Errata: "A Note on Eringen's Moment Balances"

1.) Beginning of sentence following Eq. (22) should read: " The rank–m conjugate forces $\mathfrak{s}^{(m)}, \boldsymbol{\nu}^{(m)}$, ..."

2.) Phrase following Eq. (24) should read "... functional on $\mathfrak{C}(\mathbf{0}, \mathbf{y})$."

3.) Line following Eq. (29) should read "$\mathfrak{f}(t, \mathbf{x}, \mathbf{y}) = -\mathfrak{f}(t, \mathbf{y}, \mathbf{x})$"

International Journal of Engineering Science 49 (2011) 1486–1493

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

A note on Eringen's moment balances

J.D. Goddard

Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive La Jolla, CA 92093-0411, USA

ARTICLE INFO

Article history:
Received 15 November 2010
Accepted 24 March 2011
Available online 21 April 2011

Keywords:
Micromorphic continuum
Moment balances
Virtual work
Two-scale fourier analysis
Source-flux duality

ABSTRACT

The present study provides a comparison of Eringen's [Eringen, A.C. (1970). Balance laws of micromorphic mechanics. *International Journal of Engineering Science, 8*, 819–828] general moment balances for micromorphic continua with Germain's [Germain, P. (1973). The method of virtual power in continuum mechanics. Part 2: Microstructure. *SIAM Journal on Applied Mathematics, 25*, 556–575] momentum balances based on virtual work principles, and with those derived in the present paper by a two-scale Fourier analysis of heterogeneous media. It has not been possible to establish a clear-cut correspondence between Eringen's balances and either of the latter, partly because Eringen's balances involve a mixture of surface and volume averages over microdomains.

There is disagreement between the last two methods, arising from the fact that Germain's treatment involves spatial gradients not occurring in the elementary two-scale Fourier analysis. A brief discussion is given of the possible extension of the latter to achieve agreement with the former.

As a separate matter, a construction of path-moments of density fields serves to establish a source-flux duality in continuum balances, which *inter alia* establishes a fairly direct connection between Newton's and Cauchy's laws and provides an expression for stress suggested by the statistical mechanics of point-particles.

1. Introduction

Among his extensive and widely recognized works on continuum mechanics, A.C. Eringen made numerous important contributions to the foundations and application of the theory of *micromorphic continua*.

We recall that, as an extension of the celebrated Cosserat theory, a micromorphic continuum is one whose material particles are endowed with additional degrees of freedom beyond those enjoyed by classical continua. In the case of mechanics, this includes general *microdeformation*[1] superimposed on the translational motion of the *simple* continuum (Truesdell & Noll, 1965).

In a long-running series of publications, well summarized by himself and others (Eringen, 1970, 1992; Germain, 1973), Eringen develops general balance equations for the associated micromorphic continuum. His technique consists basically of the fragmentation of a simple continuum into disjoint sub-bodies or "microdomains", in which variations in field quantities occur on scales much shorter than that of the overlying micromorphic fields.

One merit of Eringen's approach is the concrete physical interpretation it lends to certain fields as moments of familiar mechanical quantities. We take this a step further in the opening paragraphs of the following section, by recalling the equivalent moments for discrete point-particle systems (Goddard, 1998). As a further merit, Eringen's basic technique anticipates,

E-mail address: jgoddard@ucsd.edu

[1] Although Eringen often specializes to homogeneous deformation, he recognizes that the balances which are the subject of the present paper allow for arbitrary microdeformation.

0020-7225/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijengsci.2011.03.013

in many respects, the "two-scale" modeling employed by numerous workers for the homogenization of heterogeneous continua (Allaire, 1992; Kevorkian & Cole, 1996; Mielke & Timofte, 2008).

On the other hand, Eringen's results involve field variables defined by integrals over the surface of microdomains, which would seem to imply a possibly unwanted dependence on microdomain geometry. Furthermore, as pointed out by Germain (1973), it is not clear how Eringen's generalized momentum balances are related to balances obtained by other means, such as Germain's application of the principle of virtual work,[2] or other treatments of statistical mechanics and micromechanics. For example, based on the latter, Goddard (1998, 2008) has proposed a hierarchy of momentum balances involving pairwise interaction of moment stresses or "hyperstresses". As discussed below, Germain's analysis allows for a somewhat more general kinematics and conjugate hyperstress.

The present paper proceeds from a cursory review of background theory, along with certain results of Eringen (1970) and Germain (1973), to an alternative approach based on a two-scale analysis of Fourier representations and their spatial moments. As evident from several past works on the homogenization of heterogeneous media (Allaire & Conca, 1998; Kunin, 1982; Murdoch & Bedeaux, 2001), Fourier methods provide an obvious and natural tool for multiscale analysis. It is hoped that the present synthesis of methods may serve to provide, in slightly simpler notation, an enhanced appreciation of the various methods of homogenization.

As a more original effort, the final section below explores the duality between generation and flux in continuum balance equations, with results that may *inter alia* serve to diminish, if not banish, what the author views as a questionable dichotomy between Newton's and Cauchy's laws.

2. Background – Continuum balances

In this section we review the basic equations of balance and certain past work on these. As a word on notation, lowercase bold Greek and Fraktur fonts are employed for general tensors, with restricted use of lowercase bold Roman for vectors and uppercase bold Roman for second-rank tensors. The symbol $\hat{=}$ indicates equivalence between a tensor and its components relative to a general basis, designated by Latin indices. For the most part, we display contravariant components relative to a general curvilinear coordinates. Lowercase Greek indices are used to designate particulate entities and other extensive quantities and a colon is indicated to indicate the contraction or scalar product of tensors of rank greater than one. As a prelude to the discussion of continuum balances, we briefly review discrete-particle systems.

2.1. Moments in discrete-particle systems

To lend insight into certain continuum balances and their Fourier analysis to follow, we recall a related work (Goddard, 1998)[3] and consider a system of N point-particles having positions $\mathbf{x}_\alpha$ and linear momenta $\mathbf{p}_\alpha$, $\alpha = 1,\ldots,N$, each satisfying Newton's law $\dot{\mathbf{p}}_\alpha = \mathbf{f}_\alpha$, where $\mathbf{f}_\alpha$ is the sum of external plus interparticle forces. One may readily derive the Galilean-invariant moment balances (Goddard, 1998)

$$\dot{\mathfrak{p}}^{(m)} = \mathfrak{f}^{(m)}, \text{ with } \mathfrak{p}^{(m)} = \sum_\alpha \mathrm{p}'_\alpha \otimes \mathrm{x}'^m_\alpha, \ \mathfrak{f}^{(m)} = \sum_\alpha \left(\mathrm{f}'_\alpha \otimes \mathrm{x}'^m_\alpha + \mathrm{p}'_\alpha \otimes \dot{\overline{\mathrm{x}'^m_\alpha}} \right),$$

where, here as below,

$$\mathbf{z}^{m+1} = \mathbf{z}^m \otimes \mathbf{z} \hat{=} [z^{i_1} z^{i_2} \ldots z^{i_{m+1}}], \quad m = 0,1,2,\ldots, \mathbf{z}^0 := 1, \tag{1}$$

and

$$\mathbf{z}'_\alpha = \mathbf{z}_\alpha - \bar{\mathbf{z}}, \bar{\mathbf{z}} = \sum_\alpha w_\alpha \mathbf{z}_\alpha, \sum_\alpha w_\alpha = 1.$$

The w_α denote a set of constant scalar weights, usually assumed to be $m_\alpha / \sum_\beta m_\beta$, where m_α denotes particle mass. However, the above formulae remain valid under the alternative condition $\bar{\mathbf{z}} = \mathbf{0}$, $\mathbf{z}'_\alpha = \mathbf{z}_\alpha$, $\alpha = \mathbf{1},\ldots,\mathbf{N}$, yielding non-Galilean-invariant forms.

In any event, the tensors $\mathfrak{f}$ serve as generation or source of the moments $\mathfrak{p}$ and, when these are converted to densities, the above multiparticle system serves as representative of a single "particle" in a multipolar or micromorphic continuum. This is made more evident by the methods of Irving and Kirkwood (1950), Kunin (1982, 1984), where operator density or *distribution* for a particle-specific physical quantity Q and its continuous Fourier transform are given, respectively, by

$$\boldsymbol{\varrho}_Q(t,\mathbf{x}) = \sum_\alpha Q_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha\{t\}), \quad \text{and } \hat{\boldsymbol{\varrho}}_Q(t,\mathbf{k}) = \sum_\alpha Q_\alpha e^{-\mathbf{k}\cdot\mathbf{x}_\alpha} \equiv e^{-\mathbf{k}\cdot\bar{\mathbf{x}}} \breve{\boldsymbol{\varrho}}, \quad \text{with } \breve{\boldsymbol{\varrho}} = \sum_\alpha Q_\alpha e^{-\mathbf{k}\cdot(\mathbf{x}_\alpha - \bar{\mathbf{x}})}, \tag{2}$$

[2] In a related treatment of micromorphic electromagnetism (Eringen, 2006), one encounters the same surface integrals, but I have not made the effort to compare with the analysis of Maugin (1980) based on virtual work.

[3] Which presents a slightly different version, in which, incidentally, $\sum_i \dot{\mathbf{A}}_i \delta(\bar{\mathbf{x}} - \mathbf{x})$ in Eq. (26) should read $\hat{\mathbf{A}} \sum_i m_i \delta(\mathbf{x}_i - \mathbf{x})$, $\sum_i \mathbf{x}_i$ in Eq. (27) should read $\sum_i m_i \mathbf{x}_i$, $\mathbf{A}$ corresponds to $\boldsymbol{\alpha}$ of the present work, and the roles of Greek and Latin indices are reversed.

where δ is the Dirac delta. Galilean-invariant moments are then given by

$$\boldsymbol{\varrho}_{\mathrm{Q}}^{(m)} = (\imath\nabla_{\mathbf{k}})^m \breve{\boldsymbol{\varrho}}(t,\mathbf{k})\big|_{\mathbf{k}=\mathbf{0}}, \tag{3}$$

which reproduces the results given in (1).

That stated, we proceed to the analogous field theory, in which a spatial (Eulerian) description is favored over the conventional material (Lagrangian) description, with juxtaposition when necessary for comparison to other works.

2.2. Densities and balances

We consider a real tensor-valued extensive quantity $\boldsymbol{\alpha}\hat{=}[\alpha^{i_1,i_2,\ldots,i_n}]$, of rank n, endowed with a time-dependent spatial density $\boldsymbol{\varrho}_\alpha(t,\mathbf{x})\hat{=}[\varrho_\alpha^{i_1,i_2,\ldots,i_n}]$, which assigns a tensor-valued measure $d\boldsymbol{\alpha}(t,\mathbf{x}) := \boldsymbol{\alpha}(dV) = \boldsymbol{\varrho}_\alpha dV$ to an elemental spatial volume (measure) $dV(\mathbf{x}) = V(d\mathbf{x})$.

In the following we let superscript T represent the demotion of the leading index of a tensor $\boldsymbol{\psi}\hat{=}[\psi^{i_1,i_2,\ldots,i_n}]$ to trailing position, with lower or raising of indices permitted, so that $\boldsymbol{\psi}^{\mathrm{T}}\hat{=}[\psi^{i_2,\ldots,i_n,i_1}]$. To preserve the natural order of differential operators, we define ∇' as the linear operator such that

$$\nabla'\boldsymbol{\psi} = (\nabla\boldsymbol{\psi})^{\mathrm{T}} = [\psi^{i_1 i_2,\ldots,i_n}{}_{;i_{n+1}}] \tag{4}$$

where the semicolon denotes a covariant derivative, and we take

$$\mathrm{div}\boldsymbol{\psi} = \nabla'\cdot\boldsymbol{\psi} = [\psi^{i_1,i_2,\ldots,i_n}{}_{;i_n}] \tag{5}$$

Then, in the respective spatial or material description, the local or *strong* form of the $\boldsymbol{\alpha}$-balance is

$$\begin{aligned} &\partial_t\boldsymbol{\varrho}_\alpha + \mathrm{div}\boldsymbol{\varphi}_\alpha = \boldsymbol{\gamma}_\alpha, \quad \text{or } d_t\boldsymbol{\varrho}_\alpha - \mathrm{div}\boldsymbol{\tau}_\alpha = \mathfrak{g}_\alpha, \\ &\text{where,} \\ &d_t = \partial_t + \mathbf{v}\cdot\nabla', \mathfrak{g}_\alpha = \boldsymbol{\gamma}_\alpha - (\mathrm{div}\mathbf{v})\boldsymbol{\rho}_\alpha, \quad \boldsymbol{\tau}_\alpha = \boldsymbol{\varrho}_\alpha\otimes\mathbf{v} - \boldsymbol{\varphi}_\alpha, \end{aligned} \tag{6}$$

where $\boldsymbol{\gamma}_\alpha(t,\mathbf{x})\hat{=}[\gamma_\alpha^{i_1,i_2,\ldots,i_n}]$ denotes generation, $\boldsymbol{\varphi}_\alpha(t,\mathbf{x})\hat{=}[\varphi_\alpha^{i_1,i_2,\ldots,i_{n+1}}]$ a rank-$(n+1)$ flux, $\mathbf{v}(t,\mathbf{x})\hat{=}[v^i(t,\mathbf{x})]$ a material velocity, and $\boldsymbol{\tau}$ a negative "diffusion", denoted by the same symbol in Eringen (1970).

2.3. Eringen's theory

In Eringen's microdomains, certain field variables are assumed to take on the character of generalized functions or distributions, and the appropriate global or *weak* form of balance, involving a *test function* $\phi(t,\mathbf{x})$, is

$$\int_{\mathfrak{B}}[\phi(\partial_t\boldsymbol{\varrho}_\alpha - \boldsymbol{\gamma}_\alpha) - \boldsymbol{\varphi}_\alpha\cdot\nabla\phi]dV + \int_{\partial\mathfrak{B}}\phi\boldsymbol{\varphi}_\alpha\cdot\mathbf{n}dA = \mathbf{0}, \tag{7}$$

which is basically Eringen's "master balance". By employing a Taylor-series expansion of ϕ in a variable $\boldsymbol{\xi}$ representing position in a microdomain relative to a given spatial (or material), followed by averaging over $\boldsymbol{\xi}$, Eringen deduces from (7) a hierarchy of moment balances involving both surface and volume averages over microdomains.[4]

As signaled above in the Introduction, Eringen's basic theory resembles the two-scale modeling employed by others (Allaire, 1992; Kevorkian & Cole, 1996; Mielke & Timofte, 2008). In particular, it is tantamount to the *Ansatz* for distributions $\boldsymbol{\psi} = \boldsymbol{\varrho}, \boldsymbol{\varphi}, \boldsymbol{\gamma}$ and test function ϕ:

$$\boldsymbol{\psi}(t,\mathbf{x}) = \tilde{\boldsymbol{\psi}}(t,\bar{\mathbf{x}},\boldsymbol{\xi}/\epsilon), \phi(t,\mathbf{x}) = \tilde{\phi}(t,\bar{\mathbf{x}},\boldsymbol{\xi}), \quad \text{where } \boldsymbol{\xi} = \mathbf{x} - \bar{\mathbf{x}}, \tag{8}$$

where $0 < \epsilon \ll 1$ represents the characteristic ratio of microscale to macroscale, and $\bar{\mathbf{x}}$ is the provisionally fixed location of a fiducial spatial or material point that ultimately is to assume the role of position $\mathbf{x}$ in the micromorphic continuum. Thus, $|\boldsymbol{\xi}| = O(\epsilon)$ represents the microdomain as an ϵ-neighborhood of $\bar{\mathbf{x}}$, and for $\epsilon\to 0$ the field $\boldsymbol{\psi}$ generally exhibits locally unbounded variation with $\mathbf{x}$, whereas ϕ is a smooth function. In particular,

$$\nabla'\boldsymbol{\psi} = (\nabla'_{\bar{\mathbf{x}}} + \epsilon^{-1}\nabla'_{\boldsymbol{\eta}})\tilde{\boldsymbol{\psi}}, \quad \text{where } \boldsymbol{\eta} = \boldsymbol{\xi}/\epsilon, \quad \text{whereas } \nabla\phi = (\nabla_{\bar{\mathbf{x}}} + \nabla_{\boldsymbol{\xi}})\tilde{\phi}, \tag{9}$$

and so forth for higher gradients, which serves as the the basis for various two-scale perturbation schemes (Kevorkian & Cole, 1996). While it might be interesting to employ (7)–(9) to re-derive certain of Eringen's results, we adopt another approach based on Fourier techniques. As one merit of the latter, we recall that generalized functions in physical space generally possess well-behaved counterparts in Fourier space (Lighthill, 1958), as is amply demonstrated by (2).

[4] Although not considered here, a similar treatment of the time variable t leads to"two-timing".

3. Fourier transforms and moment balances

The form (9) is directly relevant to the Fourier representation of fields in unbounded bodies $\mathfrak{B}$, and, writing $\mathbf{k}\cdot\mathbf{x} = \mathbf{k}\cdot\bar{\mathbf{x}} + \boldsymbol{\kappa}\cdot\boldsymbol{\eta}$, we employ a *double* Fourier transform[5]

$$\hat{\psi}(t,\mathbf{k},\boldsymbol{\kappa}) = \int_{\mathfrak{B}} e^{-\imath\mathbf{k}\cdot\mathbf{x}}\psi(t,\mathbf{x})dV(\mathbf{x}) = \int_{\mathfrak{B}} e^{-\imath\mathbf{k}\cdot\bar{\mathbf{x}}}\check{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\kappa})dV(\bar{\mathbf{x}}), \tag{10}$$

where

$$\check{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\kappa}) = \int_{\tilde{\mathfrak{B}}} e^{-\imath\boldsymbol{\kappa}\cdot\boldsymbol{\eta}}\tilde{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\eta})dV(\boldsymbol{\eta}), \tag{11}$$

In general, $\tilde{\mathfrak{B}} = \tilde{\mathfrak{B}}(\bar{\mathbf{x}})$, as in Eringen's treatment, but we shall eventually take $\tilde{\mathfrak{B}}$ to be unbounded and independent of $\bar{\mathbf{x}}$. By analogy to the above formulae for discrete systems, we can derive local moments based on the equivalence $\nabla_{\boldsymbol{\kappa}} = -\imath\boldsymbol{\eta}$, *with*

$$(\imath\nabla'_{\boldsymbol{\kappa}})^m\check{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\kappa}) = \int_{\tilde{\mathfrak{B}}} e^{-\imath\boldsymbol{\kappa}\cdot\boldsymbol{\eta}}\tilde{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\eta})\otimes\boldsymbol{\eta}^m dV(\boldsymbol{\eta}), \tag{12}$$

for $m = 0,1,\ldots$, with local (normalized) moments

$$\overline{\boldsymbol{\psi}}^{(m)}(t,\bar{\mathbf{x}}) := \frac{\epsilon^m}{m!}\int_{\tilde{\mathfrak{B}}}\tilde{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\eta})\otimes\boldsymbol{\eta}^m dV(\boldsymbol{\eta}) = \frac{1}{m!}(\imath\epsilon\nabla'_{\boldsymbol{\kappa}})^m\check{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\kappa})\big|_{\boldsymbol{\kappa}=\mathbf{0}}, \tag{13}$$

providing a connection to Eringen's volume-average moments over sub-bodies $\tilde{\mathfrak{B}}$. As signaled above, we shall treat the latter as unbounded in the formal limit $\epsilon\to 0$, which of course avoids the issue of surface averages over $\partial\tilde{\mathfrak{B}}$ and the associated shape effects.

The normalization in (13) serves to suggest a plausible dependence on a characteristic microstructural scale ϵ. However, all the fields $\tilde{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\eta})$ may depend on ϵ, so that an abstract micromorphic continuum of grade m may be construed to emerge the limit $\epsilon\to 0$, with moments up to some arbitrary order m remaining finite.

3.1. General moment balances

With the standard Fourier transform, the balance (6) assumes the form

$$\partial_t\hat{\varrho}_\alpha + \imath\hat{\boldsymbol{\varphi}}_\alpha\cdot\mathbf{k} = \hat{\gamma}_\alpha, \tag{14}$$

and, by repeated application of the operation $\imath\nabla'_{\mathbf{k}}$, one can obtain global moment-balances. However, we can obtain analogous formulae for local moments by making use of (9) and (10) to replace (14) by:

$$\partial_t\check{\boldsymbol{\varrho}} + \overline{\nabla}'\cdot\check{\boldsymbol{\varphi}} + \imath\epsilon^{-1}\check{\boldsymbol{\varphi}}\cdot\boldsymbol{\kappa} = \check{\gamma}, \quad \text{where } \overline{\mathbf{v}}(\mathbf{t},\bar{\mathbf{x}}) = \left(\frac{\partial\bar{\mathbf{x}}}{\partial\mathbf{t}}\right)_{\mathbf{x}}, \quad \textbf{and } \overline{\nabla} = \nabla_{\bar{\mathbf{x}}}, \tag{15}$$

where $\check{\boldsymbol{\varrho}},\check{\boldsymbol{\tau}},\check{\gamma}$ are functions of $t,\bar{\mathbf{x}},\boldsymbol{\kappa}$, and where, for brevity's sake, we have suppressed subscripts α.

Generally, $\overline{\mathbf{v}}$ can be interpreted as the velocity associated with any conserved scalar w having prescribed density and flux, such that

$$\partial_t\varrho_w + \mathrm{div}\boldsymbol{\varphi}_w = 0, \quad \mathbf{v} = \mathbf{v}_w := \boldsymbol{\varphi}_w/\varrho_w, \tag{16}$$

where the density also assigns an abstract weight to each microdomain $\tilde{\mathfrak{B}}(\bar{\mathbf{x}})$.

At any rate, repeated application of $\imath\nabla'_{\boldsymbol{\kappa}}$ gives the hierarchy:

$$\left.\begin{aligned}
&\bar{d}_t\check{\boldsymbol{\varrho}}^{(m)} + \overline{\nabla}'\check{\boldsymbol{\varphi}}^{(m)} + \imath\epsilon^{-1}\check{\boldsymbol{\varphi}}^{(m)}\cdot\boldsymbol{\kappa} = \check{\gamma}^{(m)},\\
&\text{where,}\\
&\{\check{\boldsymbol{\varrho}}^{(m)},\check{\boldsymbol{\varphi}}^{(m)}\} = \imath\nabla'_{\boldsymbol{\kappa}}\{\check{\boldsymbol{\varrho}}^{(m-1)},\check{\boldsymbol{\varphi}}^{(m-1)}\} = (\imath\nabla'_{\boldsymbol{\kappa}})^m\{\check{\boldsymbol{\varrho}},\check{\boldsymbol{\varphi}}\},\\
&\text{and}\\
&\check{\gamma}^{(m)} = \imath\nabla'_{\boldsymbol{\kappa}}\check{\gamma}^{(m-1)} + \epsilon^{-1}\check{\boldsymbol{\varphi}}^{(m-1)} = (\imath\nabla'_{\boldsymbol{\kappa}})^m\check{\gamma} + \epsilon^{-1}m(\imath\nabla'_{\boldsymbol{\kappa}})^{m-1}\check{\boldsymbol{\tau}},
\end{aligned}\right\} \tag{17}$$

for $m = 1,2,\ldots$, where $m = 0$ refers to the initial balance (14). Then, the relations (13) and (17) give the local moment balances

$$\partial_t\overline{\boldsymbol{\varrho}}^{(m)} + \overline{\nabla}'\cdot\overline{\boldsymbol{\varphi}}^{(m)} = \overline{\gamma}^{(m)} + \overline{\boldsymbol{\varphi}}^{(m-1)}, \quad \text{with } \overline{\boldsymbol{\varphi}}^{(-1)} = \mathbf{0}, \tag{18}$$

for $m = 0,1,\ldots$, provided that $\check{\boldsymbol{\varphi}}^{(k)}\cdot\boldsymbol{\kappa}\to\mathbf{0}$ for $\boldsymbol{\kappa}\to\mathbf{0}$, for $k = 0,1,\ldots,m-1$.

The set (18) represents the desired hierarchy of moment-balances, the continuum analogues of discrete-particle moments proposed elsewhere (Goddard, 1998, 2008), with the standard form (6):

[5] Strictly speaking, this is the transform of $\chi_{\tilde{\mathfrak{B}}}(\boldsymbol{\eta})\check{\psi}(t,\bar{\mathbf{x}},\boldsymbol{\eta})$, where $\chi_{\tilde{\mathfrak{B}}}$ is the *characteristic function* of $\tilde{\mathfrak{B}}(\bar{\mathbf{x}})$.

$$\begin{aligned} &\partial_t \varrho_\alpha^{(m)} + \operatorname{div} \boldsymbol{\varphi}_\alpha^{(m)} = \breve{\gamma}_\alpha^{(m)} \;:=\; \gamma_\alpha^{(m)} + \boldsymbol{\varphi}_\alpha^{(m-1)}, \\ &\text{or,} \\ &d_t \varrho_\alpha^{(m)} - \operatorname{div} \boldsymbol{\tau}_\alpha^{(m)} = \breve{\mathfrak{g}}_\alpha^{(m)} \;:=\; \mathfrak{g}_\alpha^{(m)} + \boldsymbol{\varphi}_\alpha^{(m-1)}, \end{aligned} \tag{19}$$

for $m = 0, 1, \ldots$, where $\mathfrak{g}_\alpha^{(m)} = \gamma_\alpha^{(m)} - (\operatorname{div}\mathbf{v})\varrho_\alpha^{(m)}$, and $\boldsymbol{\varphi}_\alpha^{(-1)} = \mathbf{0}$, and where we have dropped overbars and restored the subscript α.

In order to secure agreement with Eringen (1970, Eqs. (3.4)–(3.6)), we must replace $\boldsymbol{\varphi}_\alpha^{(m-1)}$ in the second equation of (19) by a considerably more complicated form involving surface averages over $\partial\mathfrak{B}$ of $\boldsymbol{\tau}_\alpha^{(m)}$. At the time of this writing, it is not clear whether this replacement can be justified, even in the limit of unbounded $\mathfrak{B}$ considered in the present paper.

Although a primary focus of the present work is the set of momentum balances considered below, it is worth mentioning the mass balances, with $\boldsymbol{\alpha} = M$, $\varrho_M = \rho$, $\gamma_M \equiv 0$ in (19), and with $w = M$ in (16) defining the barycentric velocity $\mathbf{v}$. The first member of (19), with $m = 0$ is the usual mass balance, whereas the term $m = 1$ gives a balance for *mass-dipole density* $\varrho_M^{(1)}$, with source term $\boldsymbol{\varphi}_M = \rho\mathbf{v}$. The term $m = 2$, with $\varrho_M^{(2)}$ defining a moment of inertia, could presumably be brought into correspondence with Eringen's "balances of microinertia" (Eringen, 1970, Eqs. (3.10)–(3.12)), and related formulae of Germain (1973, Eqs. (46)–(47)), but space does not permit such an effort here.

3.1.1. Momentum balances

We now consider $\boldsymbol{\alpha} = \mathbf{p}$, so that, in the standard barycentric description,

$$\varrho_p = \boldsymbol{\varphi}_M = \rho\mathbf{v}, \quad \text{where } \boldsymbol{\tau}_p = \varrho_p \otimes \mathbf{v} - \boldsymbol{\varphi}_p, \quad \text{and } \gamma_p = \varrho_f, \tag{20}$$

where $\boldsymbol{\tau}_p \hat{=} [\tau_p^{ij}]$ is a stress tensor and ϱ_f an extrinsic body force density.

To make a connection to Germain (1973, *esp.* Eqs. (41) *ff.*), we note that his analysis is based on barycentric velocity, which in the present two-scale format can be written as

$$\mathbf{v}(t, \mathbf{x}) = \tilde{\mathbf{v}}(t, \bar{\mathbf{x}}, \boldsymbol{\eta}) \equiv \tilde{\boldsymbol{\varphi}}_M / \tilde{\rho}_M = \sum_{m \geqslant 0} \boldsymbol{\chi}^{(m)}(t, \bar{\mathbf{x}}) : \boldsymbol{\eta}^m, \quad \text{with } \boldsymbol{\chi}^{(0)} = \overline{\mathbf{v}}(\mathbf{t}, \bar{\mathbf{x}}), \tag{21}$$

where the tensors $\boldsymbol{\chi}^{(m)}$ are ϵ^m-times those denoted by the same symbol in Germain (1973). Germain's kinematic variables can be generated by retaining only linear terms in $\overline{\nabla}$ in the expansion of:

$$\nabla'^m \mathbf{v} \equiv (\overline{\nabla}' + \epsilon^{-1}\nabla'_{\boldsymbol{\eta}})^m \tilde{\mathbf{v}}.$$

Then, with $\boldsymbol{\chi}^{(m)}$ and $\overline{\nabla}'\boldsymbol{\chi}^{(m)}$ as kinematically independent variables, the internal power is given by

$$\sum_{m \geqslant 0} \left[\mathfrak{s}^{(m)} : \boldsymbol{\chi}^{(m)} + \boldsymbol{\nu}^{(m+1)} : \overline{\nabla}'\boldsymbol{\chi}^{(m)} \right]. \tag{22}$$

The rank-$(m+1)$ conjugate forces $\mathfrak{s}^{(m+1)}$, $\boldsymbol{\nu}^{(m)}$, which are denoted by the same letters in Germain (1973), are independent elements in the dual space. Application of the principle of virtual power then yields a hierarchy of momentum balances of the form

$$d_t \boldsymbol{\psi}^{(m)} - \operatorname{div} \boldsymbol{\nu}^{(m+1)} = \mathfrak{g}^{(m)} - \mathfrak{s}^{(m)}, \quad m = 0, 1, \ldots, \tag{23}$$

where $\mathfrak{g}$ represents external forces. Now, (23) can be brought into correspondence with (19), while preserving the character of other terms, only if

$$\boldsymbol{\nu}^{(m+1)} \equiv -\mathfrak{s}^{(m+1)} = \boldsymbol{\tau}_p^{(m)}, \quad \text{for } m = 0, 1 \ldots, \quad \text{with } \mathfrak{s}^{(0)} = \mathbf{0}.$$

It is therefore clear that the above two-scale model produces a more restricted model of a micromorphic continuum than that proposed by Germain. Since the latter involves higher-order terms in ∇, this can be visualized as the effect of macroscopic gradients on the relative motion of microdomains.

To extend the two-scale analysis, one can envisage a Fourier analysis of "mesodomains" consisting of ϵ-neighborhoods of microdomains, of exactly the same type as that carried out above for points within a given microdomain. Thus, proceeding from the tensor fields $\boldsymbol{\psi}^{(m)}$ appearing in the moment balances (19), one may derive a second hierarchy of fields $\boldsymbol{\psi}^{(m,n)}$ representing nth moments within mesodomains. Presumably, the first-gradient theory of Germain should involve only fields $\boldsymbol{\psi}^{(m,0)} \equiv \boldsymbol{\psi}^{(m)}$ and $\boldsymbol{\psi}^{(m,1)}$, with $m = 0, 1$ for the simplest micromorphic continuum. More formally, one may treat this a multi-scale expansion of the form

$$\mathbf{x} = \bar{\mathbf{x}} + \epsilon_1\boldsymbol{\eta}_1 + \epsilon_2\boldsymbol{\eta}_2 + \cdots, \quad \epsilon_1 \ll \epsilon_2 \ll \cdots,$$

with ϵ_1, $\boldsymbol{\eta}_1$ corresponding to the variables ϵ, $\boldsymbol{\eta}$ employed in the above two-scale analysis, and with multivariate Fourier transforms in $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2, \ldots$. Without further pursuit of this analysis, we turn to a basic issue surrounding balance equations.

4. Source-flux duality

As indicated by certain statistical mechanical treatments (Goddard, 1998; Irving & Kirkwood, 1950; Noll, 1955), quantities that appear as nominal source terms in microscopic balances can, under certain circumstances, be construed as the divergence of continuum-level flux. By a related technique, it is shown next that a similar construct holds for continuum balances, in accordance with the Gauss law and the related existence proof of Segev and De Botton (1991).

4.1. Path-moments of densities

Given a density $\varrho_\alpha(t,\mathbf{x})$ a continuous directed curve $\mathfrak{C}(\mathbf{a},\mathbf{b})$ running from $\mathbf{a}$ to $\mathbf{b}$, the line integral

$$\boldsymbol{\mu}_\alpha(t,\mathbf{x};\mathfrak{C}) = \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} \varrho_\alpha(t,\mathbf{x}-\mathbf{z}) \otimes d\mathbf{z}, \tag{24}$$

defines a path-moment density $\boldsymbol{\mu}_\alpha \hat{=} [\mu_\alpha^{i_1,i_2,\ldots,i_{n+1}}]$ of rank $n+1$, which obviously is a functional on $\mathfrak{C}(\mathbf{x},\mathbf{y})$. Regarded as function of the variable $\mathbf{x}$, the corresponding field $\boldsymbol{\mu}_\alpha(t,\mathbf{x})$ is readily shown to satisfy

$$\mathrm{div}\boldsymbol{\mu}_\alpha = \varrho_\alpha(t,\mathbf{x}) - \varrho_\alpha(t,\mathbf{x}-\mathbf{y}), \tag{25}$$

independently of the curve $\mathfrak{C}$. Hence, the integral around the closed curve represented by $\mathbf{y}=\mathbf{0}$ in (24), yields a solenoidal field $\boldsymbol{\mu}_\alpha$ (Goddard, 1998). We note that the Fourier representation of the above is

$$\begin{aligned} &\hat{\boldsymbol{\mu}}_\alpha(t,\mathbf{k};\mathfrak{C}) = \hat{\varrho}_\alpha(t,\mathbf{k}) \otimes \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} e^{-\imath\mathbf{k}\cdot\mathbf{z}} d\mathbf{z}, \\ &\text{with,} \\ &\imath\hat{\boldsymbol{\mu}}_\alpha\cdot\mathbf{k} = \hat{\varrho}_\alpha(t,\mathbf{k}) \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} e^{-\imath\mathbf{k}\cdot\mathbf{z}} d(\imath\mathbf{k}\cdot\mathbf{z}) = \hat{\varrho}_\alpha(t,\mathbf{k})(1-e^{-\imath\mathbf{k}\cdot\mathbf{y}}). \end{aligned} \tag{26}$$

4.2. Spatially restricted densities

If $\varrho_\alpha(t,\mathbf{x})$ represents a field that vanishes at points $\mathbf{x}$ outside a finite material body $\mathfrak{B}$, then we may choose $|\mathbf{y}|$ sufficiently large that $\mathbf{x}-\mathbf{y}$ lies outside $\mathfrak{B}$ for all $\mathbf{x}\in\mathfrak{B}$. In that case, we may disregard the term $\varrho_\alpha(\mathbf{x}-\mathbf{y},t)$ in (25). The same is true for an unbounded body $\mathfrak{B}$, provided $\varrho_\alpha(t,\mathbf{x})\to\mathbf{0}$ for $|\mathbf{x}|\to\infty$. In either case, we use the notation $\mathbf{y}=\infty$ to define such points, with $\mathfrak{C}(\mathbf{0},\infty)$ indicating the path in (24), and we designate the corresponding densities as "spatially restricted".[6]

In the case of spatially restricted densities, (25) reduces to the classical (Gauss-Maxwell) form

$$\mathrm{div}\,\boldsymbol{\mu}_\alpha = \varrho_\alpha(t,\mathbf{x}), \tag{27}$$

This imparts a duality to certain source terms in continuum balances, as they may also be expressed as divergence of flux, as shown for forces and stresses by Segev and De Botton (1991).

4.3. Stress and its localization

While physicists sometimes attribute continuum momentum balances to Newton, the modern continuum mechanics literature exhibits a pronounced deference to Euler and Cauchy.[7] As a departure from that tradition, whose axiomatic foundation is discussed elsewhere (Truesdell, 1991, Chapter III), we provisionally write the standard linear momentum balance in the form:

$$d_t\varrho_p + (\mathrm{div}\mathbf{v})\varrho_p \equiv \rho d_t\mathbf{v} = \varrho_f = \varrho_I + \varrho_E, \tag{28}$$

where $\varrho_I(t,\mathbf{x}) \hat{=} [\varrho_I^i(t,\mathbf{x})]$ is internal force density, arising from material interaction, and ϱ_E is external body-force density. If the former is spatially-restricted, then it follows immediately that we may write $\varrho_I = \mathrm{div}\,\boldsymbol{\tau}_p$, where $\boldsymbol{\tau}_p$ plays the role of a stress tensor. Moreover, with a further postulate,[8] we may express it in terms of interactions, generally non-local, between distinct material points in a continuum. In particular, we assume that

$$\varrho_I(t,\mathbf{x}) = \int_{\mathbf{y}\in\mathfrak{B}} \mathfrak{f}(t,\mathbf{x},\mathbf{y}) dV(\mathbf{y}), \tag{29}$$

[6] In a more general treatment, one could envisage spatio-temporally restricted densities, limited by various signal speeds.

[7] aptly summarized by the preeminent scholar: "In continuum mechanics Euler and Cauchy always considered only resultant forces upon bodies, and the later tradition did the same. That made reasoning on the foundation of continuum mechanics basically different from that which is used in the analytical dynamics of mass-points, which always considers systems of forces in which each point attracts or repels other points. That difference forced Cauchy to introduce contact forces and body forces *apriori*." Truesdell (1992). Those of like mind may choose to designate (28) as a strong form of Cauchy's first law.

[8] A similar form is to be found in the work of Gurtin, Williams, and Ziemer (1986, 2nd eq.), who consider general conditions for the existence of Cauchy fluxes.

where $\mathfrak{f}(t,\mathbf{x},\mathbf{y}) = -\mathfrak{f}(t,\mathbf{y},\mathbf{x}),\mathbf{y}]$ is a *two-point force density*, with

$$\mathfrak{f}(t,\mathbf{x},\mathbf{y})dV(\mathbf{x})dV(\mathbf{y}),$$

representing the vector force on, or net momentum exchange rate into, the volume element $dV(\mathbf{x})$ emanating from the element $dV(\mathbf{y})$. In various molecular theories (Goddard, 1986, 1998; Irving & Kirkwood, 1950; Murdoch & Bedeaux, 1993; Noll, 1955), $\mathfrak{f}$ can be attributed to intermolecular forces and streaming momentum exchange.

As with discrete particle systems, $\mathfrak{f}(t,\mathbf{x},\mathbf{y})$ is to be derived from kinematics by appropriate constitutive equations but, in the spirit of Newtonian mechanics, we treat it provisionally as known.

Then, by a simple changes of variables and the **xy**-antisymmetry of $\mathfrak{f}(t,\mathbf{x},\mathbf{y})$, it follows that

$$\int_{\mathbf{y}\in\mathfrak{B}} \mathfrak{f}(t,\mathbf{x}-\mathbf{y},\mathbf{y})dV(\mathbf{y}) = \int_{\mathbf{y}'\in\mathfrak{B}'} \mathfrak{f}(t,\mathbf{y}',\mathbf{x}-\mathbf{y}')dV(\mathbf{y}') = -\int_{\mathbf{y}\in\mathfrak{B}'} \mathfrak{f}(t,\mathbf{x}-\mathbf{y},\mathbf{y})dV(\mathbf{y}), \tag{30}$$

where $\mathfrak{B}' = \{\mathbf{y} : \mathbf{x}-\mathbf{y} \in \mathfrak{B}\}$. For infinite regions $\mathfrak{B}$, we can take $\mathfrak{B}' = \mathfrak{B}$, so that, whenever it exists, the integral (30) must equal zero. Assuming this to be the case, we may replace $\boldsymbol{\mu}_l(t,\mathbf{x})$, as defined by $\alpha = I$ in (24), by:

$$\boldsymbol{\tau}_p(t,\mathbf{x}) = \int_{\mathbf{y}\in\mathfrak{B}} \int_{\mathfrak{C}(\mathbf{0},\mathbf{y})} \mathfrak{f}(t,\mathbf{x}-\mathbf{z},\mathbf{y}) \otimes d\mathbf{z}\, dV(\mathbf{y}), \tag{31}$$

which has a form suggested by statistical mechanical treatments involving forces between discrete particles (Irving & Kirkwood, 1950; Noll, 1955; Goddard, 1998).

It is clear that (31) generally involves nonlocal material interactions, and in that respect it is germane to nonlocal field theories.[9] However, borrowing again from statistical mechanics (Goddard, 1998), we can formally obtain a localized version of (31), suitable for expressing contact actions across arbitrarily small portions of the boundary of a body.

4.4. Localization

In numerous situations, we expect the two-point density $\mathfrak{f}(t,\mathbf{x},\mathbf{y})$ to fall off rapidly with separation $|\mathbf{y}-\mathbf{x}|$, or, following the above two-scale analysis, to have the property

$$\begin{aligned} &\mathfrak{f}(t,\mathbf{x},\mathbf{y}) = \tilde{\mathfrak{f}}(t,\mathbf{x},\boldsymbol{\eta}), \quad \text{with } \boldsymbol{\eta} = (\mathbf{y}-\mathbf{x})/\epsilon, \quad 0 < \epsilon \ll 1, \\ &\text{where} \\ &\tilde{\mathfrak{f}}(t,\mathbf{x},-\boldsymbol{\eta}) = -\tilde{\mathfrak{f}}(t,\mathbf{x},\boldsymbol{\eta}) \quad \text{and } \tilde{\mathfrak{f}}(t,\mathbf{x},\boldsymbol{\eta}) \to \mathbf{0} \quad \text{for } |\boldsymbol{\eta}| \to \infty. \end{aligned} \tag{32}$$

Hence, we can write

$$\mathfrak{f}(t,\mathbf{x}-\mathbf{z},\mathbf{y}) = \tilde{\mathfrak{f}}(t,\mathbf{x}-\mathbf{z},[\mathbf{z}-(\mathbf{x}-\mathbf{y})]/\epsilon),$$

whence it is clear that the dominant contribution to the integral (31) comes from the region $0 \leqslant |\mathbf{z}| \leqslant R$ with $R \approx |\mathbf{y}-\mathbf{x}| = O(\epsilon)$. Then, it is easy to show that, with suitable smoothness of $\mathfrak{f}$, the integral is given up to an additive solenoidal term as the *force dipole*:

$$\boldsymbol{\tau}_p(t,\mathbf{x}) \approx \int_{\mathbf{y}\in\mathfrak{B}_\epsilon} \mathfrak{f}(t,\mathbf{x},\mathbf{y}) \otimes (\mathbf{y}-\mathbf{x})\, dV(\mathbf{y}), \tag{33}$$

with relative error ϵ, where $\mathfrak{B}_\epsilon = \mathfrak{B}_\epsilon(\mathbf{x})$ is an ϵ-neighborhood of $\mathbf{x}$. The form (33), as well as higher multipoles, are suggested by statistical mechanics [4] and embody Noll's "principle of local action" (Truesdell & Noll, 1965). While space does not allow a full elaboration on the equivalence to the standard continuum stress, the analysis suggests that statistical mechanics or micromechanical modeling could perhaps as well seek a pairwise force density, rather than a stress tensor *per se*.

5. Conclusions

As major findings of the present study, we conclude that Eringen's moment balances require some re-interpretation to establish a direct correspondence either with the type of balance derived by Germain or with that obtained by a two-scale analysis based on Fourier transforms. However, there is disagreement between the latter two methods, arising from the fact that Germain's treatment in effect includes gradient-coupling of microdomains, as reflected by spatial gradients not occurring in the elementary two-scale analysis considered above. As suggested but not implemented in the present work, it should be straightforward to extend the two-scale analysis to "mesodomains" and to achieve agreement with the results of Germain.

It is conceivable that Eringen's micromorphic continuum is more general than that proposed by Germain but, if so, it would be desirable to reconcile it with virtual work principles.

[9] Where Eringen made numerous seminal contributions (cf. Eringen, 2002). As pointed out by Maugin (1980), there is a certain matter of taste in choosing non-local vs. gradient theories, although there are well-known issues with convergence and with boundary conditions for the latter.

As a somewhat separate matter, we have explored the source-flux duality in continuum balances, which serves to establish a direct connection between Newton's and Cauchy's laws and to provide an expression for stress suggested by the statistical mechanics of Kirkwood and coworkers. It remains to verify that the localized forms involving force multipoles exhibit the contact actions of stress and hyperstress.

Note added in proof

The "peridynamics" of Silling and coworkers (cf. Lehoucq & Silling, 2008), which only recently came to the author's attention, is subsumed in the present construct, represented by (24).

Acknowledgement

This contribution dedicated to the memory of Professor A.C. Eringen is but a small token of gratitude for the privilege of a long and amicable service with him as member of the editorial board of IJES.

References

Allaire, G. (1992). Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis, 23*, 1482–1518.
Allaire, G., & Conca, C. (1998). Bloch wave homogenization and spectral asymptotic analysis. *Journal de Mathématiques Pures et Appliquées, 77*, 153–208.
Eringen, A. C. (1970). Balance laws of micromorphic mechanics. *International Journal of Engineering Science, 8*, 819–828.
Eringen, A. C. (1992). Balance laws of micromorphic continua revisited. *International Journal of Engineering Studies, 30*, 805–810.
Eringen, A. C. (2002). *Nonlocal continuum field theories*. New York: Springer.
Eringen, A. C. (2006). Micromorphic electromagnetic theory and waves. *Foundation of Physics, 36*, 902–919.
Germain, P. (1973). The method of virtual power in continuum mechanics. Part 2: Microstructure. *SIAM Journal on Applied Mathematics, 25*, 556–575.
Goddard, J. D. (1986). Microstructural origins of continuum stress fields – a brief history and some unresolved issues. In D. De Kee & P. Kaloni (Eds.), *Recent developments in structured continua. Pitman research notes in mathematics* (Vol. 143, pp. 179–208). New York: Longman/J. Wiley.
Goddard, J. D. (1998). Continuum modeling of granular assemblies. In H. J. Herrmann et al. (Eds.), *NATO ASI, physics of dry granular media* (pp. 1–24). Dordrecht: Kluwer.
Goddard, J. D. (2008). From granular matter to generalized continuum. In P. Mariano, G. Capriz, & P. Giovine (Eds.), *Mathematical models of granular matter. Lecture notes in mathematics* (Vol. 1937, pp. 1–20). Berlin: Springer.
Gurtin, M. E., Williams, W. O., & Ziemer, W. P. (1986). Geometric-measure theory and the axioms of continuum thermodynamics. *Archive of Rational Mechanics and Analysis, 92*, 1–22.
Irving, J., & Kirkwood, J. (1950). The statistical mechanical theory of transport properties. IV: The equations of hydrodynamics. *Journal of Chemical Physics, 18*, 817.
Kevorkian, J., & Cole, J. (1996). *Multiple scale and singular perturbation methods. Applied Mathematical Sciences* (Vol. 114). New York: Springer.
Kunin, I. (1982). *Elastic media with microstructure volume 1–2. Springer series in solid-state sciences* (Vol. 26 & 44). Berlin, New York: Springer-Verlag.
Kunin, I. (1984). On foundations of the theory of elastic media with microstructure. *International Journal of Engineering Studies, 22*, 969–978.
Lehoucq, R., & Silling, S. (2008). Force flux and the peridynamic stress tensor. *Journal of the Mechanics and Physics of Solids, 56*, 1566–1577.
Lighthill, M. (1958). *Introduction to Fourier analysis and generalised functions, Cambridge monographs on mechanics and applied mathematics*. Cambridge, U.K.: University Press.
Maugin, G. (1980). The method of virtual power in continuum mechanics: Application to coupled fields. *Acta Mechanica, 35*, 1–70.
Mielke, A., & Timofte, A. (2008). Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM Journal on Mathematical Analysis, 39*, 642–668.
Murdoch, I., & Bedeaux, D. (1993). On the physical interpretation of fields in continuum mechanics. *International Journal of Engineering Studies, 31*, 1345–1373.
Murdoch, I., & Bedeaux, D. (2001). Characterization of microstates for confined systems and associated scale-dependent continuum fields via fourier coefficients. *Journal of Physics A, 34*, 6495–6508.
Noll, W. (1955). Die Herleitung der Grundgleichungen der Thermomechanik der Kontinua aus der statistischen Mechanik. *Journal of Rational Mechanics and Analysis, 4*, 627–646. English Translation: Lehoucq, R., & Von Lillenfeld-Toal, A. (2010). J. Elasticity, 100(1), 5–24.
Segev, R., & De Botton, G. (1991). On the consistency conditions for force systems. *International Journal of Non-Linear Mechanics, 26*, 47–59.
Truesdell, C. (1991). *A first course in rational continuum mechanics. Pure and Applied Mathematics* (Vol. 71). Boston: Academic Press.
Truesdell, C. (1992). Cauchy and the modern mechanics of continua. *Revue d'Histoire des Sciences, 45*, 5–24.
Truesdell, C., & Noll, W. (1965). The non-linear field theories of mechanics. In S. Flügge (Ed.). *Encyclopedia of physics (Handbuch der Physik)* (Vol. III/3). Berlin, New York: Springer-Verlag.