MAE 108: Solutions HW 3

Problem 1

Ang & Tang 2.39

We have

$D$ : Number of defective panels on a given day

$A$ : Shipment accepted on a given day

- $P(D = 0) = 0.2$
- $P(D = 1) = 0.5$
- $P(D = 2) = 0.3$

a) The shipment is accepted if the supervisor finds at most one defected panel

so $P(A) = P(D \leq 1)$ and $P(\bar{A}) = P(D = 2)$

$$P(A) = P(D \leq 1) = P(D = 0) + P(D = 1) = 0.2 + 0.5 = 0.7$$

b) $P$(exactly one shipment will be rejected in 5 days)

$$= 5 \times P(D \leq 1)^4 \times P(D = 2) = 5 \times 0.7^4 \times 0.3 = 0.36$$

c) The shipment is accepted if the supervisor finds at most one defected panel, however we now take into account that not all defected panel are found. Only 80% of the them are detected and therefore rejected, $P(\bar{A}|D = 1)$. Therefore our definition of the probability of acceptance of shipment on a given day $P(A)$ changes and is dependent on the number of defective panels detected, in stead of the number of defective panels.
$D_d$: the number of defective panels detected.

Therefore

- $P(\bar{A}|D = 1) = 0.8$
- $P(A) = P(D_d \leq 1)$
- $P(\bar{A}) = P(D_d = 2)$

\[
P(A) = P(A|D = 0)P(D = 0) + P(A|D = 1)P(D = 1) + P(A|D = 2)P(D = 2)
\]

Keep in mind that $A$ is now dependent on $D_d$ when solving for the conditional probabilities and that $\bar{A}$: $D_d = 2$

\[
P(A|D = 0) = 1 - P(\bar{A}|D = 0) = 1 - 0 = 1
\]

\[
P(A|D = 1) = 1 - P(\bar{A}|D = 1) = 1 - 0 = 1
\]

This aligns with that there can not be a detection of two defected panels if there are not two defected panels to begin with. However when there are two defected panels we have:

\[
P(A|D = 2) = 1 - P(\bar{A}|D = 2)
\]

\[
= 1 - P(\bar{A}|D = 1)P(\bar{A}|D = 1) = 1 - 0.8 \times 0.8 = 0.36
\]

Hence, $P(A) = 0.2 + 0.5 + 0.36 \times 0.3 = 0.808$

There are different ways of approaching this. Another approach is:

\[
P(A) = P(D_d \leq 1)
\]

\[
= P(D_d \leq 1|D \leq 1) \cdot P(D \leq 1) + P(D_d \leq 1|D = 2) \cdot P(D = 2)
\]

By the implicit assumption, $D \leq 1$ implies $D_d \leq 1$. Then, this means that $(D \leq 1) \subset (D_d \leq 1)$ and that

\[
P(D_d \leq 1|D \leq 1) = P((D_d \leq 1) \cap (D \leq 1))/P(D \leq 1)
\]

\[
= P(D \leq 1)/P(D \leq 1) = 1
\]
\[ P(D_d \leq 1|D = 2) = 1 - P(D_d = 2|D = 2) = 1 - (0.8)^2 = 0.36 \]

Lets put this into the above equation

\[
P(A) = P(D_d \leq 1) \\
= P(D_d \leq 1|D \leq 1) \cdot P(D \leq 1) + P(D_d \leq 1|D = 2) \cdot P(D = 2) \\
= 1 \cdot (P(D = 0) + P(D = 1)) + [1 - P(D_d = 2|D = 2)] \cdot P(D = 2) \\
= 0.2 + 0.5 + 0.36 \cdot 0.3 = 0.808
\]

**Problem 2**

*Ang & Tang 2.45*

Let \( A, D, \) and \( I \) denote the respective events that a driver encountering the amber light will accelerate, decelerate, or be indecisive. Let \( R \) denote the event that s/he will run the red light.

The given probabilities and conditional probabilities are:

- \( P(A) = 0.10 \)
- \( P(D) = 0.85 \)
- \( P(I) = 0.05 \)
- \( P(R|A) = 0.05 \)
- \( P(R|D) = 0 \)
- \( P(R|I) = 0.02 \)

a) By the theorem of total probability,

\[
P(R) = P(R|A)P(A) + P(R|D)P(D) + P(R|I)P(I) \\
= 0.05 \cdot 0.10 + 0 + 0.02 \cdot 0.05 \\
= 0.005 + 0.001 \\
= 0.006
\]
b) The desired probability is \( P(A|R) \), which can be found by Bayes’ Theorem as
\[
P(A|R) = \frac{P(R|A)P(A)}{P(R)} = \frac{0.05 \times 0.10}{0.006} = 0.833
\]

c) Let \( V \) mean there exists a vehicle waiting on the other street,
- \( P(V) = 0.6 \)
- \( P(\bar{V}) = 0.40 \)

Let \( C \) denote that the driver in the other vehicle is cautious,
- \( P(C) = 0.8 \)
- \( P(\bar{C}) = 0.20 \)

The probability of collision is:
\[
P(\text{collision}) = P(\text{collision}|V)P(V) + P(\text{collision}|\bar{V})P(\bar{V})
\]
\[
P(\text{collision}|V) = P(\text{collision}|C)P(C) + P(\text{collision}|\bar{C})P(\bar{C})
\]
\[
= (1 - 0.95) \times 0.80 + (1 - 0.80) \times 0.20
\]
\[
= 0.05 \times 0.80 + 0.20 \times 0.20
\]
\[
= 0.08
\]
\[
P(\text{collision}|\bar{V}) = 0
\]

Hence,
\[
P(\text{collision}) = 0.08 \times 0.60 + 0
\]
\[
= 0.048
\]
d) 100,000 vehicles * 5% = 5000 vehicles are expected to encounter the yellow light annually. Out of these 5000 vehicles, 0.6% (i.e. 0.006) are expected to run a red light, i.e. 5000 * 0.006 = 30 vehicles. These 30 dangerous vehicles have 0.048 chance of getting into a collision (i.e. accident), hence 30 * 0.048 = \textbf{1.44} accidents caused by dangerous vehicles can be expected at the intersection per year.

**Problem 3**

*Ang & Tang 2.47*

Let $D$ denote difficult foundation problem, $F$ denote that the project is in Ford County, $I$ denote that the project in Iroquois County, and $C$ denote a project in Champaign County.

\[
P(D) = \frac{2}{3} \\
P(F) = \frac{1}{3} = 0.333 \\
P(I) = \frac{2}{5} = 0.4 \\
P(D|I) = 1.0 \\
P(D|F) = 0.5
\]

a)

\[
P(F \bar{D}) = P(\bar{D}|F)P(F) \\
= 0.5 \times 0.333 = 0.167
\]

b)

We are asked to find the following:

\[
P(C \bar{D}) = P(\bar{D}|C)P(C)
\]

Therefore we need:

\[
P(\bar{D}|C) = P(\bar{D}) = 1 - P(D) = \frac{1}{3} \\
P(C) = 1 - P(F) - P(I) \\
= 1 - 0.333 - 0.4 = 0.267
\]
Hence,

\[ P(C \bar{D}) = P(\bar{D}|C)P(C) = \frac{1}{3} \times 0.267 = 0.089 \]

c)

\[ P(I|\bar{D}) = \frac{P(\bar{D}|I)P(I)}{P(D)} = \frac{0 \times 0.4}{0.667} = 0 \]

**Problem 4**

*Ang & Tang 2.51*

Let \( C, S \) denote shortage of cement and steel bars respectively. The probabilities and condition probability are:

- \( P(C) = 0.1 \)
- \( P(S) = 0.05 \)
- \( P(S|\bar{C}) = 0.5 \times 0.05 = 0.025 \)

a)

\[
\begin{align*}
P(S \cup C) &= P(S) + P(C) - P(C|S)P(S) \\
P(\bar{C}|S) &= \frac{P(S|\bar{C})P(\bar{C})}{P(S)} = \frac{0.025 \times 0.9}{0.05} = 0.45 \\
P(C|S) &= 1 - P(\bar{C}|S) = 0.55 \\
P(S \cup C) &= 0.05 + 0.1 - 0.55 \times 0.05 = 0.1225
\end{align*}
\]

b)

\[
\begin{align*}
P(C \bar{S} \cup \bar{C}S) &= P(C \bar{S}) + P(\bar{C}S) \\
&= P(C \cup S) - P(CS) \text{ from Venn diagram} \\
&= 0.1225 - P(C|S)P(S) \\
&= 0.1225 - 0.55 \times 0.05 \\
&= 0.095
\end{align*}
\]
c) 

\[
P(S | S \cup C) = \frac{P(S(S \cup C))}{P(S \cup C)} = \frac{P(S)}{P(S \cup C)} = \frac{0.05}{0.1225} = 0.408
\]

For part d) and e) we have additional information.
Let \( U \) denote that the material was transported by truck, \( A \) denote that the material is transported by air, and \( T \) denote that the delivery was on time.

- \( P(U) = 0.6 \)
- \( P(A) = 0.4 \)
- \( P(T | U) = 0.75 \)
- \( P(T | A) = 0.9 \)

d) 

\[
P(T) = P(T | U)P(U) + P(T | A)P(A) \\
= 0.75 * 0.6 + 0.9 * 0.4 \\
= 0.81
\]

e) 

\[
P(U | \bar{T}) = \frac{P(\bar{T} | U)P(U)}{P(\bar{T})} \\
= \frac{[1 - P(T | U)] * P(U)}{[1 - P(T)]} \\
= \frac{0.25 * 0.6}{[1 - 0.81]} \\
= 0.7895
\]

Problem 5

*Ang & Tang 3.1*

Total time \( T = T_A + T_B \) its range is \( 3+4=7 \) to \( 5+6=11 \) Divide the
sample space into $A=3$, $A=4$ and $A=5$.

$$P(T = 7) = \sum_{n=3,4,5} P(T = 7|A = n) \cdot P(A = n)$$

$$= \sum_{n=3,4,5} P(B = 7 - n) \cdot P(A)$$

$$= P(B = 4) \cdot P(A = 3)$$

$$= 0.2 \cdot 0.3 = 0.06$$

Similarly

$$P(T = 8) = P(B = 5) \cdot P(A = 3) + P(B = 4) \cdot P(A = 4)$$

$$= 0.6 \cdot 0.3 + 0.2 \cdot 0.5 = 0.28$$

$$P(T = 9) = P(B = 6) \cdot P(A = 3) + P(B = 5) \cdot P(A = 4) + P(B = 4) \cdot P(A = 5)$$

$$= 0.2 \cdot 0.3 + 0.6 \cdot 0.5 + 0.2 \cdot 0.2 = 0.4$$

$$P(T = 10) = P(B = 6) \cdot P(A = 4) + P(B = 5) \cdot P(A = 5)$$

$$= 0.2 \cdot 0.5 + 0.6 \cdot 0.2 = 0.22$$

$$P(T = 11) = P(B = 6) \cdot P(A = 5)$$

$$= 0.2 \cdot 0.2 = 0.04$$

Lets check if it adds up:

$$0.06 + 0.28 + 0.4 + 0.22 + 0.04 = 1$$
Problem 6

Ang & Tang 3.3

a) Applying the normalization condition we get:

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = 1 \\
\int_{0}^{6} c \left( x - \frac{x^2}{6} \right) \, dx = 1 \\
c \left[ \frac{x^2}{2} - \frac{x^3}{18} \right]_{0}^{6} = 1 \\
c = \frac{18}{9 \times 36 - 6^3} = \frac{1}{6}
\]

b) To avoid repeating integration, let’s work with the Cumulative Distribution Function of X, which is

\[
F_X(x) = \begin{cases} 
0 & \text{for } s \leq 0 \\
\frac{1}{6} \left[ \frac{x^2}{2} - \frac{x^3}{18} \right] = \frac{9x^2-x^3}{108} & \text{for } 0 < x \leq 6 \\
0 & \text{for } s > 12
\end{cases}
\]
Since overflow already occurred, the given event is $X > 4$ (cm), hence the conditional probability

$$P(X < 5 | X > 4) = \frac{P(X < 5 \text{ and } X > 4)}{P(X > 4)} = \frac{P(4 < X < 5)}{1 - P(X \leq 4)}$$

$$= \frac{F_X(5) - F_X(4)}{1 - F_X(4)} = \frac{(9 \times 5^2 - 5^3) - (9 \times 4^2 - 4^3)}{108 - (9 \times 4^2 - 4^3)}$$

$$= \frac{100 - 80}{108 - 80} = \frac{5}{7} = 0.714$$

c) Let $C$ denote completion of pipe replacement by the next storm, where $P(C) = 0.06$. If $C$ indeed occurs, overflow means $X > 5$, whereas if $C$ did not occur then overflow would correspond to $X > 4$. Hence the total probability of overflow is

$$P(\text{overflow}) = P(\text{overflow} | C)P(C) + P(\text{overflow} | \bar{C})P(\bar{C})$$

$$= P(X > 5) \times 0.6 + P(X > 4) \times (1 - 0.6)$$

$$= [1 - F_X(5)] \times 0.6 + [1 - F_X(4)] \times 0.4$$

$$= (1 - 100/108) \times 0.6 + (1 - 80/108) \times 0.4 = 0.148$$

Problem 7
Ang & Tang 3.5

Let $F$ be the final cost (a random variable), and $C$ be the estimated cost (a constant), hence

$$X = F/C$$

is a random variable.

a) To satisfy the normalization condition,

$$\int_{1}^{a} \frac{3}{x^2} \, dx = \left[ -\frac{3}{x} \right]_{1}^{a} = 3 - \frac{3}{a} = 1$$

$$a = 3/2 = 1.5$$
b) The given event asked for is $F$ exceeds $C$ by more than 25%. That can be written as:

$$F > 1.25 * C$$

or

$$F/C > 1.25$$

It follows that its probability is:

$$P(X > 1.25) = \int_{1.25}^{\infty} f_X(x) \, dx$$

$$= \int_{1.25}^{1.5} \frac{3}{x^2} \, dx = \left[-\frac{3}{x}\right]_{1.25}^{1.5}$$

$$= -2 - (-2.4) = 0.4$$

c) The mean

$$E(X) = \int_{1}^{1.5} x^3 \, dx = [3 \ln x]_{1}^{1.5} = 1.216$$

while

$$E(X^2) = \int_{1}^{1.5} x^2 \, dx = 3(1.5 - 1) = 1.5$$

with these, we can determine the variance

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 1.5 - 1.216395324^2 = 0.020382415$$

$$\sigma_X = \sqrt{0.020382415} = 0.143$$

**Problem 8**

*Ang & Tang 3.7*
a) The event roof failure in a given year means that the annual maximum
snow load exceeds the design value, i.e. $X > 30$, whose probability is

\[
P(X > 30) = 1 - P(X \leq 30) = 1 - F_X(30)
\]
\[
= 1 - [1 - (10/30)^4]
\]
\[
= (1/3)^4 = 1/81 = 0.0123 = p
\]

Now for the first failure to occur in the 5th year, there must be four years of
non-failure followed by one failure. We already found the value of failure, $p$,
and therefore have the value of non-failure, $1 - p$. The probability of such
an event is:

\[
(1 - p)^4p = [1 - (1/81)]^4 * (1/81) = 0.0117
\]

b) Among the next 10 years, let $Y$ count the number of years in which
failure occurs. $Y$ follows a binomial distribution with $n= 10$ and $p =1/81$,
hence

\[
P(Y < 2) = P(Y = 0) + P(Y = 1)
\]
\[
= (1 - p)^n + n(1 - p)^{n-1}p
\]
\[
= (80/81)^{10} + 10 * (80/81)^9 * (1/81)
\]
\[
= 0.994
\]