{A4}$$

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