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Frictionless conveying of frictional materials

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Abstract Following a brief review of the technology and previous analyses of vibratory conveying of granular materials, a general solution is derived for a well-known rigid-slab model of slide-conveying in open and closed conduits, with Coulomb friction at the walls. The solution is applied to periodic rectangular-wave and sinusoidal forcing, and it is shown that the rectangular-wave forcing admits ideal cycles in which the kinematically optimal transport is also thermodynamically optimal, in the sense that no energy is dissipated by sliding friction.

Keywords Slide conveying · Optimal cycle · Granular materials · Coulomb friction · Vibrated granular layers

1 Introduction

Vibratory conveyors, routinely employed in industry for transport of granular materials, generally consist of an oscillating continuous conduit or trough which induces axial movement of a granular material along its surface. The main advantages of such conveyors are their simple construction, their suitability for handling hot or abrasive materials and their applicability as "dosing" equipment. Since the conduit can be totally enclosed, this type of conveying is also well suited to environmentally benign transport of dusty or hazardous materials. In "flight" or "throw" conveyors the transported material often loses contact with the trough, which generally leads to higher transport rates. Given the many parameters characterizing a vibratory conveyor and the transported material, the complete theoretical prediction of performance is a daunting task. At the present juncture, designers must rely heavily on empirical data, unfortunately scarce in the published literature.

Figure 1 presents a schematic cross-sectional views of a vibratory conveyor, with open-channel throw-conveyor shown in Fig. 1a and closed-channel slide-conveyor shown in Fig. 1b. The channel is tilted at angle α relative to the horizontal and undergoes a cyclic displacement $\mathbf{x}_c(t)$ with velocity

$$\mathbf{v}_c \equiv \dot{\mathbf{x}}_c(t) = u_c(t)\mathbf{e}_x + v_c(t)\mathbf{e}_y, \tag{1}$$

relative to a fixed system of Cartesian coordinates x, y, with basis $\mathbf{e}_x$, $\mathbf{e}_y$ affixed to the channel walls representing down-channel and cross-channel directions, respectively. The constant gravitational acceleration is given by

$$\mathbf{g}_0 = -g_0(\sin\alpha\mathbf{e}_x + \cos\alpha\mathbf{e}_y) \tag{2}$$

The elliptical cycles illustrated schematically in Fig. 1 represent one of the more common modes of cyclic displacement, with

$$\mathbf{x}_c(t) = A[\cos\beta\sin\omega t\ \mathbf{e}_x + \sin\beta\sin\omega(t+\varphi)\mathbf{e}_y], \tag{3}$$

where A is the amplitude, ω, the frequency, β, φ are constants $\in [0, 2\pi)$, with $\tan\beta$ representing amplitude ratio and φ phase lag. We recall that Nedderman and Harding [4,6] consider the linear cycle $\varphi = 0$, and Sloot and Kruyt [7] the elliptical cycle $\varphi = \pi/2$, for open-channel conveying. The nondimensional "throw number"

$$\Gamma = A\omega^2\sin\beta/g_0\cos\alpha \tag{4}$$

serves to distinguishes the two distinct modes of operation, slide-conveying and throw-conveying, accordingly as Γ is less than or greater than unity. The same parameter, with $\alpha =$

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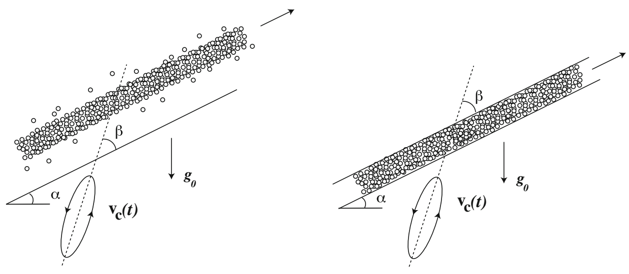

Fig. 1 **a** Open-channel throw-conveying versus. **b** Closed-channel slide-conveying

0, $\beta = \pi/2$, arises in various studies of vertically vibrated granular layers reviewed in [3,5].

The general objective in the optimal design of vibratory conveyors is maximization of the material transport rate subject to various operational constraints, such as the dynamical stability of the granular layer. This motivates the definition of a kinematic ("velocity") efficiency η, the ratio of the average velocity of the material to the maximum tangential conduit velocity. Much of past modeling as well as typical manufacturer's design charts are concerned with the functional relations of the form

$$\eta = f(\Gamma, \beta, \alpha, \varphi) \tag{5}$$

for different vibratory conveying equipment [4,6,7]. Certain investigations also consider the dependence on μ, the sliding friction coefficient between the conduit wall and the material. As suggested by the present work, energy-based definitions of conveyor efficiency are possible, serving to place conveyors on the same thermodynamic foundation as various pumps. Depending on technological circumstances, this type of efficiency could be more relevant than the typical kinematic efficiency.

Erdesz and coworkers [1,2] have performed experimental and theoretical studies of slide- and throw-conveyors. Sloot and Kruyt [7] review this later work, and they offer an improved analysis of vibratory conveying based on the rigid-slab (or in their terminology "point-mass") model. In this model, the granular mass is treated as a non-rotating flat slab that interacts with the conduit walls via Coulomb friction and inelastic impact.

Nedderman and Harding [4,6] present numerical optimization studies for slide-conveyors with unequal coefficients of static and sliding friction.

Several investigations of throw-conveying indicate that the flight phase is terminated by a completely inelastic collision with the conduit walls [7]. A thorough analysis, including the in-flight stability of throw-conveying, would necessitate a deformable continuum model of the granular mass, with accounting for the collisional rebound at a solid wall of the type recently studied for the case of vertically-vibrated layers [3]. We recall that this model trivially yields the flat-layer states described by the rigid-slab model solutions in the throw phase and furthermore, exhibits certain surface patterns observed on vertically vibrated layers. Since the present work is focused on slide conveying, the rigid-slab model is considered adequate.

2 Analysis

In the present analysis we consider the situations depicted in Fig. 1, namely, open conveyors, bounded by a single rigid conveying surface or wall at $y = 0$, and closed conveyors, bounded by both upper and lower rigid walls at $y = 0, h$, respectively. In either case, the granular material is assumed to move as a rigid slab subject to Coulomb friction on the channel walls, so that Newton's equations of motion for the velocity of the granular mass relative to the bottom wall become

$$\mathbf{v}(t) = u(t)\mathbf{e}_x + v(t)\mathbf{e}_y \tag{6}$$

is

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(t) + \mathbf{g}(t), \quad \text{with} \quad \mathbf{f}(t) = T(t)\mathbf{e}_x + N(t)\mathbf{e}_y \tag{7}$$

where T and N denote, respectively, tangential and normal force per unit mass (i.e. acceleration) exerted by the bounding surface(s), and the effective gravity is given by

$$\mathbf{g}(t) = \mathbf{g}_0 - \dot{\mathbf{v}}_c(t), \tag{8}$$

representing a periodic function of t, assumed to be piecewise continuous with mean equal to the normal gravity $\mathbf{g}_0 =$ constant.

In the present setting, the force N represents a reaction against the rigid constraint $v(t) \equiv 0$ imposed by the (non-cohesive) bounding surfaces and is given by the y-component of (7)

$$\frac{dv}{dt} = N(t) + g_y(t) \equiv 0,$$

as

$$N(t) \equiv N_c(t) := -g_y(t) = g_0 \cos\alpha + \dot{v}_c(t) \tag{9}$$

In the case of an open conveyor, the force $N \geq 0$ arises solely from the bottom wall, whereas in the case of a closed conveyor it can arise either from the bottom ($N \geq 0$) or the top ($N \leq 0$).

In the absence of sliding, the tangential force T also represents a reaction, given by the x-component of (7) as

$$T(t) = T_c(t) := -g_x(t) = g_0 \sin\alpha + \dot{u}_c, \quad \text{for} \quad u(t) \equiv 0 \tag{10}$$

According to the Coulomb criterion, sliding occurs whenever $|T| > \mu|N|$, in which case T becomes an active force, with

$$T(t) = -\mu|N(t)|\mathrm{sgn}\{u(t)\}, \tag{11}$$

where sgn denotes the set-valued signum function, with $\mathrm{sgn}(0) \in [-1, 1]$. In what follows we assume a constant coefficient of friction μ, the same for active and incipient sliding, i.e. for slipping and sticking, respectively. With non-dimensional forms based on the scaling

$$\begin{aligned} &u \to A\omega u,\ t \to \omega^{-1}t,\ g_x(t) \to A\omega^2 G_1(t),\\ &\mu g_y(t) \to -A\omega^2 G_2(t), \end{aligned} \tag{12}$$

where A and ω are, respectively, a representative amplitude and circular frequency ($= 2\pi/\text{period}$) of the cyclic channel displacement, the x-component of (7) becomes

$$\begin{aligned} &\frac{du}{dt} = G_1(t) - |G_2(t)|\mathrm{sgn}(u), \quad \text{for} \quad |G_1(t)| > |G_2(t)|,\\ &u = 0, \quad \text{otherwise} \end{aligned} \tag{13}$$

In the present setting, $G_1(t)$ and $G_2(t)$ are 2π-periodic functions of t, and the above restriction on the mean of $\mathbf{g}(t)$ can be expressed as

$$\begin{aligned} &\int_0^{2\pi} G_1(t)dt = -2\pi G\sin\alpha, \quad \text{and}\\ &\int_0^{2\pi} G_2(t)dt = 2\pi\mu G\cos\alpha, \text{ where } G := g_0/A\omega^2 \geq 0 \end{aligned} \tag{14}$$

In the case of an open conveyor, we have the additional restriction $G_2 \geq 0$.

Subject to the above restrictions on G_1, G_2, we seek 2π-periodic, piecewise-continuous solutions $u(t)$ of (13). To this end, the ODE (13) can be put in the more compact form

$$\frac{dU}{d\tau} = -\mathrm{sgn}(U + u_1), \quad \text{where } U = u - u_1, \tag{15}$$

with

$$u_1(t) = \int_0^t G_1(s)ds, \quad \text{and} \quad \tau(t) = \int_0^t |G_2(s)|ds \tag{16}$$

The relations (16) provide a parametric description, with t as parameter, of u_1 and U as functions of τ, the latter being a positive monotone function of t. Since G_1 and G_2 are periodic functions of t, the function $u_1(\tau)$ ranges over the interval $[mU(T), (m+1)U(T)]$ as τ ranges over $[mT, (m+1)T]$, for $m = 0, \pm1, \pm2, \ldots$, where $T = \tau(\omega/2\pi)$. Hence, we may restrict attention to the interval defined by $m = 0$ as representative of a typical cycle.

The obvious solution to (15) consists of three phases:

$$\begin{aligned} &u - u_1 = U = \begin{cases} -\tau, & \text{for } \tau < u_1,\\ +\tau, & \text{for } \tau < -u_1, \end{cases}\\ &\text{and}\\ &U = -u_1, \quad \text{with} \quad \left|\frac{du_1}{d\tau}\right| \equiv \frac{|G_1|}{|G_2|} \leq 1, \end{aligned} \tag{17}$$

representing, respectively, forward or positive sliding $u > 0$, backward or negative sliding $u < 0$, and sticking $u = 0$. We recall that these are sometimes referred to, respectively, as "P" (positive), "N" (negative), and "R" (rest) phases, with "F" referring to the flight-phase in throw-conveying.

3 Optimal transport

The mathematical problem of optimal transport consists generally of extremalization of a suitable functional of the solution u to (13) with respect to the forcing functions G_1, G_2. While the non-differentiability of (13) with respect to u appears to rule out standard variational methods, this state of affairs is mitigated by the availability of the exact solution (17).

We consider one of the most basic optimization problems, the maximization of net displacement over a cycle, *viz.*:

$$\begin{aligned} &\max_{\mathcal{F}} X, \quad \text{where } X\{\mathcal{F}\} = \int_0^{2\pi} u(t)dt, \quad \text{and}\\ &\mathcal{F} = \{G_1, G_2 \in \mathcal{C}\}, \end{aligned} \tag{18}$$

with u subject to (13). Here $\mathcal{C}$ denotes the class of bounded piecewise-continuous and 2π-periodic functions satisfying (14) together the further restriction $G_2 \leq 0$ for the open conveyor. In view of the first members of (14) and (16), we may substitute U for u in the definition of the objective function X in (18). We now consider certain globally-optimal or "ideal" cycles restricted only by bounds on channel acceleration followed by some brief remarks on the simply-periodic elliptical cycle illustrated in Fig. 1.

3.1 Ideal cycles

We assume the channel acceleration is bounded, such that

$$|G_2|, |G_1| \leq G_{\max}, \tag{19}$$

In the case of open conveyors, a bound on $G_{\max}$ is determined by a bound on G_2 necessary to avoid throw-conveying.

Now it is evident from (17) and the non-negativity of τ, that the maximal contribution to X from the above three phases is obtained by taking

1. $\tau \equiv 0$, hence, $G_2 \equiv 0$, for sustained forward sliding,

2. $\tau \equiv -u_1 \equiv 0$, with no sustained backward sliding, and
3. $u_1 = u_{1,\min}$, with $|G_2| \geq |G_1|$ and sustained sticking,

where $u_{1,\min}$ denotes the minimal value of u_1, obtained by taking $G_1 = -G_{\max}$ in (16).

Starting from the initial state $u(0) = 0$, the maximum growth of U or u in the forward-sliding phase is attained by taking

$$G_1(t) \equiv G_{\max}, \quad \text{and} \quad G_2 \equiv 0, \tag{20}$$

Because of (14), the state (20) can be maintained during part of the period, say $0 \leq t \leq \theta < 2\pi$ so that

$$u(t) = G_{\max} t, \quad \text{for } 0 \leq t \leq \theta \tag{21}$$

Since, in the sticking phase, we have

$$|G_2(t)| \equiv |G_1(t)|, \quad \text{and } u(t) \equiv 0, \quad \text{for } \theta \leq t \leq 2\pi, \tag{22}$$

the total displacement in a cycle is given by

$$X = G_{\max} \frac{\theta^2}{2} \tag{23}$$

At this juncture, we must distinguish between open and closed conveyors.

For closed conveyors, G_2 is unrestricted, and the quantity in (23) is maximized by choosing θ to be maximal subject to the constraint (14) on G_1, which implies that

$$\begin{aligned} &G_1(t) \equiv -G_{\max} \quad \text{for } \theta \leq t \leq 2\pi, \quad \text{with} \\ &\theta = \left(1 - \frac{G}{G_{\max}} \sin\alpha\right)\pi \end{aligned} \tag{24}$$

The first identity in (22) and the constraint (14) on G_2 are then satisfied by taking $G_2(t) \equiv \pm G_{\max}$ for respective fractions $\lambda, 1-\lambda$ of the interval $(\theta, 2\pi)$, where

$$\lambda = \frac{1}{2} R\left(1 - \frac{2\mu G \cos\alpha}{G_{\max} + G \sin\alpha}\right) \tag{25}$$

with R denoting the ramp function:

$$R(u) = \begin{cases} u, & \text{for } u \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{26}$$

Figure 2 provides a qualitative sketch of a representative cycle of the optimal G_1, G_2, for the simplest case where G_2 undergoes a single change of sign.

The acceleration cycle in Fig. 2 corresponds to a conveyor displacement consisting of a triangular loop in space, one side of which represents retraction of the conveyor in the $-x$-direction, with frictionless ($N \equiv 0$) sliding of the stationary granular mass. The other two sides represent advancement of the conveyor at $\pm$ the static angle of repose $\arctan\mu$, involving a switching of the normal-force from top to bottom wall and forward transfer of the granular mass without relative sliding. Hence, the cycle involves either sliding without friction or friction without sliding, so that frictional dissipation is zero. In that respect, our kinematically ideal cycle is also ideal in a thermodynamic sense, since we achieve the titular frictionless conveying.

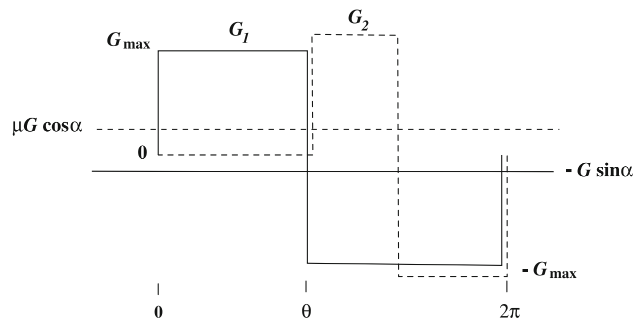

Fig. 2 Representative ideal cycle for the closed conveyor. *Solid* curve: G_1. *Dashed* curve (slightly shifted to avoid overlap): G_2

A further bit of reflection shows that the non-dimensional displacement (23) also corresponds to the magnitude of the x-displacement of the conveyor during the above retraction or advancement phases. Thus, if one identifies the amplitude A in (12) with this displacement, then $X := 1$, and (23)–(24) give

$$\theta = \frac{\sqrt{1 + 2\pi^2 G \sin\alpha} - 1}{\pi G \sin\alpha}, \quad \text{with } G_{\max} = 2/\theta^2 \tag{27}$$

The second equation serves to define the frequency ω in (12) in terms of a prescribed upper bound on $|g_x|$, and *vice versa*. In the zero-gravity limit $G \to 0$, the relation (27) gives $\theta = \pi$, along with antisymmetric form of G_1 in Fig. 2, whereas the limit $G \to \infty$ involves an impulsive conveyor acceleration, with $\theta \to 0$ and $G_{\max} \to G \sin\alpha$.

For open conveyors, $G_2 \geq 0$, and the relations (14), (21) and (22) are readily found to give the same expression as that in (24) for θ together with the further relations

$$\begin{aligned} &G_{\max} = G(2\mu\cos\alpha - \sin\alpha), \quad \text{and} \\ &X = 2\pi^2 G \frac{(\mu\cos\alpha - \sin\alpha)^2}{(2\mu\cos\alpha - \sin\alpha)} \end{aligned} \tag{28}$$

With μ and α as parameters, the first relation gives the ratio $|g_x|_{\max}/g_0$ of maximum channel acceleration to ambient gravity, whereas the second gives the ratio of optimal displacement to g_0/ω^2, irrespective of A and ω. The situation is depicted schematically in Fig. 3.

Once again, the ideal cycle involves transport without frictional dissipation. It is further evident from (28) that slide-conveying in an open conveyor is not possible in a zero-gravity environment.

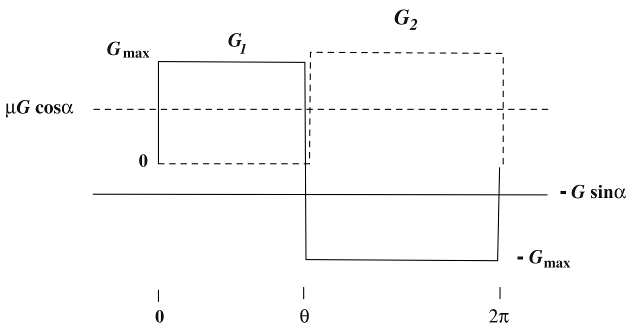

Fig. 3 Ideal cycle for the open conveyor, with $G_{\max}$ given by (28)

3.2 Elliptical cycles

For the cycle (3), one finds readily that

$$G_1 = -G \sin\alpha + \cos\beta \sin t,$$
$$G_2 = G\cos\alpha - \sin\beta \sin(t+\varphi),$$

hence

$$u_1 = -Gt\sin\alpha - \cos\beta\cos t, \tag{29}$$
$$\tau = \int_0^t |G\cos\alpha - \sin\beta\sin(t+\varphi)|dt$$

Then, by means of the general solution (17), one interesting optimization problem can be concisely stated as:

$$\max_{\beta,\varphi}\left\{\int_P (u_1-\tau)dt + \int_N (u_1+\tau)dt\right\}, \tag{30}$$

where N and P refer, respectively, to the union of t-intervals in which $\tau < \mp u_1$, corresponding to positive or negative sliding.

For open conveyors, it understood that β is restricted to angles such that G_2 is non-negative, which for $G\cos\alpha < 1$ is the interval $[0, \sin^{-1}(G\cos\alpha)]$. In that case, the absolute value signs can be removed from the integrand defining τ in (29), to yield an elementary function of t. Otherwise, the integral can be integrated piecewise, as suggested by previous works [7], leading to rather complex analytical forms. In any event, the solution to the optimization problem should be amenable to an easy numerical analysis, which will not be pursued here.

4 Conclusions

For the relative simple mode of slide conveying, the foregoing analysis provides closed-form solutions for certain elementary vibrational cycles, in which wall slip is assumed to represent the only relative motion between conveyor and transported material. This allows one to identify ideal cycles in which there is no dissipation due to sliding, a possibility that is not too surprising, given the alternative possibility of transport without slip between counter-rotating cylinders, as in calendering devices, or between parallel moving belts with sufficient normal pressure. At any rate, it is hoped that the present work will stimulate further investigation of optimal cycles for throw-conveying, which may involve collisional dissipation at conduit walls as well as in-flight instability of granular layers.

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