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A fluid-like model of vibrated granular layers: Linear stability, kinks, and oscillons

J.D. Goddard *, A.K. Didwania

Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, USA

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ABSTRACT

A continuum model is proposed for the dynamics of shallow non-cohesive granular layers driven by vertical vibration of a horizontal rigid plate *in vacuo*. The granular mass is assumed to behave as a Newtonian liquid while in flight and to exhibit a solid-like rebound from the vibrating plate. The simple version explored here involves two constant parameters, a kinematic viscosity and a coefficient of restitution.

According to the present model, the sensitive dependence of the contact dynamics on layer shape represents the primary source of instability and pattern formation, since the lack of cohesive stress rules out the Rayleigh–Taylor instability driving Faraday patterns on liquid layers. This is borne out by a linear stability analysis which indicates that flat granular layers are linearly stable against free-surface perturbations and, hence, that the instability of vibrated granular layers is nonlinear in origin.

An effort is made to capture certain aspects of the high Reynolds number nonlinear dynamics, namely spatially localized "kinks" and "oscillons", based on purely rectilinear ("antiplane") motion with constant layer thickness. Three different numerical methods were employed (1) computation of spatial amplitudes in a time-periodic solutions, (2) discretization of the underlying PDEs, with the resulting ODEs in time treated by an event-detecting integrator, and (3) a variant of the latter based on discretization of a well-known Green's function. Inaccuracies in contact detection appear to give disagreements between the different methods, such that Method (2) produced stable f/2 kinks and oscillons only if the variant (3) was employed, while Method (1) gave rise to stable oscillons but apparently unstable kinks. Also, while qualitatively similar, the numerically simulated oscillons exhibit differences from those observed experimentally.

The present findings call for further work on the numerical methods, investigation of more complex solutions with lateral motion and varicose layers, and improvements in the constitutive model for the granular layer, such as the incorporation of a velocity-dependent restitution.

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MECHANICS OF MATERIALS

1. Introduction

Granular media exhibit theoretically intriguing mechanical behavior, ranging from fluid-like to solid-like in various experimental and technological settings (Jaeger and Nagel, 1992). One remarkable example is provided by the wavy patterns on shallow granular layers driven by sinusoidal vertical motion of rigid flat plates (Fauve et al., 1989; Douady et al., 1989; Melo et al., 1994, 1995; Clément et al., 1996; Metcalf et al., 1997; Umbanhowar et al., 1996, 1998; Bizon et al., 1998; De Bruyn et al., 1998; Bizon et al., 1999; Clément and Labous, 2000; Umbanhowar and Swinney, 2000). With Γ representing peak acceleration of the plate scaled by normal gravity, a flat granular layer exhibits lift-off from the plate for $\Gamma > 1$ and appears to bounce as a flat solid layer for $\Gamma \leq 2.5$, just beyond which there emerge spatially periodic

^{*} Corresponding author. Tel.: +1 858 534 4508; fax: +1 858 534 4543. *E-mail address*: jgoddard@ucsd.edu (J.D. Goddard).

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subharmonic (f/2) square or striped surface patterns, depending on the frequency of vibration (Umbanhowar et al., 1998). These patterns undergo subsequent transitions, at larger distinct thresholds in Γ roughly independent of frequency, to hexagonal patterns and "kinks", thence to various f/4 patterns and finally to ostensibly chaotic states. A "phase diagram" in Γ and frequency is sketched out by Umbanhowar et al. (1998).

One particularly intriguing feature is a localized f/2 "oscillon" (Umbanhowar et al., 1996) appearing in a hysteretic way at the boundary between stripes and squares and representing a major focus of the present paper. Striking photographs of oscillons on layers of brass beads are given in the seminal work (Umbanhowar et al., 1996, and journal cover), and excerpts of a video photograph by members of the same research group (Umbanhowar, 2002) are shown below in Fig. 6.

The above granular patterns are frequently referred to as Faraday waves or patterns, since they resemble those arising from the classical Faraday instability (Kumar, 1996, 2000) on shallow liquid layers. However, it is worth recalling that the celebrated experiments of Faraday (1831) serve mainly to show that patterns on vibrated layers of fine powders arise from entrainment by vibratory convection of air, a finding supported by his subsequent experiments on shallow layers of pure liquids [reported in the Appendix of Faraday (1831)]. Furthermore, unlike the works cited above, many of Faraday's experiments involve flexural vibrations of the supporting plate, and the patterns he observes for heavy (sand) particles in air appear relatively unexciting.

In contrast to Faraday's liquid layers, it is clear from various experiments and particle-dynamics computer simulations (Bizon et al., 1998; Clément and Labous, 2000) that shallow granular layers generally do not maintain contact with the vibrating plate. Indeed, the wavy patterns associated with the lowest-lying instability arise from a flat-layer motion resembling that of a bouncing ball (Luck and Mehta, 1993). The importance of the resulting, generally violent impact with the bottom has already been cited by Eggers and Riecke (1999). Since the same kind of bouncing motion is assumed in the standard model for the vibratory conveying of granular solids (Colijn, 1985; Harding and Nedderman, 1990; Burmann, 1998), the issue of stability offers a practical motivation for a more comprehensive mechanical theory.

As a basis for the present model, several of the studies cited above suggest that vibrated granular layers *in vacuo* are governed mainly by inertia, gravity, and dissipative impact with the vibrating plate. According to this simplified view, the layer should be described by inviscid form of the standard (Haff) granular-gas model, in ballistic flight punctuated by solid-like inelastic rebound from the plate. This is the scenario anticipated by the gas-dynamic model of Goldshtein et al. (1996a,b), which we recall consists of the compressible Euler equations coupled with the standard granular kinetic-energy balance, such that inelastic particle collision represents the sole source of dissipation. Subsequent numerical studies by Bougie et al. (2002), based both on molecular-dynamics and the full granulargas model, as well as the later experimental observations on quasi-two-dimensional layers by Huang et al. (2006), do indeed reveal shock-like density and thermal waves following the impact of dense vibrated granular layers. In the model to be considered here, the associated multi-particle dissipation, assumed to occur on a short time scale corresponding to transmission of a shock through the rebounding granular layer, is represented simply by a global coefficient of restitution. This coefficient, denoted by *e* and to be distinguished from that describing inelastic impact between particle pairs, is treated as a given constitutive parameter, which we assume might be derived from a suitable variant on the above granular-gas models (cf. Miao et al., 2001).

As another simplification, 2D particle-dynamics simulations (Clément and Labous, 2000, see esp. their Fig. 1) suggest that changes in the overall density of the granular layer occur on an exceedingly short time scale following impact with the plate, except possibly in the gas-like states resulting from nearly elastic collisions. Snapshots from a similar type of simulation (of C. Bizon, Umbanhowar, 2002) in Fig. 1 show three different states of the granular layer, with each separated approximately by one-half period of the plate vibration and with the intermediate state showing the maximally compressed layer in contact with the plate and the other states in contact with the plate (not shown in the figures) only at their lowest extremities. These numerical simulations are confirmed qualitatively by the recent experiments of Kanai et al. (2005) on quasi-2D layers. These considerations serve in part to justify the incompressible-liquid model adopted below.

In contrast to the inviscid fluid model of Goldshtein et al. (1996a,b), the granular layer must exhibit a resistance to



Fig. 1. Movie frames extracted from particle-dynamics simulations, formerly given on the web page http://super.phys.northwestern.edu/ ~pbu/. With permission.

deformation that will regularize the discontinuous ("weak") solutions that would otherwise arise from localized impact between the layer and the plate. In principle this could be accomplished by assuming either solid-like or a fluid-like behavior of the granular layer, and, in either case, one can roughly envisage three global modes of deformation, bending, (antiplane) shearing, and stretching that would give rise to the required resistance.

We set aside the solid-like model with bending resistance envisaged by certain workers (Ugawa and Sano, 2003; Sano, 2005; Kang et al., 2007) in favor of a Newtonian fluid model like that proposed by Bizon et al. (1999). However, in contrast to the latter, our model assumes solid-like rebound at the plate, and it does not admit states of tensile stress in the granular layer. The last condition, an essential feature of non-cohesive granular media, dictates an eventual separation of the granular layer from the plate. It can be envisaged as tantamount to instantaneous "cavitation" of a volatile liquid or other "no-tension" material (Angelillo, 1993). As discussed below, this key assumption rules out the parametric Rayleigh-Taylor (Drazin and Reid, 1981) mechanism proposed by Bizon et al. (1999) as the source of Faraday patterns.

The intent of the present work is to set down the basic constitutive theory and field equations for the above model and to compare qualitatively certain solutions with prior experiment and particle-dynamics simulation of other workers. Among the possible benefits, the present effort may pave the way to a sound mechanical foundation for certain phenomenological dynamical-systems (DS) models (Aranson and Tsimring, 1998; Rothman, 1998; Venkataramani and Ott, 1998; Jeong and Moon, 1999; Jeong et al., 2000; Blair et al., 2000), models that have been remarkably successful in capturing several qualitative aspects of granular-layer dynamics. With a similar motivation, Eggers and Riecke (1999) offer a model with a more evident mechanical content, but it still relies on a phenomenological "granular diffusion", whose mechanical origins are not obvious to us.

From our point of view, purely phenomenological DS models lack the inherent capability of continuum-mechanical theories to deal with a variety of boundary-value problems, without requiring modification of the basic field equations for each new situation. A case in point is the vibratory transport of granular materials mentioned above, which involves both normal and tangential impact with inclined surfaces. In contrast to DS models, the present model would require relatively straight-forward consideration of tangential restitution combined with Coulomb friction (Goldsmith, 1960). In a similar vein, the present model could also serve to describe confined granular layers with other body forces, such as the oscillating electric fields (Aranson et al., 2000).

As a somewhat more abstract proposition, the present study may contribute to the general methodology for generating localized excitations in various media. In the fluidmechanical case, some form of nonNewtonian behavior seems to facilitate localization, as Lioubashevski et al. (1999) report the generation of oscillons on the surface of aqueous colloidal (clay) suspensions, which they attribute to a nonlinear shear thinning with frequency-dependent viscosity $v(\omega)$. The latter is also encompassed by the theory of linear viscoelasticity, and the analysis of Kumar (1999) suggests that fluid elasticity strongly modifies the Faraday instability, even without the additional requirement of shear thinning. However, nonNewtonian response may not be strictly necessary, since Arbell and Fineberg (2000) observe oscillons on water layers driven with a superposition of unequal but commensurate vibrational frequencies.

In order to elucidate various fundamental issues, the present study is focused then on the most elementary constitutive model, involving two parameters, a coefficient of restitution *e* and a kinematic viscosity *v*. While both are treated as constant, they may be assumed to depend on various parameters characterizing the periodic vibration and the granular layer. In this way, the present model can be viewed as an approximation to a more complete theory, e.g., one involving an energy balance for granular temperature, as foreseen by long-standing granular kinetic theories (Jenkins and Savage, 1983).

After a description of the continuum model in the following section, we provide an analytical treatment of the linear stability of periodically bouncing flat layers in Section 3, in order to show clearly the difference between non-cohesive granular layers and liquid layers. Then, in Section 4, we report on our numerical simulation for a special class of antiplane finite-amplitude motions as models of oscillons and kinks, and we make qualitative comparisons where possible with certain experimental observations. Section 5 gives a brief summary of major conclusions, certain inconclusive aspects of our analysis and recommendations for possible improvement.

2. The continuum model

As anticipated above, our granular layer is governed by a variant of the incompressible Navier–Stokes equations, with tacit restriction to compressive (negative semi-definite) stress tensor. In a layer bounded from below by an infinite rigid horizontal plate oscillating with vertical amplitude $A\mathbf{e}_z \sin \omega t$, the velocity $\mathbf{v}(t, \mathbf{x})$ with respect to the plate is described by:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = v \nabla^2 \mathbf{v} - \nabla P + \mathbf{g}(t) + \mathbf{a}(t, \mathbf{x}), \nabla \cdot \mathbf{v} = \mathbf{0} \bigg\}$$
(1)

where *v* denotes kinematic viscosity, $P = p(t, \mathbf{x})/\rho$ denotes pressure–density ratio, and where, with constant gravity $g_0 > 0$,

$$\mathbf{g}(t) = \mathbf{g}_0 \ (\Gamma \sin \omega t - 1) \ \mathbf{e}_z \quad \text{and} \quad \Gamma = A\omega^2/\mathbf{g}_0$$
 (2)

denote, respectively, the effective gravity and the nondimensional acceleration. The quantity $\mathbf{a}(t, \mathbf{x})$ represents the impulsive force or pressure gradient arising from impact with the bottom plate, which is designated here as the "Stosslet"¹ and defined more precisely below. The present equations (1) differ but slightly from those of Bizon et al.

¹ Derived from the well-known Stokeslet of hydrodynamics and the German "Stoß" (impact).

(1999) who, setting aside the effects of localized impact, assume an acceleration

$$\mathbf{G} = (\mathbf{g} + \mathbf{a}) \equiv G(t)\mathbf{e}_z \tag{3}$$

depending only on *t* (with sign opposite to that employed here).

The following analysis relies on the formal decomposition:

$$\left. \begin{array}{l} \mathbf{x} = \mathbf{r} + z\mathbf{e}_{z}, \quad \text{with } \mathbf{r} := x\mathbf{e}_{x} + y\mathbf{e}_{y}, \\ \nabla = \partial_{\mathbf{x}} = \nabla_{o} + \mathbf{e}_{z}\partial_{z}, \quad \text{with } \nabla_{o} := \partial_{\mathbf{r}} = \mathbf{e}_{x}\partial_{x} + \mathbf{e}_{y}\partial_{y}, \\ \mathbf{v} = \mathbf{v}_{o} + w\mathbf{e}_{z}, \quad \text{with } \mathbf{v}_{o} := u\mathbf{e}_{x} + v\mathbf{e}_{y}, \end{array} \right\}$$

$$(4)$$

where subscript *o* (orthogonal to \mathbf{e}_z) refers to the horizontal plane and \mathbf{e} to the natural cartesian basis. Then, the lower and upper boundaries of layer of thickness $h(t, \mathbf{r}) \ge 0$, are defined, respectively, by

$$z = Z(t, \mathbf{r}) = \begin{cases} \zeta(t, \mathbf{r}), \\ \zeta(t, \mathbf{r}) + h(t, \mathbf{r}) \end{cases}$$
(5)

and the surface kinematics are described by the well-known relations

$$\partial_t Z + \mathbf{v}_o \cdot \nabla_o Z = w, \quad \text{at } z = Z(t, \mathbf{r}),$$
(6)

Finally, this relation together with the condition of vanishing of surface traction, $[(\nabla \mathbf{v} + \nabla \mathbf{v})^T] \cdot \nabla(z - Z) = \mathbf{0}$, gives

$$P\nabla_{o}Z = v\{(\nabla_{o}\mathbf{v}_{o}) \cdot \nabla_{o}Z + (\nabla_{o}Z) \cdot \nabla_{o}\mathbf{v}_{o} - \nabla_{o}w - \partial_{z}\mathbf{v}_{o}\},\$$

$$P = v\{2\partial_{z}w - (\nabla_{o}w) \cdot \nabla_{o}Z - \partial_{z}\mathbf{v}_{o} \cdot \nabla_{o}Z\}$$

$$(7)$$

on the free surface (5) not in contact with the plate.

The Stosslet is defined formally, in terms of the coefficient of restitution *e*, by

$$\mathbf{a} = a\mathbf{e}_z, \quad \text{with } a(t, \mathbf{r}) = (1 + e)\dot{\zeta}^2\delta^-(\zeta) \quad \text{and} \\ \dot{\zeta} \equiv w(t, \mathbf{r}, \zeta), \tag{8}$$

with

$$\dot{\zeta}^{2}\delta^{-}(\zeta)dt = \dot{\zeta} \ \delta^{-}(\zeta)d\zeta, \quad \dot{\zeta} \ \delta^{-}(\zeta) \equiv -(1+e)\delta(t-t_{c}^{-}), \quad \text{at } \zeta(t_{c}(\mathbf{r}),\mathbf{r}) = \mathbf{0},$$
(9)

Here δ^- represents what might be called the Itô-Dirac delta function, defined so as to select the right-hand limit of the discontinuous function $\dot{\zeta}(\mathbf{t})$ at the instant of contact $t = t_c(\mathbf{r})$. Since this procedure involves the product of generalized functions, a more careful construction based on a limiting dynamical process is offered in Appendix A, which also treats the perturbed Stosslet employed below and offers a justification for the neglect of viscosity in arriving at (10).

If we neglect the viscous term in (1), the impulsive acceleration (8) gives rise to the discontinuity

$$w(t_c^+, \mathbf{r}, z) - w(t_c^-, \mathbf{r}, z) = \int_{t_c^-}^{t_c^+} (1+e)\dot{\zeta}^2 \delta^-(\zeta)dt$$
$$= -(1+e)w(t_c^-, \mathbf{r}, \mathbf{0})$$
(10)

at time $t = t_c(\mathbf{r})$ such that $\zeta(t_c, \mathbf{r}) = 0$. This relation corresponds locally to the classical Rayleigh problem of a slender solid body thrust impulsively through a viscous fluid, and it further implies a solid-like rebound from the plate:

$$w(t_c^+, \mathbf{r}, \mathbf{0}) = -ew(t_c^-, \mathbf{r}, \mathbf{0}).$$

$$(11)$$

In the special class of antiplane solutions considered below $\partial_z w \equiv 0$ and (10) reduces to (11), which serves as a boundary condition on $w(t, \mathbf{r})$.

The aforementioned Rayleigh problem suggests the necessity of a cut-off length to avoid line singularities arising from a point contacts. The analysis given in the last section of the Appendix A suggests the curvature of the contact zone as a relevant length scale but also implies that initially blunt contact zones should remain so after impact. Furthermore, the numerical discretization employed in most of this study provides a practical cut-off, with one exception discussed below in conjunction with Eq. (67).

To further minimize the number of adjustable parameters in this exploratory study *e* is treated as constant, whereas a dependence on impact velocity $\dot{\zeta}$ could give a better accounting for certain effects discussed below. Although not considered in the present study, an additional dependence on layer thickness *h*, with $e \rightarrow 0$ for a $h/d \rightarrow \infty$, where *d* is a representative particle diameter, would serve to represent the adherence of deep granular layers to surfaces undergoing small-amplitude vibration.

3. Flat layers and their stability

For a perfectly flat layer, $w = w^{(0)}(t), h = \text{const. and } (1)$ reduces to

$$\frac{dw^{(0)}}{dt} = G(t) - \frac{\partial P^{(0)}}{\partial z},$$
(12)

where G = G(t) is the acceleration defined by (3). It follow trivially from (7) for a granular layer bounded by two free surfaces that $P^{(0)} \equiv 0$ and, hence, that W(t) represents the bouncing ball with restitution *e* (Luck and Mehta, 1993; Bizon et al., 1999).

The above situation to be contrasted with the flat layer constrained to move with the bottom plate, such that $w^{(0)} \equiv 0$ and

$$\frac{\partial P^{(0)}}{\partial z} = G(t) \equiv g(t). \tag{13}$$

In this case, the condition P = 0 at the upper surface clearly implies a negative pressure at the plate whenever G(t) > 0which, while admissible for a liquid layer,² is untenable for non-cohesive solid layers, granular or otherwise. Hence, the condition G(t) = 0 signals the lift-off of a non-adherent body previously at rest on the plate (Luck and Mehta, 1993; Bizon et al., 1999).

It is precisely the hydrostatic inversion for G(t) > 0 coupled with perturbed surface elevation that drives the parametric instability of flat liquid layers (Drazin and Reid, 1981; Kumar, 2000). To make this immediately evident, we employ a conventional notation, with superscript (0),(1) representing the flat-layer base state and perturbation, respectively, to write³

² In the usual experiment P = 0 corresponds to normal atmospheric pressure, and lower pressures near the plate are not sustained long enough to cause cavitation.

³ Stability analyses of liquid layers often incorporate hydrostatic pressure directly into the stress tensor (Kumar, 1996, 2000), a stratagem of questionable merit for non-cohesive media.

$$P(t, \mathbf{r}, z) = P^{(0)}(t, z) + P^{(1)}(t, \mathbf{r}, z), Z(t, \mathbf{r}) = Z^{(0)}(t) + Z^{(1)}(t, \mathbf{r})$$
(14)

and, by means of (13),

$$P(t, \mathbf{r}, Z) = P^{(0)}(t, Z^{(0)}) + (\partial_z P^{(0)}) Z^{(1)} + P^{(1)} + \cdots \\ \equiv G(t) Z^{(1)}(t, \mathbf{r}, Z^{(0)}) + P^{(1)}(t, \mathbf{r}, Z^{(0)}) + \cdots \bigg\},$$
(15)

The term $G(t)Z^{(1)}$ in (15) represents the parametric destabilization of liquid layers, an effect which is absent from noncohesive layers.

The above considerations are directly relevant to the work of Bizon et al. (1999), who in effect replace g(t) in (3) by $H(\zeta^{(0)})g(t)$, where *H* denotes the Heaviside function with H(0) := 0. This replacement eliminates hydrostatic inversion during prolonged contact with the plate but allows it during the free-flight period. As shown below, this has dramatic implications for the stability analysis.

3.1. Periodic flat layers and linear stability

With lengths scaled by vibration amplitude A and times by inverse frequency ω^{-1} , Eqs. (1) and (5) preserve their form, with

$$v\omega/A^2 \to v, \quad g(t) \to \sin t - \Gamma^{-1}, \quad p/\rho A^2 \omega^2 \to P, \dots$$
(16)

and with v representing an inverse Reynolds number (Bizon et al., 1999). Then, the flat-layer state $u^{(0)} = \mathbf{v}^{(0)} = \mathbf{0}, w^{(0)} = W(t)$ is given by Luck and Mehta (1993):

$$W(t) = W_0 + \cos t_0 - \cos t - \frac{(t - t_0)}{\Gamma},$$

$$W_0 = W(t_0^+) = -eW(t_0^-),$$

$$\zeta = \zeta^{(0)} = (W_0 + \cos t_0)(t - t_0)$$

1 (17)

$$+\sin t_0 - \sin t - \frac{1}{2\Gamma}(t - t_0)^2 \ge 0,$$
 (18)

where t_0 represents launch time or phase relative to the time t = 0 at which acceleration of the bottom plate vanishes. The requirement of periodicity, with period $T = 2M\pi$, $M = 1, 2, \ldots$, (Luck and Mehta, 1993), implies that

$$e = \frac{1-s}{1+s}, \quad W(t_0^+) = \frac{1-s}{s} \cos t_0, \quad \text{where}$$

 $s = \frac{2\Gamma}{T} \cos t_0 \equiv \frac{1-e}{1+e}.$ (19)

The resultant profile of ζ vs. *t* for *M* = 2 is represented by a slice x = const. < 2 through the surface shown below in Fig. 2.

The relation (19) places severe restrictions on the value of restitution necessary to achieve periodicity. Since the experiments on granular layers discussed above appear generally to involve periodic states, this may suggest a variable restitution coefficient e that allows for resonant tuning to the forcing frequency. This effect is suggested by the rudimentary model of Miao et al. (2001), with a granular temperature and a global restitution depending impact velocity, and also by the finding of Clément and Labous

8 7 6 ζ 5 4 3 2 1 2.5 2 3.5 1.5 3 2.5 x $\frac{1}{2\pi}$ 1.5

Fig. 2. Elevation $\zeta(t, x)$ for kink from FDA (51)–(54) for M = 2.

(2000) that a fairly specific dependence on collision velocity of the interparticle restitution is necessary to secure agreement between particle-dynamics simulations and experiment. For the present purposes, we simply invoke the flat-layer tuning (19) as a way to define *e* for flat-layer base states and perturbations of these states.

With the preceding notational conventions, and with account taken of the equations satisfied by the flat-layer base state, the linearized disturbance equations become:

$$\partial_{t} \mathbf{v}^{(1)} + \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(1)} = \nu \nabla^{2} \mathbf{v}^{(1)} + \mathbf{a}^{(1)} - \nabla P^{(1)}$$
$$\nabla \cdot \mathbf{v}^{(1)} = \mathbf{0},$$
(20)

where $\mathbf{a}^{(1)} = a^{(1)}\mathbf{e}_{\tau}$ is a purely formal representation of the perturbed Stosslet discussed further below.

The considerations of the preceding paragraphs lead to boundary conditions:

$$\partial_t Z^{(1)} = w^{(1)}, \tag{21}$$

and

$$\left. \begin{array}{l} v \left(\nabla_o w^{(1)} + \partial_z \mathbf{v}_o^{(1)} \right) = \mathbf{0} \\ P^{(1)} = 2v \partial_z w^{(1)} \end{array} \right\}$$
(22)

on the unperturbed free surfaces

$$z = Z^{(0)}(t) = \begin{cases} \zeta^{(0)}(t), \\ \zeta^{(0)}(t) + h^{(0)}, \end{cases}$$
(23)

where

$$Z^{(1)}(t,\mathbf{r}) = \begin{cases} \zeta^{(1)}(t,\mathbf{r}), & \text{at } z = \zeta^{(0)}(t), \\ \zeta^{(1)} + h^{(1)}(t,\mathbf{r}), & \text{at } z = \zeta^{(0)}(t) + h^{(0)}, \end{cases}$$
(24)

with $h^{(0)}$ denoting the uniform thickness of the flat layer.

Following the general outlines of Kumar's analysis (Kumar, 1996) and employing the coordinate transformation $z - \zeta^{(0)}(t) - h^{(0)}/2 \rightarrow z$, with $h \equiv h^{(0)}$, one readily reduces the governing equations (20)-(23) to

$$(\partial_t - \nu \nabla^2) \nabla^2 w^{(1)} = (1 + e) \nabla_o^2 w^{(1)}(t, \mathbf{r}, \mathbf{0}) \dot{\zeta}^{(0)} \delta_-(\zeta^{(0)}), \quad (25)$$

for $-h/2 < z < h/2$, with



$$\begin{array}{l} \nu(\partial_z^2 - \nabla_o^2) w^{(1)} = \mathbf{0}, \\ (\partial_t - \nu \partial_z^2 + 3\nu \nabla_o^2) \partial_z w^{(1)} = \mathbf{0}, \\ \partial_t Z^{(1)} = w^{(1)} \end{array} \right\}$$

$$(26)$$

at $z = \pm h/2$. In contrast to liquid layers, the equations for $w^{(1)}$ are clearly decoupled from the perturbation $Z^{(1)}$ of surface elevation.

The impulsive term in (25) follows directly from the Appendix A. Since we expect ∇_o^2 to dominate ∂_z^2 near the impulse, the discontinuity in $\nabla_o^2 w^{(1)}$ implied by (25) is directly inherited from that of (20). Furthermore, for periodically bouncing flat layers the term in question represents a periodic impulse, and (25) becomes a singular Floquet equation. However, a full Floquet analysis is not necessary, and we merely apply a time-independent stability analysis over the period between impulses, with initial state represented by the usual Fourier decomposition.

By means of the standard normal-mode analysis (Kumar, 1996), with t = 0 representing the beginning of a period and with

$$\mathbf{w}^{(1)} = \hat{\mathbf{w}}(z) \exp\{\sigma t + \mathbf{k}_o \cdot \mathbf{r}\}, \quad \partial_t \to \sigma, \quad \nabla_o^2 \to -k_o^2, \quad (27)$$

Eqs. (25) and (26) reduce to the eigenvalue problem

$$(\mathscr{L} - \lambda \mathscr{M})\hat{w} = 0, \quad \text{where } \lambda = \sigma/\nu,$$

$$\mathscr{M} = \partial_z^2 - k_o^2, \quad \mathscr{L} = \mathscr{M}^2, \tag{28}$$

and

$$(\mathcal{M} + 2k_o^2)\hat{w} = 0, \quad \text{and} \quad \left(\mathcal{M} - 2k_o^2 - \lambda\right)\partial_z \hat{w} = 0,$$

at $z = \pm h/2.$ (29)

That the eigenvalue λ is real follows from the self-adjoint property of (28) and (29), which we establish as follows: With scalar product

$$\langle \phi, \psi \rangle = \int_{-h/2}^{+h/2} \phi(z)\psi(z)dz \tag{30}$$

and from the well-known result

$$\langle \phi, \mathscr{M}\psi \rangle = J\{\phi, \psi\} + \langle \mathscr{M}\phi, \psi \rangle, \quad \text{where} \\ J\{\phi, \psi\} = (\phi \partial_z \psi - \psi \partial_z \phi)|_{-h/2}^{+h/2},$$
 (31)

it follows that

$$\langle \phi, \mathscr{M}^2 \psi \rangle = J\{\phi, \mathscr{M}\psi\} + J\mathscr{M}\phi, \psi\} + \langle \mathscr{M}^2\phi, \psi \rangle. \tag{32}$$

However, (29) and the definition of $J{\phi, \psi}$ give

$$J\{\phi, \mathcal{M}\psi\} + J\{\mathcal{M}\phi, \psi\} = \lambda J\{\phi, \psi\}.$$
(33)

On multiplication of the first equation of (31) by λ and subtraction from (33), we have

$$\langle \phi, (\mathscr{L} - \lambda \mathscr{M})\psi \rangle = \langle (\mathscr{L} - \lambda \mathscr{M})\phi, \psi \rangle \quad (Q.E.D.).$$
(34)

To determine the eigenvalues, note that the general solution $\hat{w}(z)$ to (25) can be written as a linear combination of hyperbolic sines and cosines with arguments $k_o z$ and qz (Kumar, 1996), where

 $q^2 = k_o^2 + \lambda.$

The solution can further be decomposed into linearly independent solutions, one involving only sines, representing odd functions of z and corresponding to varicose modes, and the other involving only cosines, even functions and sinuous modes (cf. Kumar, 2000). Application of the boundary conditions leads, after some algebra, to the secular equations:

$$\left(\frac{\tanh\gamma\kappa}{\tanh\kappa}\right)^{\pm 1} = \frac{4\gamma}{\left(\gamma^2 + 1\right)^2},\tag{35}$$

where

 $\gamma^2 = (q/k_o)^2 = 1 + \lambda/k_o^2$, and $\kappa = k_o h/2$,

and the exponents ± 1 apply to varicose and sinuous modes, respectively.

One sees immediately, for all real $\kappa \ge 0$, that $\gamma = 1$ is a solution of both equations in (35) representing neutral states $\sigma = 0$, whereas for varicose modes there is a second solution, $\gamma = 0$ with $\sigma = -\nu k_o^2$, representing simple viscous damping. Given the behavior of tanh, it is easy to show that these are the only solutions to (35). Hence, as a major finding of the present work, we conclude that the flat-layer base states are linearly stable and therefore that pattern formation on flat layers must arise from *nonlinear* instability.

The finite-amplitude solutions considered next, which presumably originate from nonlinear instability, are seen to be the nonlinear counterparts of various solutions to (21)–(23), (25), (26) for $v \rightarrow 0$.

4. Small-*v* antiplane solutions

Inspection of (1) and (5) reveals an asymptotic solution for $v \rightarrow 0$ of the form, with

$$w = O(1), \quad \nabla_o Z = O(1), \quad \partial_z w = o(1),$$

$$\mathbf{v}_o = o(1), \quad P = O(v),$$

(36)

such that $w = w(t, \mathbf{r})$ satisfies

$$\left. \begin{array}{l} (\partial_t - v \nabla_o^2) w(t, \mathbf{r}) = g(t) + (1 + e) w^2 \delta_-(\zeta), \\ \partial_t \zeta = w(t, \mathbf{r}) \end{array} \right\}$$
(37)

with h = const.

Note that (36) differs from the standard boundary-layer scaling, with

$$V_o = O(v^{1/2}), \quad \nabla_o Z = O(v^{-1/2}), \dots,$$

since the dominant logitudinal motion is here confined to a finite distance *h*. Note further the omission of a pressure gradient $\partial_z P$ in (37), a term that would be necessary to represent singular states of prolonged contact with

$$\zeta \equiv \mathbf{0}, \quad w \equiv \mathbf{0}, \quad \partial_z P = g(t) < \mathbf{0}.$$

Such states appear to arise mainly in flat layers, as in (13), and to correspond to the "absorbing" regions of phase space identified by Luck and Mehta (1993) for the bouncing ball. To represent these states and the subsequent escape by lift-off, we adopt the artifice of Bizon et al. (1999), in effect replacing g(t) by $H(\zeta)g(t)$ in (37), together with the lift-off condition $\partial_t w = g(t) > 0$ for w = 0 (Luck and Mehta, 1993).

Motions of the type u = v = 0, $w = w(t, \mathbf{r})$, called "antiplane" in theoretical elasticity, represent a rectilinear

shearing in which inertial nonlinearity is negligible. Hence, the nonlinearity in (37) arises entirely from the Stosslet and the consequent dependence on t_c in (10), exactly as is the case with the bouncing ball of Luck and Mehta (1993). This form of solution gives rise to a constant layer thickness and represents the finite-amplitude counterpart of the sinuous modes of the linear stability analysis. The above-cited numerical simulations and experiments (Clément and Labous, 2000; Umbanhowar, 2002; Ugawa and Sano, 2003; Kanai et al., 2005) suggest the possibility of this type of motion, but also reveal a second, more complex motion involving lateral velocities with change of layer thickness, perceptible to some extent in Fig. 1.

As it stands, there exist no other characteristic length scale in the current problem, and the simple form of (37) allows us henceforth to set $v \equiv 1$ representing a coordinate rescaling $\mathbf{r} \rightarrow \sqrt{v} \mathbf{r}$.

4.1. Time-periodic solutions (TPS)

With $\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, denoting an assumed subharmonic frequency, we consider periodic solutions to (37), with $\zeta(t, \mathbf{r})$, $w(t, \mathbf{r})$ given by the real Fourier series on $(0, 2\pi/\alpha)$,

$$\zeta(t,\mathbf{r}) = \sum_{n=-\infty}^{\infty} \hat{\zeta}_n(\mathbf{r}) e^{in\alpha t}, \quad \text{with } \hat{\zeta}_n = \hat{\zeta}_{-n}^*, \tag{38}$$

$$w(t, \mathbf{r}) = \sum_{n = -\infty}^{\infty} \hat{w}_n(\mathbf{r}) e^{in\alpha t}, \quad \text{with } \hat{w}_n = in\alpha \hat{\zeta}_n, \tag{39}$$

where asterisks denotes complex conjugates, and

$$\hat{\zeta}_0(\mathbf{r}) = -\sum_{n=-\infty}^{\infty} \hat{\zeta}_n(\mathbf{r}) e^{in\alpha t_c(\mathbf{r})},\tag{40}$$

with $t_c(\mathbf{r}) \in [0, 2\pi/\alpha]$ denoting the contact time in (9) and (10) *et seq.*

The series (39) generally is only conditionally convergent owing to the discontinuity at $t = t_c(\mathbf{r})$. This is subsumed in the theory of Fourier series of generalized functions (Lighthill, 1958), according to which the Dirac delta is given by

$$\delta(t - t_c) = \frac{\alpha}{2\pi} + \frac{\alpha}{\pi} \sum_{n=1}^{\infty} \cos n\alpha (t - t_c)$$
$$= \frac{\alpha}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\alpha(t - t_c)}, \qquad (41)$$

Therefore, on substituting (39) and (41) into the first member of (37) one obtains the Fourier representation

$$\nabla_o^2 \hat{w}_n = in\alpha \hat{w}_n + \frac{\alpha}{2\pi} (1+e) w(t_c^-) e^{-in\alpha t_c} - \hat{g}_n,$$

$$n = 0, 1, 2, \dots, \qquad (42)$$

where, with δ_{ij} denoting the Kronecker delta,

$$\hat{g}_n = -\left(\frac{1}{\Gamma}\delta_{n0} + \frac{i}{2}\delta_{|n|\ m}\right) = \hat{g}^*_{-n}, \quad m := 1/\alpha,$$
(43)

is the Fourier transform of g(t). Note that (42) can also be obtained via the standard term-by-term transformation of (37), with:

$$\hat{f}_n=\frac{\alpha}{2\pi}\int_{\mathscr{P}}e^{-in\alpha t}fdt,$$

where \mathscr{P} is an arbitrary $2\pi/\alpha$ interval. The latter may be chosen to exclude the point t_c and, hence, the delta function in (37), in which case the discontinuity in w is captured by the integration of $\partial_t w$ from t_c^+ to $(2\pi/\alpha) + t_c^-$.

As indicated by (39), $\hat{w}_0 \equiv 0$, so that $w(t, \mathbf{r})$ has zero mean on any $2\pi/\alpha$ period and

$$\frac{\alpha}{2\pi}(1+e)w(t_c^-) = \hat{g}_0 = -\frac{1}{\Gamma}$$
(44)

representing the mean balance between impulse and gravity. Hence, (42) reduces simply to

$$\nabla_o^2 \hat{w}_n - in\alpha \hat{w}_n = -\frac{1}{\Gamma} e^{-in\alpha t_c(\mathbf{r})} - \hat{g}_n, \quad n = 0, 1, 2, \dots$$
 (45)

It follows from a well-known convergence property of Fourier series (Sneddon, 1995), together with (39) and (44), that

$$\sum_{n=-\infty}^{\infty} \hat{w}_{n}(\mathbf{r}) e^{in\alpha t_{c}} = \frac{1}{2} \{ w(t_{c}^{-}, \mathbf{r}) + w(t_{c}^{+}, \mathbf{r}) \}$$
$$= \frac{1}{2} (1 - e) w(t_{c}^{-}, \mathbf{r})$$
$$= -\frac{\pi (1 - e)}{\alpha \Gamma (1 + e)} = -\cos t_{0},$$
(46)

at points $t = t_c$ of discontinuity in *w*. The final equality, which follows from (19) and (46), provides an implicit equation for $t_c(\mathbf{r})$ in terms of $\hat{w}_n(\mathbf{r})$, so that (45) and (46) represent a set of nonlinear PDEs for $\hat{w}_n(\mathbf{r})$ and $t_c(\mathbf{r})$. Upon truncation of the series at $n = \pm N$, (46) obviously reduces to a polynomial of degree 2*N* for exp{ $i\alpha t_c$ }.

The TPS (38) represents a limit cycle whose stability will be elucidated by the numerical finite-difference solutions to (37) discussed below.

4.1.1. Iterative solution

Eqs. (45) and (46) are amenable to iterative solution based on the incremental form:

$$\nabla_{o}^{2} u^{k} - in\alpha u^{k} = -Q^{k} = -\frac{1}{\Gamma} \left(e^{-in\alpha t_{c}^{k}} - e^{-in\alpha t_{c}^{k-1}} \right), \quad \text{with} \\ u^{k} = \hat{w}^{k} - \hat{w}^{k-1}, \quad k = 1, 2, \dots,$$
(47)

where subscript *n* is suppressed and t_c^k is given by \hat{w}^k through (46).

Owing to the discontinuous nature of (46), it does not appear feasible to employ any standard linearization of Q^k in terms u^k , leading us to adopt instead a more tenuous extrapolation of Q^k from prior values $Q^{k-1}, Q^{k-2}, ...$ With any such approximation for Q^k , the complex Helmholtz equation (47) then can be solved via the appropriate Green's function to give the *k*th approximation w^k in terms of prior approximations $w^{k-1}, w^{k-2}, ...$ For the present work, we have employed the simplest conceivable iterations, with either $Q^k = Q^{k-1}$ or $Q^k = -Q^{k-1}$, both of which appeared to give the same final result whenever convergence was achieved.

To simulate kink or oscillon structures, we consider either linear or axisymmetric spatial dependence with $\mathbf{r} \rightarrow x$ or $\mathbf{r} \rightarrow r$, respectively. Then, a particular solution of (47) is given by elementary Green's functions in the form:

$$u^{k}(\xi) = F_{+}(\xi) \int_{\xi}^{\infty} F_{-}(\xi') \frac{Q^{k}(\xi')}{W(\xi')} d\xi' + F_{-}(\xi) \int_{0}^{\xi} F_{+}(\xi') \\ \times \frac{Q^{k}(\xi')}{W(\xi')} d\xi',$$
(48)

if |Q| is sufficiently small for $|\xi| \to \infty$, where $W(\xi) = W\{F_-, F_+\}$ is the Wronskian.

With $\beta := (in\alpha)^{1/2}$, the functions *F*, *W* are given by

$$F_{\pm}(\xi) = e^{\pm\beta\xi}, \quad \xi = x, \quad W = 2\beta, \tag{49}$$

or

$$F_{\pm}(\xi) = \begin{cases} I_0(\beta\xi) \\ K_0(\beta\xi) \end{cases}, \quad W = \xi^{-1} = r^{-1}$$
(50)

in the linear and axisymmetric cases, respectively.

4.2. FDA solutions

The problem at hand lends itself readily to numerical solution by the "method of lines" (Schiesser, 1991), in which one employs a suitable discretization in \mathbf{r} to reduce the problem to a set of ODEs in t, which then are stably solved by Runge–Kutta methods. For the present purposes, we employ the spatial discretization

$$\mathbf{r} = (i\Delta x)\mathbf{e}_x + (j\Delta y)\mathbf{e}_y, \quad i,j = 1, 2, \dots,$$

and the corresponding finite-difference approximation (FDA):

$$\zeta(t, \mathbf{r}) \to \zeta(t), \quad w(t, \mathbf{r}) \to \underline{w}(t), \dots, \quad \mathscr{L} \to \underline{L},$$
(51)

where

$$\underline{\zeta}(t) = [\zeta(t,\underline{n})], \quad \underline{w}(t) = [w(t,\underline{n})], \quad \underline{L} = [\underline{L}_{\underline{m},\underline{n}}]$$
(52)

represent column vectors and square matrix, respectively, and $\underline{n}, \underline{m}, \ldots$ are column vectors with integer components that ordinally label the discrete spatial grid points *i*, *j*.

Then, (37) can be written as the set of linear constant-coefficient ODEs

$$\frac{d\underline{w}}{dt} = v\underline{L} \ \underline{w} + g(t)\underline{\sigma}, \quad \text{and} \quad \frac{d\underline{\zeta}}{dt} = \underline{w}(t), \tag{53}$$

with rebound condition:

$$w(t_{c}^{+},\underline{n}) = -ew(t_{c}^{-},\underline{n}), \quad \text{for } \zeta(t_{c},\underline{n}) = 0, \ \underline{n} \in \mathscr{P},$$
(54)

where \mathcal{P} represents a set of one or more simultaneous contact points, while the column vector $\underline{\sigma}$ has all components equal to unity and represents spatially uniform states with $\underline{L} \ \underline{\sigma} = \underline{0}$. Thus, the solution to (37) is reduced to an eventdriven numerical simulation, with contact condition in (54) representing discrete contact events.

For later purposes, note that the linear form (53) admits analytical integration, such that, in the time period $t_c < t < t'_c$ between distinct contacts, the solution can be expressed formally in terms of matrix functions as

$$\underline{w}(t) = \exp\left\{\nu\underline{L}(t-t_c)\right\}\underline{w}(t_c^+) + \left\{W(t) - W(t_c)\right\}\underline{\sigma},
\underline{\zeta} = \left(\nu\underline{L}\right)^{-1} \left(\exp\left\{\nu\underline{L}(t-t_c)\right\} - \underline{1}\right)\underline{w}(t_c^+) + \left\{Z(t) - Z(t_c)\right\}\underline{\sigma},$$
(55)

where W, Z represent the flat-layer states:

$$W(t) = \int_0^t g(t')dt', \quad Z(t) = \int_0^t W(t')dt',$$
 (56)

with arbitrary time origin t = 0.

The matrix exponential in (55) represents a discretized Green's function, and an alternative, potentially more accurate expression for \underline{w} is obtained on replacing the first member of (55) by the discretization of

$$w(t, \mathbf{r}) = \int_{\mathscr{R}} G_o\{t - t'_c, \mathbf{r}, \mathbf{r}'\} [w_c^+(\mathbf{r}') - W(t'_c)] dA' + W(t)$$
(57)

with $t_c(\mathbf{r})$ defined by

$$\zeta(t_c, \mathbf{r}) = \mathbf{0}, \quad \text{and} \quad w_c^+(\mathbf{r}) = w(t_c^+, \mathbf{r}), \tag{58}$$

where \mathcal{R} denotes the xy plane, dA its area element,

$$w_c^+(\mathbf{r}) := w(t_c^+, \mathbf{r}), \quad t_c' = t_c(\mathbf{r}')$$
(59)

and G_o the exact Green's function for $\mathscr{L} = \partial_t - v \nabla_o^2$.

$$G_o(t, \mathbf{r}, \mathbf{r}') = \mathscr{L}^{-1} = \exp\{\nu t \nabla_o^2\}.$$
(60)

With $\nabla_o^2 \rightarrow -k_o^2$, the second expression of (60) provides various Fourier representations, one of which corresponds to a well-known Gaussian form for point impulses in the plane and to:

$$G_o(t, \mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\nu t} \exp\left(-\frac{r^2 + r'^2}{4\nu t}\right) I_0\left(\frac{rr'}{2\nu t}\right),\tag{61}$$

for circular-sheet impulses, where $r = |\mathbf{r}|$ and I_0 is standard notation for the modified Bessel function (see, e.g., Sneddon, 1995, pp. 194–198).

As discussed below, a quadrature based on (51), with discretization

$$G_o \to \underline{G}(t) = [G_{\underline{m},\underline{n}}(t)],$$
 (62)

appears to produce a more agreeable description of oscillons than the straight-forward integration of (53) to be considered next.

4.3. Computations of kinks and oscillons

A MATLABTM ODE solver with event detection ("ode113" and "fzero", respectively) were employed to treat the system (53) and (54). On detection of a zero crossing of any component or set of components of ζ , integration is stopped and then restarted after the corresponding components of \underline{w} are updated according to the rebound condition in (54).

Two special cases of localized structures were treated corresponding to kinks and oscillons (Melo et al., 1995; Umbanhowar et al., 1996, 1998). The kink consists of a narrow transition zone separating two flat layers, bouncing out of phase with one another, whereas an oscillon is a spatially localized axisymmetric perturbation of an extended flat layer. In its simplest form, a kink is a one-dimensional structures, with

$$\zeta(t,\mathbf{r}) = \zeta(t,x), \ldots, \quad \nabla_o^2 = \partial_x^2,$$

whereas oscillons have

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Fig. 3. Oscillon velocity w(t, r) - W(t) relative to surrounding flat layer from TPS.

$$\zeta(t,\mathbf{r}) = \zeta(t,r), \ldots, \quad \nabla_o^2 = r^{-1}\partial_r(r\partial_r).$$

According to the current model, these represent localized viscous zones on an otherwise inviscid layer. Under the



Fig. 4. Elevation $\zeta(t, r)$ for oscillon from TPS.

above spatial discretization, both are represented simply by vector arrays $\underline{\zeta}(t) = [\zeta_i(t)], \ldots$, with $\zeta_i(t) = \zeta(t, i\Delta s)$, where s = x or r, respectively.

4.3.1. Kinks

Various attempts to treat (45) and (46) via (47)–(49), with a discrete set of uniformly spaced grid points in $\overline{x} = x/\sqrt{v}$, met with limited success. An initial state was chosen as two flat layers satisfying (19), out of phase by 180°, and separated by a transition region given by a linear interpolation in elevation joining their parallel straight edges. This invariably evolved to a single flat layer, with one flat layer gradually encroaching on the other. By contrast, we recall that experimental kinks appear to require an additional subharmonic driving frequency for such lateral movement (Aranson et al., 1999). In the present simulation, the lateral displacement of one layer by the other could be prevented only by "pinning" both layers in their initial states. However, in this case the initially linear transition region eventually degenerated into irregular stripes, roughly parallel to the two flat-layer boundaries, whenever it was more than two or three grid points in width.

The above finding, which suggests a remarkably thin viscous interlayer of thickness $o(\sqrt{v})$, is partly borne out by the numerical solution based on (51)–(54). With the standard discretization

$$L_{ij} = \frac{1}{\Delta x^2} (\delta_{i,j-1} + \delta_{i,j+1} - 2\delta_{i,j}), \text{ for } i, j = 2, \dots, N-1$$
(63)

at internal points and with pinning of the end points to flat-layer states, the numerical solution of (51)-(54) yields the kink-like structure shown in Fig. 2 for M = 2. However, the FDA is clearly defective near the edges of the transition layer, where there is a slight but evident discontinuity in slope $\partial_x \zeta$, implying a corresponding discontinuity in $\partial_x w$ and, hence, in shear stress.

All the computations reported here are for the perioddoubled states M = 2, although we obtain similar results



Fig. 5. Polar plot corresponding to different time slices from Fig. 4, proceeding cyclically from top left to top right, followed by bottom left to bottom right.



Fig. 6. Frames extracted from a video photograph of an experimentally generated oscillon and ordered as in Fig. 5 (courtesy of Umbanhowar.)

for nearby integral values of *M*. These somewhat unsatisfying results for kinks suggests additional nonlinear effects not embodied in the basic model or else not captured by our antiplane approximation.

4.3.2. Oscillons

The relations (47), (48), (50) were applied to the system (45) and (46) with up to thirty Fourier modes. For a farfield 4π -periodic flat layer with contact time $t_0 \equiv t_c(\infty) = 0$ and velocity W(t) given by (18), this resulted in convergence to solutions $\tilde{w}, \tilde{\zeta}, \tilde{t}_c$. The computed surfaces of relative velocity $\tilde{w} - W$ shown in Fig. 3 for the case of N = 30 Fourier modes in (38) and (39) appear altogether reasonable and exhibit discontinuities associated



Fig. 7. Three cycles of $\zeta(r, t) - Z(t)$ from FDA based on Green's function (61).

with both \tilde{w} and W. There is also evidence of an overshoot resulting from the Gibbs phenomenon (Gottlieb and Shu, 1997) for Fourier series.

Unfortunately, the calculated values of contact time $\tilde{t}_c(r)$ all lie near zero, and the Fourier series obtained from

$$\tilde{\zeta}(t,r) = \int_{t_c}^t \tilde{w}(t',r)dt'$$

becomes negative at certain points (t, r). Thus, the relation (46) apparently does not allow for a practically accurate determination of $t_c(r)$. However, as evident from the second member of (37) ζ is determined only up to an arbitrary function of r, and, hence, we may take

$$w(t,r) \equiv \tilde{w}(t,r),$$

$$\zeta(t,r) = \tilde{\zeta}(t,r) + \frac{1}{2} \Big\{ \tilde{\zeta}(t_m(r),r) - |\tilde{\zeta}(t_m(r),r)| \Big\},$$
(64)

$$t_c(r) = t_m(r), \tag{65}$$

where for given r, $t_m(r)$ is the value of t at which $\zeta(t, r)$ achieves its minimum. Thus, the second member represents a shift to positive values everywhere, whenever this minimum is negative. With this modification, the computed time-periodic surface of $\zeta(t, r)$ for M = 2 is shown in Fig. 4, where it is evident that the contact time $t_c(r)$ makes a sharp transition between two essentially constant values over a distance $\Delta r \ll 1$.

An alternative iteration scheme, with t_c^k given by (65) in terms of $\zeta^k(t, r)$ by means of (64), produced results very close to those of Fig. 4. The polar plot of $\zeta(r)$ vs. $0 \le t \le 4\pi$ in Fig. 5 conveys a better appreciation of the oscillon structure implied by the information in Fig. 4. The frames are separated by approximately $\pi/4$ or 1/8 of plate-vibration cycle. The rippled states seen there are reminiscent of other models of oscillons, the DS model of Jeong and Moon (1999) and Yochelis et al. (1999), and the two-dimensional oscillons (Eggers and Riecke, 1999), the latter of which have not been found experimentally.

By way of comparison, Fig. 6 shows a sequence of frames extracted from a video photograph (courtesy of Umbanhowar) of a top view of an experimentally generated oscillon, reportedly the best and perhaps the only video record available (Umbanhowar, 2002). The interval between frames is approximately $\pi/4$ along the horizontal and $\pi/2$ in the transitions between top and bottom. Although there is a certain qualitative similarity between Figs. 5 and 6, notably in the occurrence of peaks and craters and in the collapsing peaks, Fig. 5 exhibits mirror-like symmetry between the top and bottom sequences and small-scale ripples that are not discernible in any of the experimental photographs.

To implement the FDA solution of the ODEs (53), the quasi-logarithmic transformation

$$r = e^{\xi} - 1, \quad \text{with } r^{-1}\partial_r(r\partial_r) = e^{-2\xi} \left\{ \partial_{\xi}^2 + \frac{e^{-\xi}}{1 - e^{-\xi}} \partial_{\xi} \right\}$$
(66)

which, with $\xi \sim \ln r$ for $r \gg 1$ and $\xi \sim r$ for $r \ll 1$ was employed, allowing for large steps Δr in the presumed flat-layer state at $r \gg 1$ while avoiding the singularity at r = 0 of the simple logarithmic transformation $\xi = \ln r$. With the standard discretization of ∂_{ξ} and ∂_{ξ}^2 , a numerical solution of (37) was undertaken, with the periodic solution represented by Fig. 5 taken as the initial state. It was found that the numerical solution converged extremely slowly, devolving ostensibly to a state that was only roughly periodic, with an approximate period doubling but with considerable variation in peak amplitudes of $\zeta(t, r)$ from cycle to cycle. This state of affairs suggests either an unstable limit cycle, with transition to a quasi-periodic state, or else that the numerical solution of (53) is unreliable. Given the strong influence of bottom impact, the latter appears most likely.

By way of further investigation, a modified FDA was employed in which the discretized velocity \underline{w} is determined by means of the discretized Green's function (62) and the discretized form of (57), effectively replacing the first member of (53). For convenience, the second member of (53) is integrated with the previous ODE solver and event detector, to determine contact locus (t_c , r_c). Thus, starting from one set of discretized quantities in (56), one integrates forward to the next.

To overcome numerical convergence problems arising ostensibly from large spatial gradients in G_o , a modified restitution condition was employed in which the Stosslet was diffused over a finite area, according to

$$w(t_c^+, r) = w(t_c^-, r) - (1 + e)w(t_c^-, r_c)\exp(-|r - r_c|/\rho_c),$$
(67)

with a corresponding modification of (57) and its discretized form. Partly on physical grounds and partly motivated by the last section of the Appendix A, the length scale ρ_c provisionally was identified with the radius of curvature of ζ at r_c . Since FDA estimates of this quantity were found generally to lie in the range 0.001–0.005, and since exploratory calculations showed insensitivity to the precise value in this range, ρ_c was set equal to 0.001 in all subsequent calculations.

A sample computation of elevation relative to the flat layer is presented in Fig. 7, which shows three representative cycles extracted from a much larger number of virtually identical steady cycles with period close to 4π . The same results are presented in the polar plot of Fig. 8, where the time intervals are approximately $\pi/32$, $\pi/8$, $\pi/4$,



Fig. 8. Snapshots of oscillon computed by FDA based on Green's function (61), arranged as in Figs. 5 and 6.



Fig. 9. (a) Impact phase plane and trajectory. (b) Perturbed flat layer, with (upper sketch) and without (lower sketch) perturbation in t_c , respectively.

 $\pi/2, \pi/2, \pi/16$, proceeding cyclically from upper left to lower right. As evident from both Figs. 8 and 9 the collapse of the peak appears much more abrupt than in Figs. 5 and 6.

5. Conclusions

The Abstract summarizes the major contributions of the present study. As pointed out there, our theoretical and numerical study lends considerable support to the idea that pattern formation on vibrated granular layers is driven by the inherently nonlinear interaction at the bottom boundary. Moreover, contrary to impressions conveyed by much of the literature, the absence of granular tensile stress serves to distinguish the instabilities on granular layer from the (Rayleigh–Taylor) instabilities and (Faraday) patterns on liquid layers.

Given the simplicity of the basic model employed above and the antiplane approximation, the numerical solution to the current problem turns out to be surprisingly difficult. Hence, there remain certain discrepancies between numerical solutions, notably

- 1. stable subharmonic (f/2) kinks found with FDA but not with TPS;
- 2. smooth TPS oscillons, but FDA oscillons that exhibit only approximate f/2 behavior and show irregular variations in peak height unless computed by means of the antiplane Green's function;
- 3. subharmonic f/2 oscillons from the last-mentioned calculation that display quantitatively different shapes from the f/2 TPS oscillons.

The above discrepancies are attributed to the difficulty in accurately tracking bottom contact, in a model where the related nonlinear instability is crucial. Clearly, further improvement is needed in the numerical methods.

As a separate issue, the differences between the computed oscillons and the limited experimental observations suggest that more complex solutions with "stretching modes", involving varicose layers and lateral motion, should be investigated. Given the sensitivity to bottom impact, a change in layer thickness, e.g., a small perturbation on the antiplane solutions, is expected to significantly affect the localized solutions found above. More importantly, varicosity is obviously implicated in the spatially extended waves that are much more prevalent in experiment and simulation (De Bruyn et al., 1998; Clément and Labous, 2000; Jeong et al., 2000). For example, the 2D particledynamics simulation shown in Fig. 1 involves an evident lateral "sloshing" mode, with alternation of crest and troughs in one plate cycle.

As still more fundamental issue, the basic model clearly needs some modification, such as velocity-dependent restitution, in order to justify the flat-layer tuning employed in the present study. Furthermore, the assumed kinematic viscosity v might also be allowed to depend on impact velocity, to reflect a possibly important dependence on granular temperature immediately following impact. Plausible forms for such modifications appear to be well within the reach granular kinetic theory (Jenkins and Savage, 1983), with appropriate extensions to high densities.

Appendix A. Inelastic impulse and perturbations

Here, we employ an idealized model of inelastic impact in order to define the product of generalized functions involved in (8). The mechanics of impact, involving ratedependent or hysteretic contact forces, are relevant to a number of fields (Goldsmith, 1960), particularly granular media (Luding, 1997). With ideas motivated by that literature, we consider a class of generalized functions derived from impact or "scattering" dynamics, an approach that appears complementary to the methods of "non-smooth mechanics" employed by others (cf. Moreau, 1999). We first treat ODEs, strictly applicable only to the flat layer or single particle, and then estimate the effects of layer curvature and viscosity in (1).

A.1. The flat layer

Consider the autonomous set

$$\dot{w} = \frac{dw}{dt} = f(w,\zeta) \tag{68}$$

$$\dot{\zeta} = \frac{a\zeta}{dt} = w \tag{69}$$

with specific force or acceleration $a = f(w, \zeta)$ such that an incoming initial state, with $\zeta = \zeta_- > 0, w \to w_- < 0$ for $t \to -\infty$, always results in a final state having $\zeta = \zeta_+ \ge 0, w \to w_+ \ge 0$ for $t \to \infty$, where $f(w_{\pm}, \zeta) = 0$. In the standard way, we write (69) as

$$\frac{dw}{d\zeta} = f(w,\zeta)/w \tag{70}$$

representing the phase portrait shown in Fig. 9(a), involving incoming and outgoing branches, with restitution

$$e = -w_+/w_-.$$
 (71)

The restitution generally depends on ζ_-, w_- , but we are concerned here with the limit process defined by $\epsilon \downarrow 0$, with

$$f(\boldsymbol{w},\boldsymbol{\zeta}) = f_{\boldsymbol{\epsilon}}(\boldsymbol{w},\boldsymbol{\zeta}) = \frac{1}{\boldsymbol{\epsilon}} F(\boldsymbol{w},\boldsymbol{\zeta}/\boldsymbol{\epsilon}), \tag{72}$$

where F(w,z) is a bounded O(1) function with bounded O(1) *z*-derivative $\partial F/\partial z$. In this case, the dynamics occurs on time and space scales $O(\epsilon)$, and (70) can be written in terms of a stretched coordinate $\overline{\zeta} = \zeta/\epsilon$ as

$$\frac{dw}{d\bar{\zeta}} = F(w,\bar{\zeta})/w \tag{73}$$

with initial asymptote $w \to w_-$ for $\overline{\zeta} \to \infty$, and with restitution $e(w_-)$ given by the final asymptote $w \to w_+$ for $\overline{\zeta} \to \infty$.

By means of the solution $w\{\bar{\zeta}\}$ to (73), we then have $w_+ - w_- \equiv -(1 + e)w_-$ which serves to define a generalized function $\dot{\zeta}\delta_-(\dot{\zeta},\zeta)$, such that

$$\frac{dw}{dt} = \frac{d\zeta}{dt} = (1+e)\dot{\zeta}^2\delta_{-}(\zeta)$$
(74)

produces the same jump $w_+ - w_-$.

A.2. Velocity-independence

The special case of a velocity- or energy-independent restitution involves severe restrictions on $f(w, \zeta)$, which are clarified by letting $\chi = w/w_{-}$, so that (73) becomes

$$\frac{d\chi}{d\bar{\zeta}} = \frac{F(\chi \ w_{-}, \bar{\zeta})}{\chi \ w_{-}^2} \tag{75}$$

with $\chi \to 1$ for $\overline{\zeta} \to \infty$. Obviously, the solution to (75) and the final asymptote $\chi \to -e$ for $\overline{\zeta} \to \infty$ must depend generally on the parameter w_- appearing in (75), except for very special forms of *f*. One notes immediately that a sufficient condition for independence of w_- is that $f(w, \zeta)$ be homogeneous of degree two in *w*, a notable example being

$$f(w,\zeta) = w^2 h\{\operatorname{sgn}(w),\zeta\}$$
(76)

allowing for different dependence on position ζ on the incoming and outgoing branches of the type often postulated for inelastic impact (Goldsmith, 1960).

The relation (76) suggests the notation employed in (8) but may not represent the only possibility. For example, the linear viscoelastic ("spring-dashpot") model:

$$f = \omega_0^2 \zeta - \frac{1}{\tau} w, \tag{77}$$

with (elastic) frequency ω_0 and (viscous) relaxation time τ , has constant restitution e, as (Luding, 1997) shows. In a more direct proof, note that (70) can be integrated by elementary methods to give

$$(w - \lambda_{+}\zeta)^{1-\beta}(w - \lambda_{-}\zeta)^{1+\beta} = \text{const.},$$
(78)

where

$$\lambda_{\pm} = \frac{1 \pm \beta^{-1}}{2\tau}, \text{ with } \beta^{-1} = \sqrt{1 + 4(\omega_0 \tau)^2},$$
 (79)

are the roots of $\lambda^2 + \lambda/\tau - \omega_0^2$. The singular solutions $w_{\pm}(\zeta) = \lambda_{\pm}\zeta$ of (70) represent asymptotes for $\zeta \to \infty$, with

$$e = -\frac{w_{-}}{w_{+}} = \frac{1-\beta}{1+\beta}.$$
 (80)

Hence, on relaxing the implicit restriction to bounded w_{\pm} , (77) provides a counterexample to (76). However,

the generalized function of interest arises from the stiff limit:

$$\lambda_{\pm} \sim \pm \omega_0, \quad \beta \sim (2\omega_0 \tau)^{-1}, \quad w_{\pm} \sim \overline{\zeta}, \quad \text{for } \epsilon \equiv (\omega_0 \tau)^{-1} \to 0,$$
(81)

which implies elastic impact, with $e \sim 1 - \epsilon \rightarrow 1$ (Luding, 1997). An interesting question, not explored here, is whether (77) can represent an "inner" elastic approximation for small ϵ to an "outer" inelastic force of the form (76).

A.3. Perturbations

For velocity-independent restitution, it is clear that perturbed impacts, for which $w_- = w_-^{(0)} + w_-^{(1)}$ and $w_+^{(0)} = -ew_-^{(0)}$, must also satisfy $w_+^{(1)} = -ew_-^{(1)}$. This can be represented by a Stosslet perturbation

$$a^{(1)} = (1+e)\dot{\zeta}^{(1)}\dot{\zeta}\delta_{-}(\zeta), \tag{82}$$

so that (20) gives the corresponding form for a granular layer:

$$w^{(1)}(t_c^+, \mathbf{r}, z) - w^{(1)}(t_c^-, \mathbf{r}, z) = -(1+e)w^{(1)}(t_c^-, \mathbf{r}, 0),$$
(83)

where

$$t_c(\mathbf{r}) = t_c^{(0)} + t_c^{(1)}(\mathbf{r}).$$
(84)

The perturbation $t_c^{(1)}$ generally involves perturbations both in dwell time and arrival time. For the former, one may define a dwell time by $\theta_- + \theta_+$, where

$$heta_{\pm}=\pm\int_{0}^{\pm}igg[rac{1}{w(\zeta)}-rac{1}{w_{\pm}}igg]d\zeta,$$

given the existence of the combined integrals taken over the solution branches discussed above. However, it is evident that $\theta = O(\epsilon)$ for $\epsilon \to 0$ and, hence, that perturbations in this quantity generally should be negligible compared to the perturbation in arrival time determined from the perturbed contact condition:

$$\zeta(t_c, \mathbf{r}) = \zeta^{(0)}(t_c, \mathbf{r}) + \zeta^{(1)}(t_c, \mathbf{r})$$
(85)

$$=\partial_t \zeta^{(0)}(t_c^{(0)}, \mathbf{r}) t_c^{(1)}(\mathbf{r}) + \zeta^{(1)}(t_c^{(0)}, \mathbf{r}) + \dots = \mathbf{0}.$$
 (86)

Within the framework of an infinitesimal theory it is reasonable to employ the pre-impulsive value of the discontinuous function $\partial_t \zeta^{(0)}$ and to require that the perturbed impact precede the unperturbed impact, so that $t_c^{(1)} \leq 0$, as illustrated by the left-hand side of Fig. 5(b). With these restrictions, (86) leads to a *nonlinear* relation:

$$t_c^{(1)}(\mathbf{r}) = -R\{-1/w^{(0)}(t_c^{(0)}, \mathbf{r})\}R\{-\zeta^{(1)}(t_c^{(0)}, \mathbf{r})\},$$
(87)

where R(s) = sH(s) denotes the ramp function. Thus, our accounting for perturbed arrival time takes us beyond the realm of linear stability theory.

A.4. Curvature and viscous effects

To illustrate the salient points, it suffices to consider the axisymmetric form of the antiplane PDEs (37) near an isolated point of impact $t = t_c$, r = 0, say. Also, we take as sin-

gular body force the special form (76), without dependence of h on sgn(w), such that

$$\partial_t w - v r^{-1} \partial_r (r \partial_r) w = \frac{w^2}{\epsilon} h\left(\frac{\zeta}{\epsilon}\right),$$
(88)

where the non-singular body force g(t) are omitted. We assume the layer is locally flat in the vicinity of contact, such that the Taylor series

$$\zeta(t,r) = \zeta_0(t) + \zeta_2(t)r^2 + \dots,$$

$$w(t,r) = w_0(t) + w_2(t)r^2 + \dots,$$
(89)

apply. Then, with the identical scaling employed above, one obtains from the Taylor series representation of (88) the ODEs

$$\begin{aligned} \frac{d}{d\overline{t}}\overline{\zeta}_i &= w_i(t), \quad i = 0, 1, \dots, \quad \text{with } \overline{t} = (t - t_c)/\epsilon, \\ \frac{d}{d\overline{t}}w_0 &= w_0^2 h(\overline{\zeta}_0) + 4\epsilon v w_2, \\ \frac{d}{d\overline{t}}w_2 &= 2w_2 w_0 h(\overline{\zeta}_0) + \frac{w_0^2}{4} h'(\overline{\zeta}_0)\overline{\zeta}_2, \dots, \end{aligned}$$

where h'(s) denotes a derivative with respect to *s*. Clearly, the effect of viscosity on the basic Stosslet is $O(\epsilon)$ provided w_2 remains O(1).

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