Micromorphic balances and source-flux duality

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Abstract. This is a further note on the (Guass-Maxwell) force-flux construct proposed previously (Goddard, J.D., A note on Eringen's moment balances, *Int. J. Eng. Sci.*, in the press, 2011). Motivated in part by its promise as a homogenization technique for constructing micromorphic continua, the present work is focused rather on some additional representations and on novel applications, such as the derivation of dissipative thermodynamic potentials from force-flux relations.

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INTRODUCTION

As one one approach to understanding the physical origins of a micromorphic continua, Eringen [1] assumes that micromorphic fields arise from moments over microdomains made up from a more primitive continuum. In a recently published paper, this author [2] gives a comparison of Eringen's [1] moment balances with P. Germain's [3] generalized momentum balances based on virtual work principles and also with those derived by a two-scale Fourier analysis of heterogeneous media, but it was not possible to establish a precise correspondence between any of the three methods. With no attempt to resolve that problem, the present note is focused on a source-flux construct introduced in [2][1]. This construct establishes a fairly direct connection between Newton's and Cauchy's laws and provides an expression for Cauchy stress suggested by the statistical mechanics of point-particles [4]. Furthermore, it appears to offer a useful mathematical foundation for the "peridynamics" of Stilling and coworkers [5].

Setting aside the relation of the source-flux construct to micromorphic continua, we elaborate further on this representation, considering a possible application to the derivation of dissipative potentials of the type given by Edelen [6, 7].

SOURCE-FLUX DUALITY

Path moments of density fields

As in [2], we generally employ lower-case bold Greek symbols for tensors of arbitrary rank, with the symbol $\triangleq$ indicating in brackets [ ] their components for arbitrary curvilinear coordinates, and we employ standard curvilinear-tensor notation. Then, given a density field $\boldsymbol{\varrho}_\alpha(t,\mathbf{x}) \triangleq [\varrho_\alpha^{i_1,i_2,\dots,i_n}(t,\mathbf{x})]$ of a rank-n tensor-valued quantity $\boldsymbol{\alpha} \triangleq [\alpha^{i_1,i_2,\dots,i_n}]$, and a continuous directed curve $\mathfrak{C}(\mathbf{a},\mathbf{b})$ running from $\mathbf{a}$ to $\mathbf{b}$, the path integral

$$\boldsymbol{\mu}_\alpha(t,\mathbf{x};\mathfrak{C}) = \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} \boldsymbol{\varrho}_\alpha(t,\mathbf{x}-\mathbf{z})\otimes d\mathbf{z}, \quad \left(i.e. \quad \mu_\alpha^{i_1,i_2,\dots,i_{n+1}}(t,\mathbf{x};\mathfrak{C}) = \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} \varrho_\alpha^{i_1,i_2,\dots,i_n}(t,\mathbf{x}-\mathbf{z})dz^{i_{n+1}}\right) \tag{1}$$

yields a rank-$(n+1)$ flux[2] as functional $\boldsymbol{\mu}_\alpha \triangleq [\mu_\alpha^{i_1,i_2,\dots,i_{n+1}}]$ on $\mathfrak{C}(\mathbf{0},\mathbf{y})$, which satisfies

$$\operatorname{div}\boldsymbol{\mu}_\alpha = \boldsymbol{\varrho}_\alpha(t,\mathbf{x}) - \boldsymbol{\varrho}_\alpha(t,\mathbf{x}-\mathbf{y}),\ \forall\mathfrak{C}(\mathbf{0},\mathbf{y}),\ \text{where } \operatorname{div}\boldsymbol{\mu} \triangleq [\mu^{i_1,i_2,\dots,i_n,i_{n+1}}{}_{;n+1}], \tag{2}$$

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[1] To make for a reasonably self-contained discussion, a number of formula from that work are repeated here.
[2] The term "flux" is used loosely to designate quantities that are analogous to Maxwell's dielectric displacement.

with semicolon in subscript representing a covariant derivative. The closed curve, with $\mathbf{y} = \mathbf{0}$, obviously gives a solenoidal field, and the representation (1) is evidently unique only up to an additive solenoidal field.

For the case of *spatially restricted* densities $\varrho_\alpha(t,\mathbf{x})$ that vanish at points $\mathbf{x}$ outside a finite material body (or otherwise have finite spatial support) $\mathfrak{B}$ [2], we may choose

$$\mathbf{y} \in \mathfrak{U} = \{\mathbf{u} : \varrho_\alpha(t, \mathbf{x} - \mathbf{u}) \equiv \mathbf{0}, \forall \mathbf{x} \in \mathfrak{B}\} \tag{3}$$

e.g. we could take $|\mathbf{y}| = \infty$. We then obtain the classical (Gauss-Maxwell) form

$$\operatorname{div} \boldsymbol{\mu}_\alpha(t,\mathbf{x}) = \varrho_\alpha(t,\mathbf{x}), \tag{4}$$

which represents the aforementioned duality between density ϱ and flux $\boldsymbol{\mu}$ in various continuum balances.

For example, the standard linear momentum balance becomes [2]:

$$\rho \dot{\mathbf{v}} = \varrho_f, \tag{5}$$

where $\rho = \rho(t,\mathbf{x})$ denotes mass density, $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$ material velocity, and $\varrho_f = \varrho_f(t,\mathbf{x})$ force density. Assuming further that $\varrho_f = \varrho_I + \varrho_E$, where ϱ_E is external body-force density and $\varrho_I(t,\mathbf{x})$ is internal force density arising from material interactions, we may make use of the above results to write:

$$\varrho_I = \operatorname{div} \boldsymbol{\tau}, \text{ with } \boldsymbol{\tau} = \boldsymbol{\tau}(t,\mathbf{x}) = \int_{\mathbf{z} \in \mathfrak{C}(\mathbf{0},\infty)} \varrho_I(t, \mathbf{x} - \mathbf{z}) \otimes d\mathbf{z}, \tag{6}$$

where $\boldsymbol{\tau}$ plays the rôle of a stress tensor.

As pointed out earlier [2], the representations (5)-(6) serve to blur the distinction between Newton's and Cauchy's laws that is routinely invoked in much of the continuum-mechanics literature.

It remains to establish the precise connection of the construct (1) to the classical Gauss-Maxwell relations, as employed in various theories of electricity, magnetism and gravitation. Without undertaking a comprehensive treatment, it seems plausible that one might define the the flux in the region outside a finite body $\mathfrak{B}$ as a solenoidal continuation of the flux given by (4), by means of appropriate matching on $\partial\mathfrak{B}$. However, some analog of potential theory would no doubt be required to achieve uniqueness.

Pairwise internal forces and localization of stress

With a further postulate of generally non-local pairwise interaction between distinct material points in a continuum [2], one may express the internal forces as:

$$\varrho_I(t,\mathbf{x}) = \int_{\mathbf{y} \in \mathfrak{B}} \mathfrak{f}(t,\mathbf{x},\mathbf{y}) dV(\mathbf{y}) \tag{7}$$

where $\mathfrak{f}(t,\mathbf{x},\mathbf{y}) = -\mathfrak{f}(t,\mathbf{y},\mathbf{x})$ is a *two-point force density*, with $\mathfrak{f}(t,\mathbf{x},\mathbf{y})dV(\mathbf{x})dV(\mathbf{y})$ representing the vector force on, or net momentum exchange rate into, $dV(\mathbf{x})$ emanating from $dV(\mathbf{y})$. As pointed out elsewhere [2], the representation (7) is implicit in certain continuum-mechanical axioms regarding internal forces [8] and explicit in the recent "peridynamics" of Silling and coworkers [5][3]. As pointed out in the same work [2], it is possible to obtain a localized form of (7) as the *force dipole*:

$$\boldsymbol{\tau}(t,\mathbf{x}) \approx \mathbf{T}(t,\mathbf{x}) = \int_{\mathbf{y} \in \mathfrak{B}_\epsilon} \mathfrak{f}(t,\mathbf{x},\mathbf{y}) \otimes (\mathbf{y} - \mathbf{x})\, dV(\mathbf{y}), \tag{8}$$

with relative error ϵ, where is a small parameter measuring the range of the internal force $\mathfrak{f}(t,\mathbf{x},\mathbf{y})$ and $\mathfrak{B}_\epsilon = \mathfrak{B}_\epsilon(\mathbf{x})$ is an ϵ-neighborhood of $\mathbf{x}$.

When symmetric, the tensor $\mathbf{T}$ in (8) represents Cauchy stress, and remains to be seen whether the higher-order stresses in micromorphic continua can be represented by higher-order moments of $\mathfrak{f}(t,\mathbf{x},\mathbf{y})$ and to establish the connection to micromorphic kinematics and virtual work principles.

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[3] See also `http://en.wikipedia.org/wiki/Peridynamics`.

ADDITIONAL FORMS AND APPLICATIONS

Straight-line paths

Given the underlying non-uniqueness, which involves an additive closed path representing a solenoidal field, we may choose the path $\mathfrak{C}$ in (1) to be the straight line, with $\mathbf{z} = r\mathbf{e}$, $0 \le r \le R = |\mathbf{y}|$, $d\mathbf{z} = \mathbf{e}dr$, such that

$$\boldsymbol{\mu}_\alpha(t,\mathbf{x}) = \left[\int_0^R \boldsymbol{\varrho}_\alpha(t,\mathbf{x}-r\mathbf{e})dr\right]\otimes\mathbf{e},\ \text{with } R = R(\mathbf{e}) \in \{r : \boldsymbol{\varrho}_\alpha(t,\mathbf{x}-r\mathbf{e}) = \mathbf{0}, \forall\mathbf{x}\in\mathfrak{B}\}, \tag{9}$$

where $\mathbf{e}$ is a unit vector and where, in the simplest version, we may take $R(\mathbf{e}) \equiv \infty$. Straight-line paths have been employed in related works [9, 10, 6], reflecting a certain recognition of the underlying non-uniqueness.

In the present context, the above result can be generalized to include the following linear combination of straight-line paths:

$$\boldsymbol{\mu}_\alpha(t,\mathbf{x}) = \int_\Omega\left[\int_0^{R(\mathbf{e})} \boldsymbol{\varrho}_\alpha(t,\mathbf{x}-r\mathbf{e})dr\right]\otimes\mathbf{e}w(\mathbf{e})d\Omega(\mathbf{e}),\ \text{with } \int_\Omega w(\mathbf{e})d\Omega(\mathbf{e}) = 1, \tag{10}$$

where $d\Omega(\mathbf{e})$ is the standard solid-angular measure on the unit sphere Ω, and $w(\mathbf{e})$ is a weighting function.

The special case of a discrete pair of straight-line paths is represented by the Dirac measure

$$w(\mathbf{e}) = y_1\delta(\mathbf{e}-\mathbf{e}_1) + y_2\delta(\mathbf{e}-\mathbf{e}_2),\ \text{with } y_1 + y_2 = 1, \tag{11}$$

which can be viewed as two straight-line portions of a closed path, part of which lies outside $\mathfrak{B}$.

Dissipative Potentials

The present construct has other possible applications, including an alternative approach to Edelen's [6, 7] dissipative thermodynamic potential. Thus, with $\mathbf{x}$ referring now to an element in the n-dimensional vector space of thermodynamic forces and $\boldsymbol{\varrho}(\mathbf{x})$ to an element in the dual space of fluxes, denoted, respectively, by the conventional symbols $\mathbf{X}, \mathbf{J}$ in [6, 7], we require that the moment in (1) reduce to the form

$$\boldsymbol{\mu} \mathrel{\hat{=}} [\mu^{ij}] = \Phi(\mathbf{x})\mathbf{I}, \quad \text{where} \quad \mathbf{I} \mathrel{\hat{=}} [g^{ij}],, \tag{12}$$

Φ is the scalar potential, and the idemfactor $\mathbf{I}$ is represented by the metric tensor g^{ij}. For a Euclidean space, this reduction is valid only if

$$\Phi(\mathbf{x}) = \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} \boldsymbol{\varrho}(\mathbf{x}-\mathbf{z})\cdot d\mathbf{z} := \int_{\mathbf{z}\in\mathfrak{C}(\mathbf{0},\mathbf{y})} \varrho_j(\mathbf{x}-\mathbf{z})dz^j, \tag{13}$$

However, the integral will be independent of path $\mathfrak{C}(\mathbf{0},\mathbf{y})$ if and only if $\varrho_{i;j} = \varrho_{j;i}$, which is the symmetry condition given as Eq. (3.4) of Edelen [7][4] . It then follows from the n-dimensional generalization of (2) that

$$\mathrm{grad}\Phi := \partial\Phi(\mathbf{x})/\partial\mathbf{x} = \boldsymbol{\varrho}(\mathbf{x}) - \boldsymbol{\varrho}(\mathbf{x}-\mathbf{y}) \tag{14}$$

As opposed to the general form (1), we may now take $\mathbf{y} = \mathbf{x}$ to give $\boldsymbol{\varrho}$ as the gradient of Φ, provided $\boldsymbol{\varrho}(\mathbf{0}) = \mathbf{0}$. Moreover, (13) can be reduced to an integral over a straight-line path giving Edelen's [6, 7] result:

$$\Phi(\mathbf{x}) = \int_0^1 \mathbf{x}\cdot\boldsymbol{\varrho}(\tau\mathbf{x})d\tau = \int_0^1 x^j\varrho_j(\tau\mathbf{x})d\tau \tag{15}$$

[4] Although not explored here, that symmetry would seem to imply that the thermodynamically orthogonal flux denoted by $\mathbf{U}$ in [6] must be identically zero.

CONCLUSIONS

As pointed out in the previous article [2], further work is needed to reconcile the different micromorphic models obtained from various techniques based on moments, an issue not addressed in the present brief note. Suffice it to say that the proposed force-flux construct may be useful for clarifying the above question, by providing expressions for the moments associated with micromorphic continua [2], or by developing fully non-local theories in fields such as rarefied gas dynamics and radiation dynamics, as envisaged by the “peridynamics" of Silling and coworkers [5].

As illustrated by the present note, the same construct may have other applications, such as providing an alternative route to Edelen’s derivation of dissipative thermodynamic potentials.

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