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Cite this article: Goddard J, Kamrin K. 2019 Dissipation potentials from elastic collapse. *Proc. R. Soc. A* **475**: 20190144.
http://dx.doi.org/10.1098/rspa.2019.0144

Received: 7 March 2019
Accepted: 9 May 2019

Subject Areas:
mechanics

Keywords:
dissipative systems, nonlinear Onsager symmetry, elastic failure, energy landscape

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Dissipation potentials from elastic collapse

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Generalizing Maxwell's (Maxwell 1867 IV. *Phil. Trans. R. Soc. Lond.* **157**, 49–88 (doi:10.1098/rstl.1867.0004)) classical formula, this paper shows how the dissipation potentials for a dissipative system can be derived from the elastic potential of an elastic system undergoing continual failure and recovery. Hence, stored elastic energy gives way to dissipated elastic energy. This continuum-level response is attributed broadly to dissipative microscopic transitions over a multi-well potential energy landscape of a type studied in several previous works, dating from Prandtl's (Prandtl 1928 Ein Gedankenmodell zur kinetischen Theorie der festen Körper. *ZAMM* **8**, 85–106) model of plasticity. Such transitions are assumed to take place on a characteristic time scale T, with a nonlinear viscous response that becomes a plastic response for $T \to 0$. We consider both discrete mechanical systems and their continuum mechanical analogues, showing how the Reiner–Rivlin fluid arises from nonlinear isotropic elasticity. A brief discussion is given in the conclusions of the possible extensions to other dissipative processes.

1. Introduction

As a generalization of the Rayleigh dissipation function, the concept of dissipation potential goes back at least to the plastic potential of von Mises [1], with later extensions to viscoplasticity, variational principles and convex duality by Hill [2,3] and Moreau [4]. The subsequent and largely overlooked work of Edelen [5,6] establishes rigorously the existence of dissipation potentials for strictly dissipative systems, thereby justifying phenomenological treatments like those cited above [7]. However, unlike elastic potentials that are equivalent to stored elastic energy, dissipation potentials,

dubbed 'pseudo-potentials' by Maugin [8], are not in general equal to rates of dissipation and only become proportional to those rates in the case of homogeneous potentials [7].

Thus we are led to search for a deeper energetic significance, and the object of the following article is to show how dissipation potentials can be related to the dissipation arising from an idealized form of continual elastic failure and recovery. We employ the term 'collapse' to denote solid-like failure or fracture and also to denote fluid-like relaxation from some transient elastic state.

We first consider a discrete system and then extend the results to continuum mechanics. While our treatment can be viewed as mechanical in nature, it can with slight modification be extended to other dissipative processes such as networks of chemical reactions accompanied by diffusive transport of mass and heat [9].

The present work represents a nonlinear generalization of the classic formula of Maxwell [10] for relaxing elasticity, with viscosity given as the product of elastic modulus and relaxation time, as discussed further in the following. While Maxwell considers plastic deformation of solids, his formula for linear systems does not apply to the limit of vanishing relaxation time with rate-independent plastic response, a lacuna remedied by the present work.

2. Discrete systems

We consider here systems described by generalized velocities $\mathfrak{v} = [\mathfrak{v}_i]$ and conjugate forces $\mathfrak{f} = [\mathfrak{f}^i]$ with $i = 1, \ldots n$, which represent dual spaces equivalent to $\mathbb{R}^n$. As in the following, we indicate components of vectors and tensors in brackets [ ] (brackets which are also used below to denote the arguments of linear functions). We employ Fraktur font for abstract vectors and forces and Roman for vectors and tensors in $\mathbb{R}^3$, with boldface for these quantities and lowercase for components.

Thus, given any past history of velocity or force the current values are given by generally invertible relations $\mathfrak{f}(\mathfrak{v})$ and $\mathfrak{v}(\mathfrak{f})$ with dissipation

$$\mathfrak{D} = \mathfrak{f} \cdot \mathfrak{v} = \mathfrak{v} \cdot \mathfrak{f} = \mathfrak{f}^i \mathfrak{v}_i = D(\mathfrak{v}) = D^*(\mathfrak{f}) \geq 0, \quad \text{for } |\mathfrak{f}||\mathfrak{v}| > 0, \tag{2.1}$$

where $|\ |$ denotes a suitable norm and the dot product represents pairing or 'scalar product' and sums are taken over repeated indices. One of the present authors [7] has presented an alternative to Edelen's derivation of the formula connecting dissipation potential to dissipation rate.[1] If (2.1) is restricted to strict inequality, we obtain a 'strongly dissipative' or 'hyperdissipative' system in which force and fluxes are given by complementary dissipation potentials,

$$\mathfrak{f} = \partial_{\mathfrak{v}} \psi_{\mathrm{d}}(\mathfrak{v}) \quad \text{and} \quad \mathfrak{v} = \partial_{\mathfrak{f}} \varphi_{\mathrm{d}}(\mathfrak{f}), \quad (\text{i.e. } \mathfrak{f}^i = \partial_{\mathfrak{v}_i} \psi_{\mathrm{d}} \text{ and } \mathfrak{v}_i = \partial_{\mathfrak{f}^i} \psi_{\mathrm{d}}), \tag{2.2}$$

as perfect counterparts to the potentials or free energies of elastic systems. The corresponding Maxwell-type relations between cross derivatives represent Edelen's *nonlinear Onsager symmetry*. Dissipative systems of this type have received attention in several past and recent studies, e.g. [3,9,12].

We distinguish at the outset our macroscopic description of applied kinematics or forces from a detailed microscopic description of dynamics based on specific micromechanical models such as those considered in previous models of plasticity, fracture and viscous flow, e.g. [13–19]. Thus, the specification of the generalized velocity $\mathfrak{v}$ or conjugate force $\mathfrak{f}$ represent, respectively, kinematically 'hard' or 'soft' control of the microscopic motion, and we begin by considering the former.

We note that certain previous models invoke viscous or other forms of dissipation to describe the eventual microscopic loss of elastic energy. However, from the present vantage point, this merely reflects the collapse of elasticity on yet smaller time scales. In other words, we propose to attribute *all* dissipative processes to the relaxation and thermalization of elastic energy on

[1] A key relation, eqn (9) of [7] is cited by Germain [11] (ch. 7, eqn (25)) but seems unduly restricted to homogeneous or approximately homogeneous potentials. The alternative forms for the potentials that are subsequently proposed by Germain represent Ziegler's 'normality' rule which is only valid for homogeneous potentials [7].

sufficiently small time scales Furthermore, this applies to the entropic elasticity of ideal gases or ideal rubbery solids, where finite rates of deformation lead inevitably to viscous dissipation.

(a) Velocity potential

We consider a generalized displacement represented by $\mathfrak{u} = [u_i] \in \mathbb{R}^n$ possessing a conjugate elastic force $\mathfrak{f}_e(t) = \partial_{\mathfrak{u}} \psi_e(\mathfrak{h}, \mathfrak{u}) \in \mathbb{R}^n$ given in terms of an elastic potential $\psi_e(\mathfrak{h}, \mathfrak{u})$ that is equal to stored elastic energy. Here, $\mathfrak{u} = \mathfrak{u}(t)$ and $\{\mathfrak{v}(t') : t' \leq t\}$ represents the history of generalized velocity $\mathfrak{v} = \dot{\mathfrak{u}} = \mathrm{d}\mathfrak{u}/\mathrm{d}t$, and, in the following, we decompose this history into the present value $\mathfrak{v}(t)$ and the *relative history* $\mathfrak{h} = \{\mathfrak{v}_t(t') : t' < T\}$ with $t < T$, where $\mathfrak{v}_t(t') = \mathfrak{v}(t') - \mathfrak{v}(t)$ and $T > 0$ is an upper limit to be specified. Also, we shall employ the symbol $\mathfrak{h} = \mathfrak{h}(t)$ to denote the latter and, in a certain abuse of terminology, refer to it as 'the history'. We represent functionals depending on $\mathfrak{h}$ by the notation employed for ordinary functions. This translational decomposition and the subsequent decomposition of Fréchet derivatives is analogous to the multiplicative decomposition of Coleman [20], which is adopted in our treatment of continuum mechanics in §3. In the following, we shall treat the history $\mathfrak{h} = \mathfrak{h}(t)$ as the union of a fixed past history $\mathfrak{h}_0 = \mathfrak{h}(t)$ for $t < 0$ and the history $\mathfrak{h} \backslash \mathfrak{h}_0$ for $0 < t < T$, where we employ standard notation $\backslash$ for set difference.

The dependence of elasticity on $\mathfrak{h}_0$ serves to represent microscopic processes leading to inelastic stiffening or fracture that modifies subsequent elastic response. In many instances, the microscopic processes can be described by thermo-statistical transitions over a non-convex energy landscape represented by a multi-well potential. This topography involves multiple values of $\mathfrak{u}$ at which ψ_e is a local minimum representing states of repose with vanishing force $\mathfrak{f}_e = \mathbf{0}$ [13,17–19,21–24]. As in most of those studies, we restrict our analysis to isothermal systems. Otherwise, we could in principle account for temperature changes by means of an appropriate energy balance with dissipation included and with changes in temperature serving to modify the history $\mathfrak{h}$.

The existence of a dissipation potential $\psi_d(\mathfrak{h}, \mathfrak{v})$ with dissipative force $\mathfrak{f}_d = \partial_{\mathfrak{v}} \psi_d(\mathfrak{h}, \mathfrak{v})$ is guaranteed for strongly dissipative systems, and we propose to construct $\psi_d(\mathfrak{h}, \mathfrak{v})$ from the above elastic potential. Thus, with instantaneous storage of elastic energy $\mathfrak{f}_e \cdot \mathfrak{v}$ equal to the instantaneous rate of dissipation $\mathfrak{f}_d \cdot \mathfrak{v}$ for all velocities $\mathfrak{v}$, we obtain a strictly dissipative system with dissipative force equal to elastic force:

$$\mathfrak{f}_d = \partial_{\mathfrak{v}} \psi_d(\mathfrak{h}, \mathfrak{v}) = \partial_{\mathfrak{u}} \psi_e(\mathfrak{h}, \mathfrak{u}) = \mathfrak{f}_e, \tag{2.3}$$

where partial derivatives are taken at constant $\mathfrak{h}(t)$ and the displacement $\mathfrak{u}$ in (2.3) is a functional of $\mathfrak{h}$ and a function of the present velocity $\mathfrak{v}(t)$ given by

$$\left.\begin{aligned} \mathfrak{u}(t) = \mathfrak{u}(\mathfrak{h}, \mathfrak{v}, t) = \int_0^t \mathfrak{v}_t(t')\,\mathrm{d}t' + \mathfrak{u}_0 + t\mathfrak{v}(t), \\ \text{with } \mathfrak{u}(T) = \mathfrak{x}(\mathfrak{h}, \mathfrak{v}, T) = \mathfrak{w} + T\mathfrak{v}(t), \\ \text{where } \mathfrak{w} = \mathfrak{w}(\mathfrak{h}) = \mathfrak{u}_0 + \int_0^T \mathfrak{v}_t(t')\,\mathrm{d}t' \text{ and } 0 \leq t \leq T, \end{aligned}\right\} \tag{2.4}$$

Here, $\mathfrak{u}_0(\mathfrak{h}_0)$ is an initial displacement and $T = T(\mathfrak{h}_0)$ a 'relaxation' time arising from microscopic dynamics, both to be discussed below.

We note in passing the relation to the Clausius–Duhem inequality for isothermal processes:

$$\dot{\psi} = \mathfrak{f}_e \cdot \mathfrak{v} - \mathfrak{D}, \quad \text{where } \mathfrak{f}_e = \partial_{\mathfrak{u}} \psi_e, \quad \text{and} \quad \mathfrak{D} \geq 0, \tag{2.5}$$

$\mathfrak{D}$ denoting dissipation rate and ψ Helmholtz free energy. On the one hand, when $\mathfrak{D} \equiv 0$ we obtain $\dot{\psi} = \dot{\psi}_e = \mathfrak{v} \cdot \partial_{\mathfrak{u}} \psi_e$, whereas when $\mathfrak{D} \equiv \mathfrak{v} \cdot \partial_{\mathfrak{v}} \psi_d$ the Helmholtz free energy does not change. Therefore, we may henceforth replace the corresponding elastic energy at given $\mathfrak{u}$ by dissipated energy.

By a translation of time and displacement, we have replaced time t by relative time $t - t_0$, where t_0 is the time at which $\mathfrak{u} = \mathfrak{u}_0$ and $\mathfrak{u}_0$ is displacement from a previous equilibrium state. It is evident that the second equation of (2.4) represents the decomposition of the displacement $\mathfrak{u}$ into a displacement resulting from a constant velocity $\mathfrak{v}$ equal to the present value $\mathfrak{v}(t)$ plus an integral representing history $\mathfrak{h}\backslash\mathfrak{h}_0$.

The approximate solution proposed in the following section depends on a quasi-steady approximation for $\mathfrak{v}$ that requires the smallness of some norm of the history $\mathfrak{v}_t(t')$, $t' < t$.

In particular, for sufficiently smooth $\mathfrak{v}(t')$, we may substitute the Taylor series expansion about $t' = t$, with superscript (n) on $\mathfrak{v}(t)$ denoting the nth time derivative,

$$\mathfrak{v}_t(t') = (t' - t)\mathfrak{v}^{(1)}(t) + \frac{1}{2!}(t' - t)^2\mathfrak{v}^{(2)}(t) + \cdots \tag{2.6}$$

into the formula (2.4) for $\mathfrak{w}$ to obtain the series

$$\mathfrak{w} = \mathfrak{u}_0 + \frac{(T - t)^2 - (-t)^2}{2!}\mathfrak{v}^{(1)} + \frac{(T - t)^3 - (-t)^3}{3!}\mathfrak{v}^{(2)} + \cdots, \tag{2.7}$$

in which the coefficient of $\mathfrak{v}^{(n)}$ lies in $(-T^{n+1}, T^{n+1})/(n + 1)!$ whenever $t \in (0, T)$. Hence, we have the result

$$\mathfrak{u}(T) = \mathfrak{u}_0 + T\mathfrak{v}(t) + O(T^m|\mathfrak{v}^{(m)}|), \tag{2.8}$$

where m is the value of $n = 2, 3, \ldots$, if any, for which $T^n|\mathfrak{v}^{(n)}|/n!$ is largest. More generally, it is easy to show that the condition

$$\left|\int_0^T \mathfrak{v}_t(t')\,\mathrm{d}t'\right| \ll T|\mathfrak{v}(t)|$$

is equivalent for continuous $\mathfrak{v}(t)$ to the condition

$$\left|\overline{\mathfrak{v}} - \mathfrak{v}(t)\right| \ll |\mathfrak{v}(t)|, \quad \text{where } \overline{\mathfrak{v}} = \overline{\mathfrak{v}}(T) = \frac{1}{T}\int_0^T \mathfrak{v}(t)\,\mathrm{d}t. \tag{2.9}$$

We shall have occasion in the following to employ the approximation based on the neglect of the integral in the last equation of (2.4). Note that it fails for $\mathfrak{v}(t) \equiv \mathbf{0}$ and becomes exact for constant $\mathfrak{v}(t) \neq \mathbf{0}$.

(b) General formula and approximation

It follows from the second relation in (2.4) that $\partial_{\mathfrak{v}}\mathfrak{u}(T)|_{\mathfrak{h}} = \partial_{\mathfrak{v}}\mathfrak{x}(\mathfrak{h}, \mathfrak{v}, T)|_{\mathfrak{h}} = T\mathbf{I}$, where $\mathbf{I}$ is the idemfactor. Hence, we provisionally take

$$\psi_{\mathrm{d}}(\mathfrak{h}, \mathfrak{v}) = \frac{1}{T(\mathfrak{h}_0)}[\psi_{\mathrm{e}}(\mathfrak{h}, \mathfrak{u}(T)) - \psi_{\mathrm{e}}(\mathfrak{h}_0, \mathfrak{u}_0)], \tag{2.10}$$

where $\mathfrak{u}(T) = \mathfrak{x}(\mathfrak{h}, \mathfrak{v}, T)$ is given by the second relation in (2.4), with $\mathfrak{w} = \mathfrak{u}(T)|_{\mathfrak{v}=\mathbf{0}}$. Then, with the relation (2.10) interpreted as the mean rate of elastic energy dissipated in time period T, we obtain (2.3) upon application of the chain rule with $\mathfrak{h}$ held constant.

Hence, with the previously assumed equality of elastic and dissipative forces we have

$$\mathfrak{f}_{\mathrm{d}}(\mathfrak{h}, \mathfrak{v}) = \mathfrak{f}_{\mathrm{e}}(\mathfrak{h}, \mathfrak{w} + T\mathfrak{v}), \quad \text{where } \mathfrak{v} = \mathfrak{v}(t). \tag{2.11}$$

The condition $\mathfrak{v} \cdot \mathfrak{f}_{\mathrm{d}} = \mathfrak{v} \cdot \mathfrak{f}_{\mathrm{e}}(\mathfrak{w} + T\mathfrak{v}) > 0$, for $|\mathfrak{v}| > 0$, is necessary for dissipativity and convexity of ψ_{d} with respect to $\mathfrak{v}$, and this is guaranteed for $\mathfrak{w} = \mathbf{0}$ in any region where the elastic potential ψ_{e} is a convex function of $\mathfrak{u}$. To explore the further ramifications, we write

$$\mathfrak{v} \cdot \mathfrak{f}_{\mathrm{e}}(\mathfrak{h}, \mathfrak{w} + T\mathfrak{v}) \equiv \frac{1}{T}[(\mathfrak{w} + T\mathfrak{v}) \cdot \mathfrak{f}_{\mathrm{e}}(\mathfrak{h}, \mathfrak{w} + T\mathfrak{v}) - \mathfrak{w} \cdot \mathfrak{f}_{\mathrm{e}}(\mathfrak{h}, \mathfrak{w} + T\mathfrak{v})] > 0. \tag{2.12}$$

In regions where ψ_{e} is convex, the first term on the right side of this relations is positive, and convexity of ψ_{d} requires that $\mathfrak{w} \cdot \mathfrak{f}_{\mathrm{e}}(\mathfrak{h}, \mathfrak{w} + T\mathfrak{v}) \leq 0$ for $T \neq 0$. If this is to be satisfied for arbitrary $\mathfrak{v}$ for $T \neq 0$, we must take $\mathfrak{w} = \mathbf{0}$. However, inspection of the definition of $\mathfrak{w}$ in (2.4), where $\mathfrak{u}_0$ depends on $\mathfrak{h}_0$ and the integral depends on $\mathfrak{h}(t)\backslash\mathfrak{h}_0$, we are required to take both these terms to be

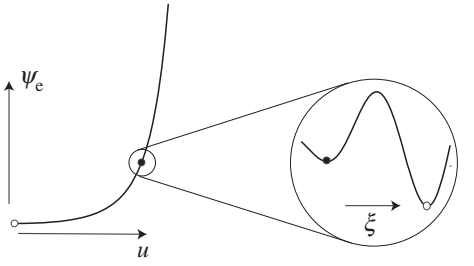

Figure 1. Macroscopic elastic potential on the left, with corresponding microscopic energy landscape on the right.

equal to zero. In general, the integral is not equal to zero but, according to the analysis given above is $O(T^m)$ for some $m > 1$ or is otherwise negligible compared to $T\mathfrak{v}(t)$. With this approximation, (2.10) reduces to the approximate form

$$\psi_d(\mathfrak{h}, \mathfrak{v}) = \frac{\psi_e(\mathfrak{h}, T\mathfrak{v}(t)) - \psi_e(\mathfrak{h}_0, \mathbf{0})}{T}, \quad \text{for } T \neq 0, \tag{2.13}$$

where $T = T(\mathfrak{h}_0)$ represents the time to achieve complete elastic failure and a new relative minimum of ψ_e, with $\mathfrak{f} = \mathfrak{f}_e = \mathbf{0}$. Note that this approximation provides an exact solution for constant $\mathfrak{v}(t)$ and is also valid for $\mathfrak{v} = \mathbf{0}$. Without loss of generality, we can set $\psi_e(\mathfrak{h}, \mathbf{0}) = 0$ in order to simplify (2.13). It is clear from the resulting expression that the convexity of ψ_d is determined by that of ψ_e.

(c) Graphic representations

Figure 1 provides a one-dimensional conceptual view of a convex macroscopic elastic potential $\psi_e(\mathfrak{u})$ together with an associated microscopic potential energy landscape $\hat{\psi}_e(\boldsymbol{\xi})$, represented by a non-convex double-well potential. Here, $\boldsymbol{\xi}$ is an associated microscopic coordinate specifying position on the microscopic potential energy landscape and describing relaxation from the high energy state via transition over the (activation) energy barrier separating the hypothetical elastic state at $\mathfrak{u} = \mathfrak{u}(T) \approx \mathfrak{u}_0 + T\mathfrak{v}$ from the relaxed state at $\mathfrak{u}_0$. The actual final state is represented by the open circle and corresponds to complete dissipation of the external work. It should be emphasized that the curve on the right represents a microscopic trajectory of the system only up to a point of (spinodal) instability, from which the detailed system dynamics governs the actual transition to a final stable state, as is common with bifurcations from statically unstable states in various dynamical systems.

Also, because the elastic energy $\psi_e(\mathfrak{u}(T)) - \psi_e(\mathfrak{u}_0)$ is completely released, the curve on the left should be regarded as dissipated rather than stored elastic energy. The equality of left- and right-hand energy levels is tantamount to the Hill–Mandel condition for homogenization [25] (concept of a continuum). However, the connection between $\mathfrak{u}$ and $\boldsymbol{\xi}$ depends ultimately on the details of a homogenization that presumably would give the former in terms of an appropriate average of the latter, which represents a much smaller microscopic displacement scale (or strain scale in the continuum models considered below).

In the case, where elastic failure or 'rupture' has no influence on the subsequent elastic properties, representing complete 'healing', the curves in figure 1 are invariant. If, in addition, the velocity $\mathfrak{v}$ is independent of time, the state represented by the solid dot is also invariant with $\mathfrak{u} = \mathfrak{v}T$, whereas otherwise the dot moves along the left-hand curve, with a possibly altered right-hand curve. On the other hand, if elastic failure influences the subsequent elastic response, e.g. by partial healing, the curves change with history $\mathfrak{h}$ and, hence, with time t. Moreover, the left-hand curve may itself be non-convex or evolve with loading history to a non-convex form which would engender a non-convex dissipation potential and the associated dissipative instabilities.

We note that lateral expansion of the left-hand curve implies a decrease of elastic energy for given $\mathbf{u}$, with elastic 'softening' that can be attributed to dissipation. On the other hand, while a lateral contraction implies an increase of elastic energy associated with elastic 'stiffening' (or strain hardening in the continuum models discussed below) this augmented elastic energy is non-recoverable, since the response from any elastic state is completely dissipative according to the present model. Accordingly, we should interpret ψ in (2.5) as isothermally recoverable elastic energy.

In a more general and precise way, we may envisage a prolongation of the curve on the right-hand side of figure 1 to represent an extended, 'bumpy' hyper-surface representing the microscopic energy landscape that underlies the smooth macroscopic surface depicted by the left-hand curve. We note that Puglisi & Truskinovsky [17, fig. 9] portray the combination of these curves as a single bumpy curve for a small system with a small number ($N = 7$) of linearly connected mechanical elements. Thus, their treatment appears to involve stored and perhaps mechanically non-recoverable elastic energy that may serve to represent plastic strain hardening.

In the present treatment, a point on the left-hand curve can be regarded as the homogenization of the 'representative volume element' (RVE) represented by the right-hand curve for large N. Hence, $T(\mathfrak{h})$ represents the mean time for transition from some unstable point elastic state to a new relaxed state on the microscopic energy landscape. As indicated by previous studies, the path taken near some particular 'col' or saddle-point will depend on the system dynamics.

We recall that Freund, Suzuki and colleagues [18,19,26] have proposed a Kramers-type [21] thermally activated transition over a potential barrier associated with rupture of chemical bonds. Whereas a quantum mechanical transition, envisioned, e.g., by classical transition-state theory [22], seems more appropriate for the breakage of chemical bonds, thermal transition provides a plausible model for processes such rupture of Van der Waals bonds, the visco-plastic yield and flow of glasses and non-Newtonian fluids [23,24,27], and the unfolding of proteins [19] discussed below.

(d) Rate-independent (plastic) limit

In the limit $T = 0$, we no longer require $\mathfrak{w} \equiv \mathbf{0}$ to satisfy dissipativity, and (2.10) reduces in the limit (by l'Hôpital's rule) to

$$\begin{aligned}\lim_{T\to 0} \psi_{\mathrm{d}} &= \lim_{T\to 0} \frac{1}{T}[\psi_{\mathrm{e}}(\mathfrak{h}, \mathbf{u}(T)) - \psi_{\mathrm{e}}(\mathfrak{h}_0, \mathbf{u}_0)] \\ &= \lim_{T\to 0} \frac{1}{T}[\psi_{\mathrm{e}}(\mathfrak{h}, \mathbf{u}_0 + T\mathfrak{v}) - \psi_{\mathrm{e}}(\mathfrak{h}_0, \mathbf{u}_0)] \\ &= \mathfrak{v} \cdot \partial_{\mathbf{u}} \psi_{\mathrm{e}}|_{\mathbf{u}=\mathbf{u}_0} = \mathfrak{v} \cdot \mathfrak{f}_0, \quad \text{where } \mathfrak{f}_0 = \mathfrak{f}_{\mathrm{e}}|_{\mathbf{u}=\mathbf{u}_0},\end{aligned} \tag{2.14}$$

which represents a rate-independent force $\mathfrak{f}_{\mathrm{d}} = \mathfrak{f}_0$ like that predicted by Prandtl's [13] classical one-dimensional 'toy' model of rate-independent plasticity and the recent elaboration by Puglisi & Truskinovsky [17]. To obtain a proper plasticity model for higher dimensions, we must assume that the director $\mathbf{u}_0/|\mathbf{u}_0|$ is given in terms of $\mathfrak{v}/|\mathfrak{v}|$, with the simplest model $\mathbf{u}_0/|\mathbf{u}_0| = \mathfrak{v}/|\mathfrak{v}|$ representing a kind of material isotropy. Although this assumption introduces a dependence on velocity on the right side of (2.10), it is readily verified that this does not contribute to the dissipation, even though the assumption is not necessary for $T > 0$.

Note that the limit $T = 0$ is represented in figure 1 by a lateral collapse of the microscopic energy landscape, with an abrupt drop-off from the high-energy to the low-energy state. Note also that this completely dissipative limit is to be distinguished from the standard (von Mises) elastoplasticity or fracture mechanics [14], which involve elastic effects in both inelastic loading and elastic unloading from a limiting elastic locus of yield or damage. Finally, note that $T(h)$ is the sole quantity that must be specified by microscopic dynamics when $T > 0$, whereas in the limiting case $T(h) = 0$ it is $\mathbf{u}_0(h)$ that must be specified by these dynamics. We further note that $\mathbf{u}_0$ now represents a point of instability rather than a relaxed state.

As a further important point, we note that the transition between elastic and dissipative states may often involve a sort of phase transition [14]. However, viscoelastic fluids may exhibit a transient elastic response on short time scales which eventually gives way to viscous response. Thus, it is worth recalling in the present notation that the celebrated viscoelastic model of Maxwell [10][2]

$$\frac{\mathrm{d}\mathfrak{f}}{\mathrm{d}t}+\frac{\mathfrak{f}}{T}=C\frac{\mathrm{d}\mathfrak{u}}{\mathrm{d}t} \tag{2.15}$$

is fluid-like in character, exhibiting elastic response with elastic (spring) constant C for rapidly varying $\mathfrak{u}(t)$ with finite relaxation time T but exhibiting viscous response with coefficient CT for quasi-steady motions or for small T. One sees immediately that the limit $T\to 0$ with CT constant gives a viscous fluid without restriction on the magnitude of $\mathfrak{v}=\mathrm{d}\mathfrak{u}/\mathrm{d}t$ but that it represents a singular perturbation by neglecting $T\,\mathrm{d}\mathfrak{f}/\mathrm{d}t$. The same is true of the nonlinear extension of (2.15) represented by the present work, in which $C\mathfrak{v}$ is in effect replaced by a nonlinear term $\partial_{\mathfrak{u}}\psi_{\mathrm{e}}(T\mathfrak{v})/T$ with a non-quadratic elastic potential.

In closing this subsection, we note that one can envisage higher-order approximations in T with $\psi_{\mathrm{d}}=\psi_{\mathrm{d}}(\mathfrak{v},T\dot{\mathfrak{v}},T^2\ddot{\mathfrak{v}},\ldots)$ in (2.10), with dots denoting time derivatives, and with corresponding approximation in the continuum models discussed below in §3. To the extent that such approximations involve a (poly)convex function of its arguments that represent strictly dissipative response, the subsequent expansions of ψ_{d} in powers of T might serve to justify the current model as a quasi-steady approximation. However, this representation should be distinguished from certain 'retarded-motion' or 'slow-motion' approximations of continuum viscoelasticity [28]. In particular, we recall that the latter may lead to models with rest-state instability [29], which we believe might plausibly be attributed to lack of convexity at a given order in T of such approximations.

(e) Force potential

Based on the above analysis and the Legendre–Fenchel dual (complementary) elastic potential $\varphi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f})$ of $\psi_{\mathrm{e}}(\mathfrak{h},\mathfrak{u})$, which is assumed to satisfy

$$\varphi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f})+\psi_{\mathrm{e}}(\mathfrak{h},\mathfrak{u})=\mathfrak{f}\cdot\mathfrak{u},\quad\text{with }\mathfrak{u}=\partial_{\mathfrak{f}}\varphi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f}), \tag{2.16}$$

where $\mathfrak{h}^*=\{\mathfrak{f}(t'):t'<t\}$ denotes the past history of force, we can now make use of (2.13) and (2.16) to obtain the dual dissipation potential

$$\begin{aligned}&\varphi_{\mathrm{d}}(\mathfrak{h}^*,\mathfrak{f})=\varphi_{\mathrm{d}}(\mathfrak{h}^*,\mathfrak{f}(t),t)=\mathfrak{f}\cdot\mathfrak{v}-\psi_{\mathrm{d}}(\mathfrak{h},\mathfrak{v})=\frac{1}{T}\left\{\varphi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f})-\mathfrak{f}\cdot\left[\int_0^T\mathfrak{v}_T(t')\,\mathrm{d}t'+\mathfrak{u}_0\right]\right\},\\&\text{with }\mathfrak{v}(t')=\partial_{\mathfrak{f}}\varphi_{\mathrm{d}}(\mathfrak{h}^*,\mathfrak{f}(t')),\quad T=T(\mathfrak{h}^*),\quad\text{and}\quad\mathfrak{u}_0=\partial_{\mathfrak{f}}\phi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f})|_{t=0},\end{aligned} \tag{2.17}$$

where we have once more taken $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{f}$ to be the same for both the dissipative system and its elastic progenitor. The integral in (2.17) is a functional of $\mathfrak{h}^*$ as indicated by the fourth equation of (2.17), and it is easy to show that $\mathfrak{v}=\partial_{\mathfrak{f}}\varphi_{\mathrm{d}}|_{\mathfrak{h}^*}$. Thus, (2.17) provides the dissipative analogue of complementary elastic energy, the subject of a recent comprehensive review of Cappechi *et al.* [30].

It is further easy to show that the limiting behaviour for small T is

$$\varphi_{\mathrm{d}}(\mathfrak{f},t)\sim\frac{\varphi_{\mathrm{e}}-\mathfrak{f}\cdot\mathfrak{u}_0}{T}=\frac{\varphi_{\mathrm{e}}-\mathfrak{f}\cdot\partial_{\mathfrak{f}}\varphi_{\mathrm{e}}|_{t=0}}{T}\quad\text{for }T\to 0. \tag{2.18}$$

To avoid a singularity in φ_{d} we must choose $\mathfrak{f}\cdot\mathfrak{u}_0=\varphi_{\mathrm{e}}(\mathfrak{h}^*,\mathfrak{f})$, representing a restriction on $\mathfrak{f}$ for given $\mathfrak{u}_0$, and vice versa. This restriction serves to define an elastic limit or locus of instability in force or displacement.

[2]While ostensibly motivated by the kinetic theory of gases, Maxwell's digresses to the viscoelastic and plastic behaviour of solids.

3. Continuum mechanics

As an important example of continuum models, consider an isothermal non-polar elastic medium with specific elastic strain energy (Helmholtz free energy) $\psi_e(\mathbf{F},\mathbf{x})$, where $\mathbf{F}(\mathbf{x}) = \partial\mathbf{x}/\partial\mathbf{x}^\circ$ is the gradient of the deformation from reference position $\mathbf{x}^\circ \in V^\circ \subseteq \mathbb{R}^3$ in a stress-free reference configuration to the current position $\mathbf{x}(t,\mathbf{x}^\circ) \in V \subseteq \mathbb{R}^3$ of a material particle. Assuming frame-indifference, we take $\psi_e(\mathbf{F},\mathbf{x}) = \psi_e(\mathbf{E},\mathbf{x})$, where $\mathbf{E}$ is one of any number of frame-indifferent Lagrangian strains, with which we identify the generalized displacement as $\mathfrak{u} = \mathbf{E}$, with generalized velocity $\mathfrak{v} = \dot{\mathbf{E}}$, and conjugate stress $\mathfrak{f} = \mathbf{S}$, with $\mathbf{S}\cdot\dot{\mathbf{E}} = \det(\mathbf{F})\boldsymbol{\sigma}\cdot\mathbf{D}$, where $\boldsymbol{\sigma}$ and $\mathbf{D}$ denote, respectively, Cauchy stress and rate of deformation. Then, $\mathbf{S}_e = \partial_{\mathbf{E}}\psi_e$, where ψ_e is energy per unit reference-state volume.

The choice of $\mathbf{E}$ as function of the more basic right Cauchy–Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is somewhat arbitrary and depends on the particular application, as discussed in appendix A and illustrated by the examples considered below. For example, we recall that the Hencky strain $\mathbf{E} = \log\mathbf{U}$, where $\mathbf{U} = \sqrt{\mathbf{C}}$ is the right stretch, is conjugate to the rotated Kirchhoff stress $\mathbf{S} = \det(\mathbf{U})\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$ [31], where $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ is the finite rotation.

It is generally recognized that for quasi-steady deformations one has the following elastic minimum principle for $\mathbf{x} = \mathbf{x}(\mathbf{x}^\circ)$ with external body-force potential $\phi(\mathbf{x})$:

$$\min_{\mathbf{x}(\mathbf{x}^\circ)} \int_{V^\circ} [\psi_e(\mathbf{E}(\mathbf{x})) - \phi(\mathbf{x})]\,\mathrm{d}V^\circ(\mathbf{x}^\circ), \tag{3.1}$$

which, with fixed $\mathbf{x}(\mathbf{x}^\circ)$ on ∂V°, gives the equations of equilibrium. Without considering the variant appropriate to various mixed boundary-value problems, we note that present study establishes a connection of the above principle to its dissipative analogue involving a minimum over $\mathbf{v} = \dot{\mathbf{x}}$, with $\psi_e(\mathbf{E})$ replaced by $\psi_d(\mathbf{D})$ and ϕ by $\mathbf{v}\cdot\nabla\phi$ [7, eqn (56)].

Most of the above formulae for discrete systems carry over *mutatis mutandi* to the continuum-elastic system. In particular, the dissipation potential is now obtained as the following analogue of (2.13):

$$\psi_d(\mathfrak{h},\dot{\mathbf{E}}(t)) = \frac{\psi_e(\mathfrak{h}, T\dot{\mathbf{E}}(t)) - \psi_e(\mathfrak{h}_0,\mathbf{0})}{T}, \quad \text{for } T \neq 0, \tag{3.2}$$

where $\mathfrak{h}$ represents the history of $\dot{\mathbf{E}}$, ψ_e is energy per unit volume, and we once again may take $\psi_e(\mathfrak{h},\mathbf{0}) = 0$. In a similar way, (2.14) becomes the rate-independent rheology

$$\lim_{T\to 0} \psi_d = \dot{\mathbf{E}}\cdot\partial_{\mathbf{E}}\psi_e|_{\mathbf{E}=\mathbf{E}_0} = \dot{\mathbf{E}}\cdot\mathbf{S}_e|_{\mathbf{E}=\mathbf{E}_0}, \tag{3.3}$$

where $\mathbf{E}_0$ represents a certain critical strain for failure. Finally, it is a fairly straightforward matter to obtain the stress-dependent dissipation potential $\varphi_d(\mathbf{S})$ by means of the foregoing analysis. For the present purposes, we adopt a frame-indifferent definition of convexity based on the epigraph of $\psi_e(\mathbf{E})$ as the criterion for convexity or non-convexity of ψ_d, noting that may generally lead to different conditions for uniqueness of solution than criteria based on the notion of polyconvexity [32].

4. Examples

Here, we consider a few examples illustrating some obvious and other less obvious applications of (3.2).

(a) Maxwell–Onsager connection

As one of the most direct applications we consider the collapse on time scale T of anisotropic linear elasticity, with fourth-rank elastic tensor $\mathcal{C}$, so that

$$\left.\begin{aligned} \psi_e &= \tfrac{1}{2}\mathbf{E}\cdot\mathcal{C}[\mathbf{E}] = \tfrac{1}{2}\mathcal{C}^{ijkl}E_{ij}E_{kl} \\ \text{and} \qquad \therefore\ \psi_d &= \tfrac{1}{2}\dot{\mathbf{E}}\cdot\boldsymbol{\mathcal{R}}[\dot{\mathbf{E}}] = \tfrac{1}{2}\mathcal{R}^{ijkl}\dot{E}_{ij}\dot{E}_{kl}, \quad \text{with } \boldsymbol{\mathcal{R}} = T\mathcal{C} \end{aligned}\right\} \tag{4.1}$$

where $\mathcal{R}$, the Onsager resistance, is the inverse of the standard Onsager conductance $\mathcal{L}$. The relation (4.1) is a generalization of Maxwell's [10] formula derived from (2.15), and it represents a generalization of the isotropic formula proposed by Howard [15, pp. 39 ff.] for viscosity arising from the rupture and healing of bio-molecular networks.[3] The most general linear isotropic form of (4.1) gives the Navier–Stokes equation as a relaxation of Navier's elasticity.

While we have suppressed notation for dependence of $\mathcal{C}$ on the history $\mathfrak{h}$, it is plausible that starting from an anisotropic network with a given material symmetry one might generate a succession of networks with differing symmetries and anisotropic viscosities.

It is also worth noting that (4.1) is a special case of a nonlinear relation involving secant moduli $\mathcal{R}(\dot{\mathbf{E}})$ and $\mathcal{C}(\mathbf{E})$. Hence, the nonlinear drag formula presented elsewhere [12] can be regarded as the result of collapsing elastic resistance under sustained flow. We consider next the special case of isotropic elasticity.

(b) Time scale T in various models of transition over potential barriers

As pointed out above, the definition of the time scale T depends on the assumed microscopic dynamics, as illustrated by the following.

(i) Plasticity model of Puglisi & Truskinovsky (P-T) [17]

This one-dimensional model of a periodic potential energy landscape is based on equating a viscous dissipative force to an elastic force derived from the derivative of an assumed elastic energy with respect to a dissipative displacement. Adapted to our notation, equations (2.13)–(2.14) of P-T read

$$\psi_{\mathrm{e}}(\mathfrak{u},\mathfrak{h}) = \frac{C}{2}(\mathfrak{u}-\mathfrak{u}_{\mathrm{d}})^2 + \mathfrak{f}_{\mathrm{M}}\mathfrak{u}_{\mathrm{d}} - k\delta\cos\left(\frac{\mathfrak{u}_{\mathrm{d}}}{\delta}\right), \tag{4.2}$$

whose derivative gives the P-T balance of forces as

$$\eta\mathfrak{v}_{\mathrm{d}} = \eta\dot{\mathfrak{u}}_{\mathrm{d}} = C(\mathfrak{u}-\mathfrak{u}_{\mathrm{d}}) - \mathfrak{f}_{\mathrm{M}} - k\sin\left(\frac{\mathfrak{u}_{\mathrm{d}}}{\delta}\right), \tag{4.3}$$

where $\mathfrak{u}_{\mathrm{d}} = \mathfrak{h}$, the dissipative displacement, represents the history effect, which P-T pre-emptively identify with a plastic displacement $\alpha = \mathfrak{u}_{\mathrm{p}}$. Their formulation of (4.2) assumes linear elasticity with elastic constant C and linear viscous dissipation with viscosity η. One can identify a time constant by the Maxwell protocol as $T = \eta/C$ in the ODE (4.3) and rewrite it as

$$T\dot{\mathfrak{u}}_{\mathrm{d}} = \mathfrak{u} - \mathfrak{u}_{\mathrm{d}} - F_{\mathrm{M}} - K\sin\left(\frac{\mathfrak{u}_{\mathrm{d}}}{\delta}\right), \quad \text{where } F_{\mathrm{M}} = \frac{\mathfrak{f}_{\mathrm{M}}}{C},\ K = \frac{k}{C}, \tag{4.4}$$

which represents an ODE for $\mathfrak{u}_{\mathrm{d}}$ given $\mathfrak{u}(t)$. Note that P-T considers the case of constant $\mathfrak{v} = \dot{\mathfrak{u}}$. Otherwise, (4.4) could be integrated, generally numerically, for given fixed parameters δ, Σ, K. The limit $T \to 0$ represents a singular perturbation with corresponding solution giving plastic displacement $\mathfrak{u}_{\mathrm{p}} = \mathfrak{u}_{\mathrm{d}}$ as the periodically repeated jumps between branches of a multivalued function $\mathfrak{u}_{\mathrm{p}}(\mathfrak{u})$.

(ii) Kramers-type transitions over a single energy barrier

With a view towards unfolding of a protein under applied tension, Dudko *et al.* [19] give a model for transition time $\tau(F)$ as function of pulling force F for transition over a barrier on a two-dimensional energy landscape, and, as shown in their figure 3, $T \to 0$ for large F. To fit into the present paradigm, we can interpret $\tau^{-1}(F)$ as a generalized velocity as function of generalized force $\mathfrak{f} = F$ derived from a suitably defined elastic potential ϕ_{e}.

[3]It also provides one answer to the question posed many years ago by the late GK Batchelor as to the connection between linear Onsager symmetry and linear elastic symmetry [33].

This is illustrated by the phenomenological one-dimensional model discussed by Dudko *et al.* [26] where their equation (3) can be written in the current notation as

$$\mathfrak{v} = \partial_{\mathfrak{f}}\phi_{\mathrm{d}}(\mathfrak{f}), \quad \text{where } \phi_{\mathrm{d}}(\mathfrak{f}) = \frac{1}{T}\phi_{\mathrm{e}}(\mathfrak{f}), \quad \text{with } \phi_{\mathrm{e}} = \frac{1}{\xi}\exp\{E[1 - \nu\xi\mathfrak{f}/E]^{1/\nu}\},$$

where, in terms of the (Kramers) kinetic coefficient of Dudko *et al.* $\mathfrak{v} = -k(\mathfrak{f})$, with $T = 1/k(0)$. Where ξ represents a non-dimensional reaction coordinate and E a non-dimensional activation energy presumably rendered non-dimensional by the standard product $k_{\mathrm{B}}\Theta$ of Boltzmann constant k_{B} and absolute temperature Θ.

The later model of Suzuki & Dudko [19] involves a dissipative transition over a two-dimensional energy barrier and they derive an approximate solution given by their equation (7) in the present format by

$$\mathfrak{v} = \frac{1}{T}\partial_{\mathfrak{f}}\phi_{\mathrm{e}}(\mathfrak{f}), \quad \text{where } \phi_{\mathrm{e}} = \int_0^{\mathfrak{f}} q(F)\exp\{-E[(1 - q^3(F)\}]\,\mathrm{d}F,$$

where $\mathfrak{f}$ and $\mathfrak{v}$ are pulling force and parallel velocity (denoted, respectively, by F and $1/\tau(F)$ in [19]), and $q(F)$ and T (respectively, $\Delta q(F)$ and $\tau_0 = \tau(0)$ [19]) are given in terms of the various parameters defining the two-dimensional geometry of the local energy landscape.

(c) Reiner–Rivlin fluids from nonlinear isotropic elasticity

In frame-indifferent (hyper)elasticity, the strain energy ψ_{e} depends on the anisotropic invariants of a general Lagrangian strain $\mathbf{E}(\mathbf{C})$, where any invertible isotropic function is admissible [34]. This leads to an expression for the conjugate stress $\mathbf{S}$ as a derivative of a potential ψ_{e} depending on the anisotropic invariants of $\mathbf{E}$. According to (3.2) this further leads to a non-linear anisotropic viscous model with $\mathbf{E}$ replaced by $T\dot{\mathbf{E}}$.

Then, by the chain rule, the strain rate $\dot{\mathbf{E}}$ will in general be a linear function of $\mathbf{D}$ depending on $\mathbf{C}$. The dependence on $\mathbf{C}$ generally engenders an anisotropic dependence on $\mathbf{D}$ even if the original elasticity is isotropic. By contrast, while ostensibly inspired by non-linear elasticity, the model of Reiner [35] (designated later as Reiner–Rivlin), involves a strictly isotropic dependence on $\mathbf{D}$. It is shown in appendix A that this isotropy is achieved for fairly general isotropic strain measures provided $\psi_{\mathrm{e}}(\mathfrak{h}, \mathbf{E})$ is an isotropic function of its arguments.

(d) Granular and glassy media

In his celebrated work on the subject, Reynolds [36] considered the limiting case of an assembly of rigid frictionless sphere with reversible volume-coupled change of shape governed by dilatancy. By equating to zero the work of volumetric expansion against a confining pressure plus the work of shearing, Reynolds provides an expression for an 'apparent' coefficient of friction given by the ratio of shear stress to pressure governed by dilatancy. With Reynolds dilatancy interpreted as a rigid (generally non-holonomic) kinematic constraint, certain quasi-static numerical simulations tend to support the ideas of Reynolds [37]. However, the potential flaw in his argument and in the quasi-static numerical simulations is the neglect of instability of locally dilated states with subsequent dynamics leading to a rapid dissipative collapse. This is suggested by the classical work of Rowe [38] and illustrated by his schematic one-dimensional model in figure 2, where τ denotes shear stress and p confining pressure.

The more recent dynamic simulations of Peyneaux & Roux [39], which include collisional dissipation between rigid frictionless spheres, tend to confirm that continual shearing leads to a rate-independent dissipative stress without global dilatancy. Hence, it is plausible that this state of affairs is consistent with the current theoretical notion of collapsing elasticity, in a model closely related to previous models of plastic flow, e.g. [13,17].

In particular and according to the Reynolds picture, we assume that any assembly of rigid particles subject to constant confining pressure will exhibit a kind of elastic response in shear,

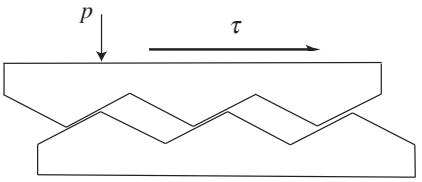

Figure 2. Energy landscape of an idealized unstable frictionless slideblock, after [38].

based on the volume expansion against the confining pressure owing to the dilatancy constraint, at least up to some critical shearing deformation at which the assembly becomes unstable. Although this is likely to occur at local shear strains of order unity, we will simplify matters by adopting globally the small strain analysis of Reynolds, with reference configuration represented by some state of maximum packing density. Thus, our finite strain $\mathbf{E}$ reduces to the usual infinitesimal strain, with $\dot{\mathbf{E}} = \mathbf{D}$, and we denote by primes their deviator parts, with

$$\mathbf{E} = \mathbf{E}' + \epsilon_V \mathbf{I} \quad \text{and} \quad \mathbf{D} = \mathbf{D}' + \dot{\epsilon}_V \mathbf{I}, \tag{4.5}$$

with dilatancy relation $\epsilon_V = \epsilon_V(\mathbf{E}')$ and with stresses conjugate to deviatoric and isotropic parts given, respectively, by $\mathbf{S}' = \boldsymbol{\sigma}'$ and $-p = -\mathrm{tr}(\boldsymbol{\sigma})/3$.

Because of the additional constraint of a given confining pressure, the relevant elastic potential is now that of Gibbs and, with Helmholtz free energy equal to zero, the volumetric Gibbs free energy is given by $\psi_\mathrm{e} = p\epsilon_V(\mathbf{E}')$, relative to the state of maximum density. Assuming that the time scale T for dissipative collapse is negligibly small, we obtain immediately from (2.14)

$$\psi_\mathrm{d} = \mathbf{D}' \cdot \partial_{\mathbf{E}'} \epsilon_V(\mathbf{E}')\big|_{\mathbf{E}'=\mathbf{E}'_0}, \tag{4.6}$$

where $\mathbf{E}'_0$ is a representative strain at which the dilated assembly becomes unstable.

As it stands, (4.6) is applicable to anisotropic crystalline arrays for which one can obtain exact expressions for $\epsilon_V(\mathbf{E}')$, as pointed out by Reynolds [36]. However, it is unlikely that such ordered arrays remain stable on continual shearing without tending to some disordered isotropic state. For such a state, it is reasonable to assume that $\epsilon_V = \epsilon_V(\gamma)$, where $\gamma = \sqrt{\mathrm{tr}(\mathbf{E}'^2)}$ so that (4.6) reduces to the simpler form of rate-independent plasticity:

$$\psi_\mathrm{d} = p\mathbf{D}' \cdot \frac{\mathbf{E}'}{|\mathbf{E}'|}\partial_\gamma \epsilon_V(\gamma)\Big|_{\gamma_0} = p|\mathbf{D}'|\partial_\gamma \epsilon_V(\gamma)\big|_{\gamma_0}, \quad \text{with } \boldsymbol{\sigma}' = p\frac{\mathbf{D}'}{|\mathbf{D}'|}\frac{\mathrm{d}\epsilon_V}{\mathrm{d}\gamma}\Big|_{\gamma=\gamma_0} \tag{4.7}$$

where γ_0 represents a critical shear strain for dilatant instability.

We have assumed that $\mathbf{E}'$ and $\mathbf{D}'$ are co-directional in (4.7), in keeping with the above assumption for discrete systems with the isotropic form of (2.14). The relation (4.7) is simply the classical result of Reynolds with a strain cut-off γ_0 at which the assembly of frictionless particles becomes unstable. A specification is clearly required of γ_0 and $\mathrm{d}\epsilon_V/\mathrm{d}\gamma|_{\gamma=\gamma_0}$.

Note that, with $\mathrm{tr}(\mathbf{D}) = \mathrm{d}\dot{\epsilon}_V/\mathrm{d}|\mathbf{D}'| = \mathrm{d}\epsilon_V/\mathrm{d}\gamma$, the total stress power given by Reynolds, $\boldsymbol{\sigma}' \cdot \mathbf{D}' - p\,\mathrm{tr}(\mathbf{D})$, is nominally zero on the micro-scale up to the point of instability, where stored free energy $p\epsilon_V$ is completely dissipated by the shear stress $\boldsymbol{\sigma}'$ given by (4.7).

As a somewhat standard, elementary extension of (3.2) to frictional spheres, one obtains the resulting formula for stress by adding a Coulomb coefficient μ to $\mathrm{d}\epsilon_V/\mathrm{d}\gamma|_{\gamma=\gamma_0}$, which is expressed somewhat differently in the analysis of Rowe [38]. The dilatancy $\mathrm{d}\epsilon_V/\mathrm{d}\gamma|_{\gamma=\gamma_0}$ now depends on $\mathfrak{h}$ and vanishes at the so-called critical state.

In the above example, collisional dissipation (imperfect restitution) and Coulomb friction can again be viewed as the small-scale failure of elasticity, with the latter represented, e.g. by the elastic-asperity model of Bureau *et al.* [16].

We note that (4.7) may have relevance to the rapid shearing of glassy liquids, where rough estimates of the critical dilatancy can be made based on the $\boldsymbol{\sigma}'/p$ ratio (4.7) [40]. Collisional

dissipation in the granular medium is of course replaced by the eventual thermalization of collision-induced intermolecular vibrations. The distinction between this rate-independent plastic response and the viscous response of ordinary liquids is explained by the occurrence in the latter of a characteristic time T for thermal transitions over potential barriers [23,24].

(e) Non-local continuum models

Although we shall not explore the details here, it is plausible that most of the above ideas will carry over to non-local models with dissipative models derived from non-local hyperelastic models. An example is provided by the weakly non-local or 'gradient' model proposed recently for granular materials [41] which involves a grain-inertial relaxation time that can be identified as the time constant T of the present work. The analogy to higher-gradient (Mindlin–Toupin) elasticity noted in that work is now subject to another interpretation. In the same way, the fully non-local granular-fluidity model of Henann & Kamrin [42], which gives the deformation rate as a functional of stress, can also be interpreted in terms of an elastic model with strain given as functional of stress. It should be noted that the basic visco-plastic model involved these works can be linearly resolved into additive plastic term with $T = 0$ and viscous terms with $T > 0$, providing one example of heterogeneous time constants.

5. Conclusion

The abstract summarizes the main results of this study. We have not considered the difficult questions of the homogenization necessary to obtain continuum-level response from microscopically unstable processes, a topic which has received considerable attention elsewhere [43]. Apart from questions of uniqueness, our work does suggest a possibility of employing the homogenization of heterogeneous elastic systems to obtain dissipation potentials for their dissipative counterparts. For example, an elastic system with heterogeneous time scales, some approaching zero, would give rise to a visco-plastic response of the kind mentioned above in §4d.

The present work may also provide a basis for the principle of minimum dissipation potential derived from the analogous principle for hyperelastic systems.

Finally, there is an interesting question of the possible relevance to dissipative systems of chemical and biochemical reactions in the presence of diffusional transport, such as those considered elsewhere [9].

Data accessibility. This article has no additional data.

Competing interests. We declare we have no competing interests.

Funding. The second author acknowledges support from grant NSF CBET-1253228 from the US National Science Foundation.

Acknowledgements. The authors wish to thank Prof. Prashant Purohit for the reference to Howard [15]. The first author acknowledges a 2017–2018 visiting professorship at the École Supérieure de Physique et de Chimie Industrielles, Paris, where part of this paper was drafted.

Appendix A. From isotropic elasticity to Reiner–Rivlin fluid

As pointed out elsewhere [34], any Lagrangian strain $\mathbf{E}$ given as an invertible function $\mathbf{E} = \mathfrak{E}(\mathbf{C})$ of the (positive-definite) Cauchy–Green tensor $\mathbf{C}$ is generally admissible as frame-indifferent Lagrangian measure of local deformation. With brackets [ ] denoting linear transformations (represented by the action of fourth-rank tensors), it follows by the chain rule that the corresponding rate of strain is an invertible linear function $\dot{\mathbf{E}} = \mathfrak{G}(\mathbf{C})[\mathbf{R}^T\mathbf{D}\mathbf{R}]$ of $\mathbf{D}$ given by the Fréchet derivative $\mathfrak{F}(\mathbf{C}) = \partial_{\mathbf{C}}\mathfrak{E}$ as

$$\mathfrak{G}(\mathbf{C})[\mathbf{K}] = 2\mathbf{U}\mathfrak{F}(\mathbf{C})\mathbf{U}[\mathbf{K}], \tag{A 1}$$

where $\mathbf{K}$ is an arbitrary second-rank tensor, $\mathbf{U}=\sqrt{\mathbf{C}}$ is the right stretch, and $\mathbf{R}=\mathbf{F}\mathbf{U}^{-1}$ the finite rotation. The conjugate stress $\mathbf{S}$ must satisfy [31]:

$$\left.\begin{aligned} &\det(\mathbf{F})\boldsymbol{\sigma}\cdot\mathbf{D}=\mathbf{S}\cdot\dot{\mathbf{E}}=\mathbf{S}\cdot\mathfrak{G}[\mathbf{R}^T\mathbf{D}\mathbf{R}]\equiv\mathfrak{G}^*[\mathbf{S}](\mathbf{R}^T\mathbf{D}\mathbf{R})\\ \text{and}\qquad &\therefore\ \mathbf{S}=\det(\mathbf{F})\mathfrak{G}^{*-1}[\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}],\end{aligned}\right\}\tag{A 2}$$

where $\mathfrak{G}^*$ is the adjoint of $\mathfrak{G}$. Note that $\det(\mathbf{F})=\det(\mathbf{C})/2$ can be regarded as a function of $\mathbf{E}$ that has no effect on isotropy but can be viewed as representing the effect of density in Reiner's dilatant fluid.

Now, the expressions in the first paragraph above for $\mathbf{E}$ and $\dot{\mathbf{E}}$ give

$$\dot{\mathbf{E}}=\mathfrak{H}(\mathbf{E})[\mathbf{R}^T\mathbf{D}\mathbf{R}],\quad\text{where }\mathfrak{H}(\mathbf{E})[\mathbf{K}]=\mathfrak{G}(\mathfrak{E}^{-1}(\mathbf{E}))[\mathbf{K}].\tag{A 3}$$

Therefore, according to (2.13), ψ_{d} is given by $\psi_{\mathrm{e}}(\mathbf{E})$ with $\mathbf{E}$ replaced by $T\dot{\mathbf{E}}$. With the latter given by (A 3) we replace $\mathbf{E}$ there as well with $T\dot{\mathbf{E}}$, which leads to a generally implicit equation for the latter in terms of $\mathbf{R}^T\mathbf{D}\mathbf{R}$.

The above procedure leads to the rotated stress $\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R}$ as function of rotated deformation rate $\mathbf{R}^T\mathbf{D}\mathbf{R}$. However, in the case, where the functions $\mathfrak{E}$ and ψ_{e} are isotropic it follows that Cauchy stress $\boldsymbol{\sigma}$ is given by an isotropic function of $\mathbf{D}$.

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