Errata: A general micromorphic theory of kinematics and stress in granular media

- Page 1, end of Column 1 and Page 2, beginning of Column 1: Read: "... *higher-gradient* ("Toupin-Mindlin") material (Mindlin 1964)... *micromorphic* ("Cosserat-Green-Rivlin-Eringen") material ..."

- Eq. 10 should read:

$$\dot{w} = \dot{W}/V = \frac{1}{V}\sum_{k=1}^{E} V_k \hat{\mathbf{T}}_k : \hat{\mathbf{L}}_k,$$

- Eq. 17: "$\tilde{\mathbf{u}}_k \leftrightarrow \tilde{\mathbf{f}}_k$" should read "$\mathbf{u}_k \leftrightarrow \mathbf{f}_k$"

- Page 4, Column 1, Paragraph 1, last word: "model" should read "models".

- Page 4, Column 1, Paragraph 2, Line 2: "files" should read "field"

- Page 4, Column 1, Paragraph 2, Penultimate line: "of for" should read "of"

A general micromorphic theory of kinematics and stress in granular media[0]

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This is a brief overview and critique of principles and techniques for the homogenization of granular media, with emphasis on the quasi-static mechanical behavior. A brief review is given of the micrormorphic continuum, which is adopted here as a general model of such media. A novel formulation is given of the graph- and energy-theoretic principles underlying granular micromechanics, revealing the connections to structural mechanics and general network analysis. A novel energy-based method is proposed for homogenization. Based on this method, polynomial representations for particle displacements and forces provide the relevant gradients and moment stresses for micromorphic continua. A qualitative assessment is given of micrormorphic effects in typical granular media, based on a consideration of particle and contact-zone length scales. For the usual geomaterial, a micropolar (generalized Cosserat) model, already adopted by prominent workers in the field, should suffice to represent the continuum mechanics, whereas the more general micromorphic model may be needed for soft cellular materials.

1 MICROMORPHIC CONTINUA

One goal of the present work is to establish the micromorphic continuum as a plausible model for granular and cellular media. After a brief introduction to micromorphic continua, we summarize certain aspects of micromechanics and homogenization. The notation is similar to that employed in a previous work (Goddard 1998), where bold symbols are employed for space tensors , lowercase symbols for vectors $\boldsymbol{\varphi} = \varphi^\alpha \mathbf{g}_\alpha$, uppercase for higher-order tensors $\mathbf{L} = L^{\alpha\beta\cdots}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \otimes \ldots$, etc. where $\otimes$ denotes a tensor product, and Greek superscripts and subscripts refer to a basis $\mathbf{g}_\alpha, \alpha = 1, 2, 3$, derived from appropriate spatial coordinates. For the present purposes, the latter may be taken as orthogonal Cartesian. A colon is employed denote the exhaustive, ordered contractions of tensors of rank two and higher, such that, for $n \geq m$, $(L^{\alpha_1\cdots\alpha_n}\ldots) : (M^{\beta_1\cdots\beta_m}\ldots) := L^{\alpha_1\cdots\beta_1\cdots\beta_m}\ldots M_{\beta_1\ldots\beta_m}\ldots$, and we employ superscript T to denote transposition of the right-most tensor component with all the preceding, so that $(L^{\alpha_1\alpha_2\ldots\alpha_n}\ldots)^T := (L^{\alpha_n\alpha_1\alpha_2\cdots}\ldots)$. The standard notation $\mathbf{Lx} (= L^\alpha_\beta x^\beta \mathbf{g}_\alpha)$ is employed for linear transformations of vectors via second-rank tensors.

With $\mathbf{a}^n$ denoting the n-fold symmetric tensor product $\otimes^n \mathbf{a}$, the Taylor series expansion for the velocity (or infinitesimal displacement) $\mathbf{v}$,

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_o + \mathbf{L}_1 \colon \mathbf{r} + \mathbf{L}_2 \colon \mathbf{r}^2 + \ldots, \tag{1}$$

$$\text{with} \quad \mathbf{r} = \mathbf{x} - \mathbf{x}_o, \; \mathbf{L}_n = \tfrac{1}{n!}(\boldsymbol{\nabla}^n \otimes \mathbf{v})^T_o, \tag{2}$$

provides the well-known expansion for global stress-power density in a simple continuum:

$$\dot{w} = \tfrac{1}{V}\int_V \mathbf{T} \colon \mathbf{L} dV = \textstyle\sum_n \dot{w}_n, \tag{3}$$

$$\text{with} \quad \dot{w}_n := \mathbf{T}_n \colon \mathbf{L}_n, \; \mathbf{T}_n := \int_V \mathbf{T} \otimes \mathbf{r}^n dV \tag{4}$$

where $\mathbf{T}$ is Cauchy stress and $\mathbf{L} = (\boldsymbol{\nabla} \otimes \underline{\mathrm{v}})^T$ (first) velocity gradient. The $\mathbf{T}_n$ are moment stresses representing the generalized forces and the work associated with the kinematical quantities $\mathbf{L}_n$(Green & Rivlin 1964; Goddard 1998). Equation (4) serves to establish an equivalence between a non-homogeneous *simple* continuum and a homogeneous *multipolar* continuum i.e. a continuum endowed with intrinsic moment stresses. For such a continuum, we can discern two important special cases:

1. The $\mathbf{L}_n$ are identical with higher gradients of the local velocity field, as defined in (2), or

2. they are intrinsic "particulate" fields, say $\mathbf{L}^p_n(\mathbf{x}, t)$, given by more general constitutive equations.

The first represents a *higher-gradient* ("Cosserat-Mindlin") material (Cosserat & Cosserat 1909; Mindlin 1964), while the second represents a more general, *micromorphic* ("Mindlin-Green-Rivlin-Eringen") material (Mindlin 1964; Green & Rivlin 1967; Eringen 1968). By means of the mathematical "fragmentation" of a simple continuum into discontinuous subdomains, Eringen and coworkers (Eringen 1999) derive micromorphic field theories, which resemble those obtained by various statistical mechanical studies of systems of deformable particles (Eringen 1999). The same type of multipolar balance

[0]From Garcia-Rojo, R. et al. (eds), *Powders and Grains (2005)*, Vol. 1, pp. 129-134, Taylor & Francis, 2005.

laws arise even for the simplest case of point-like particles (Goddard 1998).

The micromorphic continuum is a special case of a multipolar continuum endowed with a *polyad* of deformable vectors or ”directors” attached to each material particle (Green & Rivlin 1967), with $\mathbf{L}_n^p$ representing 3^n such vectors. The simplest (”grade one”) micromorphic continuum is characterized by deformable triad of vectors and, hence, a single second-rank (velocity-gradient) tensor $\mathbf{L}^p$ attached to each material point, which serves to represent an homogeneous microstructural (”particle”) deformation and rotation. In the special case of a micropolar (Cosserat) continuum, $\mathbf{L}^p = \mathbf{W}^p = -(\mathbf{W}^p)^T$ and $\boldsymbol{\omega}^p$ =vec$(\mathbf{W}^p)$ represent a (particle) spin generally distinct from the global spin $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$.

The stress moments $\mathbf{T}_n$ in a multipolar continuum satisfy a hierarchy of balances of the form (Eringen 1999)

$$\boldsymbol{\nabla}\cdot\mathbf{T}_{n+1}^T + \mathbf{T}_n = \mathbf{G}_{n+1}, \text{ for } n = 0, 1, \ldots, \quad (5)$$

where $\mathbf{T}_0 := \mathbf{0}$, and the $\mathbf{G}$’s represent extrinsic body moments plus accumulation of intrinsic multipolar momenta. The latter are not made explicit here, since they are negligible in the quasi-static limit.

2 MICROMECHANICS

Granular micromechanics involves both extrinsic modes or degrees of freedom, associated with motion of particle centroids, and intrinsic modes associated with particle deformation. Although the two are coupled through particle contacts, we focus attention first on the extrinsic modes.

2.1 *Graph-Theory of Extrinsic Modes*

In a schema dating back to the early works of Satake (Satake 1993; Kruyt 2003a), we let particle centroids define the the nodes or vertices $j = 1, 2, \ldots, N$ defined by the associated Delaunay triangulation (Bagi 1996; Goddard 1998; Goddard 2004). This defines an abstract (connected simple) graph $\mathcal{G}$, the *contact network*, with edges or branches $i = 1, 2, \ldots, E$ representing nominal nearest-neighbors and defining contacts or virtual contacts.

For the associated matrix formulation, let underlined lowercase quantities denote column arrays (abstract vectors) associated with edges and nodes, while superscript * denotes transposition (vector-space dual). e.g. $\underline{\varphi} = [\varphi_i]^* = [\varphi_1, \ldots, \varphi_N]^*$ denotes a $1 \times N$ row of scalars, $\underline{\boldsymbol{\varphi}} = [\boldsymbol{\varphi}_i]^* = [\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_E]^*$, a $1 \times E$ row of space vectors, etc., while underlined uppercase denotes the associated linear transformations or matrices, e.g. $\underline{A} = [A_{ij}], \underline{\mathbf{A}} = [\mathbf{A}_{ij}]$, etc. Assignment of directions to the edges of the above graph yields a directed graph (Berge 1973; Goddard 2002), with $(E \times N)$ incidence matrix $\underline{D} = [D_{ij}]$:

$$D_{ij} = \begin{cases} +1, \text{if edge } i \text{ enters node} j, \\ -1, \text{if edge } i \text{ leaves node } j, \\ \;\;0, \text{otherwise} \end{cases} \quad (6)$$

The matrices $\underline{D}$ and transpose $\underline{D}^*$ represent difference-operators, which we designate, respectively, as the *differential* and the *codifferential*. Thus, $\underline{D}\underline{\varphi}$ yields differences along edges of nodal ”potentials” represented by $\underline{\varphi}$, while $\underline{D}^*\underline{j}$ yields nodal accumulations from flows along edges (Goddard 2002). Thus, substitution $\underline{\varphi} \to \underline{\mathbf{x}}$, where $\underline{\mathbf{x}} = [\mathbf{x}_i]^*$ represents the particle centroids, yields the array of branch- or edge vectors $\underline{\mathbf{l}} = [\mathbf{l}_i]^* = \underline{D}\underline{\mathbf{x}}$, while the relative velocities (or displacements) along edges are given by $\underline{\mathbf{u}} = \underline{D}\underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ denotes nodal velocities (or infinitesimal displacements). Therefore, the quasi-static equilibrium of forces and the compatibility of relative displacements are given by:

$$\underline{D}^*\underline{\mathbf{f}} = \underline{\mathbf{0}}, \quad \underline{D}_\times^*\underline{\mathbf{u}} = \underline{\mathbf{0}}, \text{ where } \underline{D}_\times = \text{null}(\underline{D}^*), \quad (7)$$

null denoting the *null space* (or *kernel*) of a linear transformation, with *cross differential*, $\underline{D}_\times$ given by any $E \times M$ matrix whose columns are a basis for the null space of $\underline{D}^*$. The rank of $\underline{D}$ is $N - 1$ (Berge 1973), representing $N - 1$ independent force balances in (7), and the rank of $\underline{D}_\times$ is $M = E - N + 1$, which follows from Euler’s celebrated formula. Space does not permit a discussion of duality associated with (7), which is covered in various graph-theoretic treatments of structural mechanics (Shai 2001). However, in contrast to the usual engineering structures, the graphs for granular media or other mobile cellular assemblies are often transitory, reflecting abrupt topological rearrangement engendered by finite deformation.

2.2 *Extrinsic power and virtual work*

The power (or incremental work) is given by the scalar product:

$$\dot{W} = (\underline{\mathbf{f}}, \underline{\mathbf{u}}) := \underline{\mathbf{f}}^* \cdot \underline{\mathbf{u}} = \sum_{k=1}^{E} \mathbf{f}_k \cdot \mathbf{u}_k \quad (8)$$

The condition $\dot{W} = 0$, gives the analog of Tellegen’s theorem (Shai 2001) for electrical circuits, requiring external forcing of some particles. Otherwise, this represents a virtual-work principle, implying that force equilibrium is satisfied for any compatible displacement $\underline{\mathbf{u}} = \underline{D}\underline{\mathbf{v}}$:

$$(\underline{\mathbf{f}}, \underline{D}\underline{\mathbf{v}}) = (\underline{D}^*\underline{\mathbf{f}}, \underline{\mathbf{v}}) = 0, \;\; \forall \underline{\mathbf{v}} \;\Rightarrow \underline{D}^*\underline{\mathbf{f}} = \underline{\mathbf{0}}, \quad (9)$$

with a dual relation guaranteeing compatibility. The replacement $[\dot{W}, \mathbf{v}, \mathbf{u}, \mathbf{f}] \to [V, \mathbf{x}, \mathbf{l}, \mathbf{a}]$ in (8)-(9), where $\mathbf{a}$ denotes area vectors discussed below, yields an intriguing geometric formula that is relevant to granular dilatancy.

3 ENERGY-BASED HOMOGENIZATION

In another interpretation of (8), $\dot{W}$ represents boundary forces and continuum-level stress power density (3), with

$$\dot{w} = \dot{W}/V = \sum_{k=1}^{E} V_k \hat{\mathbf{T}}_k : \hat{\mathbf{L}}_k, \tag{10}$$

where $\hat{\mathbf{T}}_k = V_k^{-1}\mathbf{f}_k \otimes \mathbf{l}_k$ and $\hat{\mathbf{L}}_k = V_k^{-1}\mathbf{u}_k \otimes \mathbf{a}_k$, with $V_k = \mathbf{l}_k \cdot \mathbf{a}_k$. Here $\mathbf{a}_k$ is a vector area and V_k a volume defined uniquely for branch vector $\mathbf{l}_k$ by means of the associated Delaunay triangulation (Goddard 2004). The tensor products $\hat{\mathbf{T}}_k$ and $\hat{\mathbf{L}}_k$ represent contributions of branch k to the global volume, and the volume-average (Cauchy) stress and velocity gradient, defined by

$$V = \sum_{k=1}^{E} V_k, \;\; \mathbf{T} = \frac{1}{V}\sum_{k=1}^{E} V_k \hat{\mathbf{T}}_k, \;\; \mathbf{L} = \frac{1}{V}\sum_{k=1}^{E} V_k \hat{\mathbf{L}}_k \tag{11}$$

As with other heterogeneous media. the granular power $\dot{w}$ in (10) is generally not given by $\mathbf{T}:\mathbf{L}$ determined from (11), owing to inhomogeneity arising from macroscopic gradients and random microscopic fluctuations. In keeping with the continuum form (1), and following previous works, we assume that $\mathbf{f}_k, \mathbf{u}_k$ are known (e.g. from a micromechanical calculation) and consider the representation of $\mathbf{u}_k$:

$$\mathbf{u}_k = \tilde{\mathbf{u}}_k + \mathbf{u}'_k, \text{ with } \tilde{\mathbf{u}}_k = \tilde{\mathbf{L}}\mathbf{l}_k + \tilde{\mathbf{L}}_2 \colon \mathbf{l}_k^2 + \ldots + \tilde{\mathbf{L}}_m \colon \mathbf{l}_k^m \tag{12}$$

where the polynomial in $\mathbf{l}_k$ is attributed to macroscopic gradients and $\mathbf{u}'_k$ to random fluctuations.

To specify the parameters $\tilde{\mathbf{L}}_n$ in (12) for some subset of m branch vectors $\mathbf{l}_k$, several authors advocate strict polynomial fits, with a maximal value of m, or else some other "best" fit of (12) to continuum kinematics (Liao et al. 1997; Suiker et al. 2001; Jenkins & Koenders 2004). Given the well-known pathology ("overfitting") of polynomial fits of random data, and in view of the importance of energy, we take the position that a "best" fit should rather be based on minimization of an appropriate norm of stress-power fluctuations plus some norm of the variation implied by (12)[1].

As a prototypical linear method, consider

$$\sigma^2 = \sum_{k=1}^{m} [\mathbf{f}_k \cdot (\mathbf{u}_k - \tilde{\mathbf{u}}_k)]^2 + Q, \;\; \partial\sigma^2/\partial\tilde{\mathbf{L}}_n = 0, \tag{13}$$

where Q denotes a quadratic form in the $\tilde{\mathbf{L}}_n$. This leads to a set of linear equations for $\tilde{\mathbf{L}}_n, n = 1, \ldots, m$, with corresponding estimate for stress power:

$$\tilde{\dot{w}} = \tilde{\mathbf{T}} : \tilde{\mathbf{L}} + \tilde{\mathbf{T}}_2 \colon \tilde{\mathbf{L}}_2 + \ldots + \tilde{\mathbf{T}}_m \colon \tilde{\mathbf{L}}_m \tag{14}$$

[1] closely related to the so-called "generalized additive models" (Wood 2004)

where, as the analog of (4) and in a form proposed elsewhere (Goddard 1998), the moment stresses are given by the force moments (multipoles):

$$\tilde{\mathbf{T}}_n = \frac{1}{V}\sum_{k=1}^{m} \mathbf{f}_k \otimes \mathbf{l}_k^n, \;\; n = 1, \ldots, m \tag{15}$$

irrespective of the $\tilde{\mathbf{L}}_n$. One obtains a dual for (12)-(15) by means of the substitution

$$\tilde{\mathbf{u}}_k \leftrightarrow \tilde{\mathbf{f}}_k, \;\; \tilde{\mathbf{L}}_k \leftrightarrow \tilde{\mathbf{T}}_k, \;\; \mathbf{l}_k \leftrightarrow \mathbf{d}_k := \mathbf{a}_k / V_k \tag{16}$$

where $\mathbf{d}_k^n \colon \mathbf{l}_k^n = 1$, and (15) provides the estimate $\tilde{\mathbf{L}}_n$.

Replacing (13) by $\sigma^2 = \sum_k (\tilde{\mathbf{f}}_k \cdot \tilde{\mathbf{u}}_k - \mathbf{f}_k \cdot \mathbf{u}_k)^2 + Q$, where Q is a quadratic form in $\tilde{\mathbf{L}}_k, \tilde{\mathbf{T}}_k$, and employing the same polynomial representations of $\tilde{\mathbf{u}}_k, \tilde{\mathbf{f}}_k$ in terms of $\mathbf{l}_k, \tilde{\mathbf{d}}_k$, respectively, we obtain a more general, simultaneous estimate of stresses and gradients. Bilinearity of Q in $\tilde{\mathbf{u}}_k, \tilde{\mathbf{f}}_k$ allows for an interpretation in terms of a real (as opposed to abstract) energy.

3.1 *Intrinsic moments and overall continuum fields*

In the usual model of granular media, particle contact forces are localized on extremely small (Hertzian) contact zones, and the effective particle stress for a given particle $\mathbf{T}^p$ is given by (11), with k referring to contact points, V replaced by particle volume V^p, branch vector $\mathbf{l}_k$ replaced by contact moment arm $\mathbf{r}_k$, and $V_k = V^p$, constant for all k. The last member of (11) gives a similar but less exact estimate of particle velocity (or displacement) gradient $\mathbf{L}^p$, on letting $\mathbf{u}$ denote difference in velocity (or displacement) between contact point k and particle centroid, and on taking $\mathbf{a}_k = a_k \mathbf{n}_k$, where $\mathbf{n}_k$ is the unit normal to the local contact tangent plane, with $a_k = V^p / \mathbf{r}_k \cdot \mathbf{n}_k$ denoting an effective contact area. This description of intrinsic particle kinematics represents a type of finite-element approximation, whereas the exact treatment of the micromechanics requires constitutive equations and field equations for the particle interior, subject to localized tractions on the particle surface and followed by appropriate averaging over particle volume.

At any rate, it is clear that the localized surface stresses provide the coupling of the intrinsic particle deformation to the degrees of freedom represented by motion of particle centroids. This paramount aspect of granular mechanics may be obscured by micromechanical analyses that put particle rotation, a property of finite grains, on the same footing as the motion of particle centroids.

With an appropriate replacement of (12) and (13)-(15), one obtains higher-order micromorphic effects, represented by $\mathbf{T}_n^p, \mathbf{L}_n^p, \;\; n > 1$. The ever-increasing dependence on particle length scales is manifestly clear. In a similar vein, we expect that higher-order contact moments will exhibit a similar dependence on the dimension of contact zones.

Given the above estimates of continuum-level moments, the following *Ansatz* is suggested by, but not rigorously derived from the above-cited works of Eringen and coworkers:

$$\mathbf{X} = \nu^c \mathbf{X}^c + \nu^p \mathbf{X}^p + \mathbf{X}^s, \tag{17}$$

where $\mathbf{X} = \langle \mathbf{T}_n \rangle$ or $\langle \mathbf{L}_n \rangle, n = 1, 2 \ldots$, represent volume (or surface) averages, with $\langle \mathbf{X}_n \rangle^i = O(1)$ for for $\nu^i \to 0$. The superscript c refers to a continuum-level contribution arising from the relative motion of particle centroids; p to a contribution arising from the internal structure of particles, regarded as pieces of a continuous medium; and s to a contribution arising from singular surfaces. ν^p denotes particle volume fraction and ν^c void fraction, given by $1 - \nu^p$ in the usual granular medium. Typical singular surfaces involve interfacial slip, such as cracks, or other kinematic discontinuities, or interfacial tension and other (multipolar) stress jumps (Eringen 1999). The relation (17) appears to cover various limiting case, e.g. $\nu^p \to 0, 1$, and Mindlin's special case $\mathbf{X}^c \to \mathbf{X}^p$ (Mindlin 1964), often used for multipolar-elastic model.

Whenever the $\mathbf{L}_n^p$ are derivable as gradients of a volume-average velocity files $\mathbf{v}^p$, say, the $\langle \mathbf{T}_n \rangle^i,\ i = c, p$ yield a correct value for stress-power over regions in which the velocity field $\mathbf{v}^i$ has boundary values given by a polynomial of maximal degree n, whereas $\langle \mathbf{L}_n \rangle^i$ yields the power of for moment-stress fields whose surface moments all vanish except for $\mathbf{T}_n^i \mathbf{n} = \overline{\mathbf{T}}_n^i \mathbf{n}$, where $\overline{\mathbf{T}}_n^i$ is constant. Otherwise, the power $\dot{w}_n^i$ generally is not equal to $\langle \mathbf{T}_n \rangle^i : \langle \mathbf{L}_n \rangle^i$.

In the micropolar limit, the gradient $\mathbf{L}^p$ reduces to the antisymmetric spin anticipated above in Section 2, while the symmetric part of $\mathbf{T}^p$ reduces to a statically indeterminate work-free stress, and the antisymmetric part represents a couple. Quasi-static equilibrium dictates that this couple vanish, in the absence of contact couples or external body couples (Bardet & Vardoulakis 2001). Since contact couples in simple materials can only arise as moments of stress distributions over finite areas, it becomes clear that such effects will be important only for pairs of particles with multiple or extended contact zones. For assemblies of nearly-rigid particles typified by Hertzian contact, one can therefore anticipate that higher-order spatial gradients $\mathbf{L}_n,\ n > 1$ are required for non-zero $\mathbf{T}_n,\ n > 1$, as borne out by numerous analyses (Liao et al. 1997; Suiker et al. 2001; Kruyt 2003b).

4 CONCLUSIONS

The micromorphic continuum provides an attractive model of granular and cellular media, with the micropolar limit being appropriate for nearly rigid grains. For the latter, asymmetric stress and couple stress must arise from higher gradients of particle spin, whereas for soft granular and cellular media, these can also arise from finite contact moments. The extension to granular dynamics appears straightforward (Eringen 1999)). Shear bands and the propagation of short-wavelength shear waves, which have received considerable attention in granular mechanics, remain of course a plausible testing ground for micromorphic theories.

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