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The internal model principle for periodic disturbances with rapidly time-varying frequencies

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SUMMARY

In this paper, we study the limitations of scheduling an internal model to reject disturbances with a time-varying frequency. Hence, any adaptive method that uses scheduling and frequency estimation is also limited in the same manner. The limitations of scheduling are to investigate by posing and solving the problem of rejecting periodic disturbances from a multichannel system when the frequencies of the periodic disturbances are changing rapidly in time by designing an scheduled controller that satisfies the internal model principle. The periodic disturbances are modeled by a sum of sinusoids and the frequencies of the disturbances are used for scheduling the controller. It is shown that a controller that regulates input additive disturbances may not regulate the same disturbances added to the output of the system. This is in contrast to the classical case where the frequency of the disturbances is constant. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

It has been several decades since Davison [1], Francis and Wonham [2], Johnson [3] and others presented their work on the robust servomechanism problem. Since this time, the internal model principle has been used frequently to design controllers with asymptotic tracking and disturbance rejection properties. However, the majority of these results have been stated for disturbances that satisfy a linear time-invariant differential equation with known coefficients. Some results exist for disturbances that satisfy time-varying differential equations [4], but this problem was studied in the context of optimal control and thus the values of the coefficients are needed for all time. Having future and past knowledge of the disturbance differential equation is very restrictive and there are many application where this is infeasible. An alternative path is to use scheduling control which only requires current and past information regarding the coefficients of the time-varying differential equation that the disturbance satisfies.

When applied to a scheduling problem it has been observed in applications that the internal model principle seems to hold. That is, if the internal model is scheduled at the same rate as the disturbance, then favorable tracking and disturbance rejection properties have been observed. For example, in [5] an adaptive repetitive controller is designed by estimating the frequency of the disturbance and updating the compensator. In [6], adaptive repetitive control is used to suppress

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vibrations. In [7], an equivalence between time-varying internal models and adaptive feedforward control is shown. In [8], the internal model is updated to cancel an disturbance with an unknown frequency. In [9], the adaptive internal model principle is discussed.

In this paper, we develop the concept of a scheduled controller that satisfies the internal model principle for rapidly time-varying disturbance dynamics. It is shown that disturbances on the input and output of the plant must be considered separately, which is in contrast to the previous results. Regulation of input additive disturbances is shown to be possible even when allowing the disturbances to vary arbitrarily fast. It is also shown that a controller that completely rejects input disturbances might not reject output additive disturbances. These results are of particular interest to engineers working with optical or magnetic disk drives [5], tape drives, rotating mechanical systems, noise control, and vibrations control [6, 10], where disturbances may enter at the output of the system and can be time varying in nature.

2. BACKGROUND AND PRELIMINARIES

2.1. Linear parameter-varying systems

In this section, we define linear parameter-varying (LPV) systems and review some basic concepts related to this topic. For more information, the interested reader is referred to [11].

Definition 1
Given a compact set $\Omega \subset \mathbb{R}^{n_\rho}$, the *parameter variation set* $\mathscr{F}_\rho$ denotes the set of all piecewise continuous functions from $\mathbb{R}^+$ to Ω with a finite number of discontinuities in any finite interval.

Definition 2
Let $A_K(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_x \times n_x})$, $B_K(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_x \times n_u})$, $C_K(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_y \times n_x})$, $D_K(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_y \times n_u})$, and Ω be given, where $\chi(U, V)$ is the space of continuous functions from U to V. An n_x^{th}-order *linear*, *parameter-varying system* is any system that satisfies

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

where $\rho \in \mathscr{F}_\rho$, $x, \dot{x} \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$.

When considered as a function of time $\rho(t) \in \mathscr{F}_\rho$, the set of piecewise continuous functions from $\mathbb{R}^+$ to Ω with a finite number of discontinuities in any finite interval. However, at any instance in time $\rho \in \Omega$, a collection of vectors.

Definition 3
Suppose that the parameter set Ω is given by

$$\begin{aligned} \Omega &= \mathrm{Co}(\zeta_1, \zeta_2, \ldots, \zeta_{N_\rho}) \\ &= \left\{ \sum_{k=1}^{N_\rho} \alpha_i \zeta_i : \alpha_i \geqslant 0, \sum_{k=1}^{N_\rho} \alpha_i = 1 \right\} \end{aligned}$$

then the system is called a *polytopic linear parameter-varying system* (PLPV).

2.2. Problem formulation

The problem that we are considering is shown in Figure 1. In this figure, the plant G is a linear continuous-time system with n_u inputs, n_y outputs, n_G states, and the state-space realization is given by

$$G : \begin{bmatrix} \dot{x}_G(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_G & B_G \\ C_G & D_G \end{bmatrix} \begin{bmatrix} x_G(t) \\ u(t) \end{bmatrix}, \tag{1}$$

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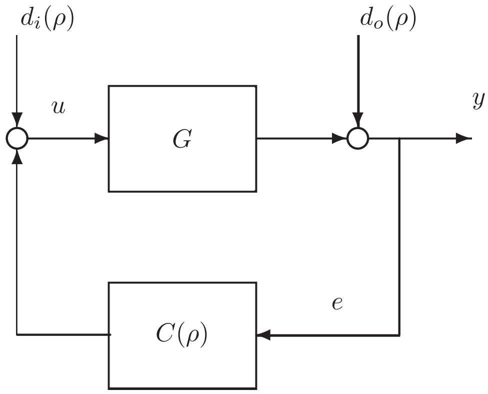

Figure 1. Problem formulation for multichannel periodic regulation. The scheduled MIMO controller $C(\rho)$ is used to reject time-varying sinusoids added to the input and output channels of the MIMO plant G.

where $A_G \in \mathbb{R}^{n_G \times n_G}$, $B_G \in \mathbb{R}^{n_G \times n_u}$, $C_G \in \mathbb{R}^{n_y \times n_G}$, $D_G \in \mathbb{R}^{n_y \times n_u}$. The input disturbance $d_i \in \mathbb{R}^{n_u}$, the output disturbance $d_o \in \mathbb{R}^{n_y}$, and the scheduled controller $C(\rho)$ are an LPV system given by

$$C(\rho)\colon \begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_C(\rho) & B_C(\rho) \\ C_C(\rho) & D_C(\rho) \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix},$$

where $A_C(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_C \times n_C})$, $B_C(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_C \times n_y})$, $C_C(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_u \times n_C})$, $D_C(\cdot) \in \chi(\mathbb{R}^{n_\rho}, \mathbb{R}^{n_u \times n_y})$, and $\rho \in \mathscr{F}_\rho$. At each instance in time, the parameter ρ belongs to the compact set $\Omega \subset \mathbb{R}^{n_\rho}$.

For the purposes of investigating the properties of the closed-loop system, we will denote the output sensitivity function as $S_o := (I - GC)^{-1}$ and the input sensitivity function as $S_i := (I - CG)^{-1}$.

Each element of the input disturbance d_i and the output disturbance d_o is assumed to satisfy

$$\begin{aligned} \dot{x}_d &= \operatorname{diag}\left(\begin{bmatrix} 0 & \omega_1(t) \\ -\omega_1(t) & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_2(t) \\ -\omega_2(t) & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \omega_{N_d}(t) \\ -\omega_{N_d}(t) & 0 \end{bmatrix} \right) x_d, \\ &\qquad \omega_i(t) \in [\underline{\omega}_i, \overline{\omega}_i] \\ &= A_d x_d \qquad (2) \\ y_d &= C_d x_d \qquad (3) \end{aligned}$$

such that $0 < \underline{\omega}_i < \overline{\omega}_i < \infty$. The parameter vector ρ is then given by

$$\rho(t) = [\omega_1(t) \;\; \omega_1(t) \;\; \ldots \;\; \omega_{N_d}(t)]^{\mathrm{T}},$$

and throughout the paper it will be assumed that the parameter vector is measurable at the current time. Thus, every input and every output channel of the plant is subjected to a sum of time-varying periodic disturbance given by

$$y_d(t) = \sum_{i=1}^{N_d} a_i \sin(\alpha_i(t)) + b_i \cos(\alpha_i(t)),$$

$$\alpha_i(t) = \int_0^t \omega_i(\tau)\, \mathrm{d}\tau.$$

The plant is considered to be known and the problem is to design the scheduled controller $C(\rho)$ to achieve output regulation for all periodic input disturbances d_i and output disturbances d_o satisfying (2) in each channel.

For notational simplicity, we will assume that d_i and d_o are produced by

$$\begin{aligned} \dot{x}_i &= A_{H_i}(\rho)x_i, \quad \dot{x}_o = A_{H_o}(\rho)x_o, \\ d_i &= C_{H_i}x_i, \quad d_o = C_{H_o}x_o, \end{aligned} \tag{4}$$

where

$$A_{H_i} = \mathrm{diag}\left(\underbrace{A_d, A_d, \ldots, A_d}_{n_u \text{ times}}\right), \quad A_{H_o} = \mathrm{diag}\left(\underbrace{A_d, A_d, \ldots, A_d}_{n_y \text{ times}}\right),$$

$$C_{H_i} = \mathrm{diag}\left(\underbrace{C_d, C_d, \ldots, C_d}_{n_u \text{ times}}\right), \quad C_{H_o} = \mathrm{diag}\left(\underbrace{C_d, C_d, \ldots, C_d}_{n_y \text{ times}}\right).$$

These systems have the general form given by

$$\dot{x}(t) = A(\rho)x(t),$$

where $\rho(t) \in \mathscr{F}_\rho$ is a piecewise continuous function with a finite number of discontinuities in any finite interval that maps $\mathbb{R}^+$ to Ω, and Ω is a compact space.

It should be noted that when ω_i, $i = 1, \ldots, N_d$, are known constants, this problem is a subset of the general servocompensator problem considered by Davison [1, 12], Francis and Wonham [2], and more recently by de Roover *et al.* [13]. However, in this paper we will extend the results to include compensation for time-varying systems.

2.3. Quadratic stability

In order to study the stability of the closed-loop system in the presence of rapidly changing disturbance dynamics we will use the concept of quadratic stability.

Definition 4 (*Quadratic Stability*)
The LPV system $\dot{x}(t) = A(\rho)x(t)$ is *quadratically stable* if there exists a matrix $P > 0$ such that

$$PA(\rho) + A(\rho)^{\mathrm{T}}P < 0 \quad \forall\ \rho \in \Omega.$$

Proposition 1
For a PLPV system quadratic stability reduces to

$$PA(\zeta_i) + A(\zeta_i)^{\mathrm{T}}P < 0 \quad \forall\ i,$$

where the same matrix $P > 0$ should satisfy the LMI condition for each vertex of Ω.

For quadratic stability of feedback systems, it is useful to investigate block triangular systems. For this purpose we introduce the following lemma.

Lemma 1 (*Xie and Eisaka* [*14, Lemma 2*])
Consider the block matrix $Q(\theta)$, where

$$Q(\theta) = \begin{bmatrix} Q_{11}(\theta) & 0 \\ Q_{21}(\theta) & Q_{22}(\theta) \end{bmatrix}.$$

Suppose the matrices $Q_{11}(\theta)$ and $Q_{22}(\theta)$ are quadratically stable, as defined in Definition 4, and $Q(\theta)$ is a continuous-bounded function then $Q(\theta)$ is quadratically stable.

2.4. Servocompensator results for constant frequency

Francis and Wonham [2] showed that the purpose of the internal model principle is to place closed-loop blocking zeros where the unstable poles of the disturbance are located. This placement of the closed-loop blocking zeros gives a controller that asymptotically rejects periodic disturbances (with constant frequency) in the presence of non-destabilizing parametric uncertainty. A single-input-single-output application of this principle requires that the controller denominator polynomial can be factored into the disturbance polynomial and another part for stability [10].

From [2, 12], we have the following regarding linear MIMO systems that are subjected to disturbances that satisfy a linear time-invariant differential equation.

Theorem 1 (*Davison* [*1, Lemma 1*])
The necessary and sufficient conditions that there exist an internal stabilizing controller for the plant G, with a realization given by (1), such that $\lim_{t\to\infty}\|y(t)\|=0$ for all input and output disturbances satisfying (2) in each channel with $\omega_i(t)=c_i\in\mathbb{R}$ is that the following all hold:

(i) (A_G, B_G) is stabilizable.
(ii) $\operatorname{rank}\left(\begin{bmatrix} A_G-\lambda I & B_G \\ C_G & D_G \end{bmatrix}\right)=n_G+n_{y_G} \quad \forall\ \lambda\in\sigma(A_d)$.
(iii) (A_G, C_G) is detectable.

A controller that satisfies the above will have a realization [2, 12] given by

$$C(s)=\left[\begin{array}{cc|c} A_{C11} & 0 & B_{C1} \\ A_{C21} & A_{C22} & B_{C2} \\ \hline C_{C2} & C_{C2} & D_C \end{array}\right] \tag{5}$$

or equivalently

$$C(s)=\left[\begin{array}{cc|c} A_{C11} & A_{C12} & B_{C1} \\ 0 & A_{C22} & B_{C2} \\ \hline C_{C2} & C_{C2} & D_C \end{array}\right], \tag{6}$$

where A_{C11} in the first realization or A_{C22} in the second is a model of the disturbance such that the minimal polynomial of the disturbance divides A_{C11} or A_{C22} at least n_{y_G} times.

3. SERVOCOMPENSATOR DESIGN FOR TIME-VARYING FREQUENCY

3.1. Necessary conditions for regulation

Since the class of disturbances being considered here contains the constant frequency case, the basic necessary condition for the existence of a parameter-varying servocompensator is easily obtained from the constant case.

Corollary 1
The necessary conditions that there exist an internal stabilizing controller for the plant G, with a realization given by (1), such that $\lim_{t\to\infty}\|y(t)\|=0$ for all input and output disturbances satisfying (2) in each channel is that Theorem 1 holds for each fixed $\omega\in[\underline{\omega}_i,\overline{\omega}_i]\forall i$.

It should be noted that the result mentioned in Corollary 1 is not sufficient in the case that the frequency is time varying. The remainder of the paper will clearly prove this point, where it is shown that the phase delay applied to the control signal as it passes through the plant response is the reason for this phenomenon. Notice, in addition, that this is much more restrictive than the constant frequency case in that the plant cannot have zeros at any frequency in $[\underline{\omega}_i,\overline{\omega}_i]$ for each i.

3.2. Sufficient conditions for regulation

In this section, we consider the asymptotic regulation of input and output disturbances that satisfy (2) separately.

3.2.1. Input disturbances.

Lemma 2
Consider the state-space realization for the controller $C(\rho)$ given by

$$C(\rho)=\left[\begin{array}{cc|c} A_{C11}(\rho) & 0 & B_{C1}(\rho) \\ A_{C21}(\rho) & A_{H_i}(\rho) & B_{C2}(\rho) \\ \hline C_{C1}(\rho) & C_{H_i}(\rho) & 0 \end{array}\right] \tag{7}$$

the input disturbance $d_i(t)$ satisfying

$$\dot{x}_i = A_{H_i}(\rho)x_i, \tag{8}$$

$$d_i = C_{H_i}x_i \tag{9}$$

and the state-space realization for the plant given by (1).

Suppose that the gains $A_{C11}(\rho)$, $A_{C21}(\rho)$, $B_{C1}(\rho)$, $B_{C2}(\rho)$, and $C_{C1}(\rho)$ are chosen such that the feedback system of G and $C(\rho)$ is stable in the presence of the time-varying scheduling parameter $\rho \in \mathscr{F}_\rho$, then

$$\lim_{t\to\infty} \|y(t)\| = 0,$$

where $y = G(u+d_i)$ and $u = C(\rho)y$.

Proof
To consider observability of the disturbance states, consider the A and C matrices of $S_i(\rho)H_i$. Define $R_i(\rho) := I - D_C(\rho)D_G$, and consider the following realization for the controller:

$$C(\rho)=\left[\begin{array}{cc|c} A_{C11}(\rho) & A_{C12}(\rho) & B_{C1}(\rho) \\ A_{C21}(\rho) & A_{C22}(\rho) & B_{C2}(\rho) \\ \hline C_{C2}(\rho) & C_{C2}(\rho) & D_C(\rho) \end{array}\right].$$

Then, we obtain (dropping ρ)

$$\left[\begin{array}{cccc} Z_{11} & Z_{12} & Z_{13} & B_G R_i^{-1} C_{H_i} \\ Z_{21} & Z_{22} & Z_{23} & B_{C1} D_G R_i^{-1} C_{H_i} \\ Z_{31} & Z_{32} & Z_{33} & B_{C2} D_G R_i^{-1} C_{H_i} \\ 0 & 0 & 0 & A_{H_i} \\ \hline R_i^{-1} D_C C_G & R_i^{-1} C_{C1} & R_i^{-1} C_{C2} & R_i^{-1} C_{H_i} \end{array}\right],$$

where

$$Z_{11} = A_G + B_G R_i^{-1} D_C C_G, \quad Z_{12} = B_G R_i^{-1} C_{C1}, \quad Z_{13} = B_G R_i^{-1} C_{C2},$$

$$Z_{21} = B_{C1} C_G + B_{C1} D_G R_i^{-1} D_C C_G, \quad Z_{22} = A_{C11} + B_{C1} D_G R_i^{-1} C_{C1},$$

$$Z_{23} = A_{C12} + B_{C1} D_G R_i^{-1} C_{C2}, \quad Z_{31} = B_{C2} C_G + B_{C2} D_G R_i^{-1} D_C C_G,$$

$$Z_{32} = A_{C21} + B_{C2} D_G R_i^{-1} C_{C1}, \quad Z_{33} = A_{C22} + B_{C2} D_G R_i^{-1} C_{C2}.$$

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Now, apply the constant similarity transformation

$$T=\begin{bmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & I\\ 0 & 0 & 0 & I \end{bmatrix}$$

and if we choose $C_{C2}(\rho)=C_{H_i}$, $A_{C22}(\rho)=A_{H_i}(\rho)$, and $A_{C12}(\rho)=0$, then we obtain

$$\left[\begin{array}{cccc} Z_{11} & Z_{12} & Z_{13} & 0\\ Z_{21} & Z_{22} & Z_{23} & 0\\ Z_{31} & Z_{32} & Z_{33} & 0\\ 0 & 0 & 0 & A_{H_i}(\rho)\\ \hline R_i(\rho)^{-1}D_C(\rho)C_G & R_i(\rho)^{-1}C_{C1}(\rho) & R_i(\rho)^{-1}C_H & 0 \end{array}\right]. \tag{10}$$

This implies that x_i is unobservable and hence the input–output mapping does not depend upon the disturbance states. If the remaining 3×3 block matrix Z is made stable, then all signals will decay to the origin. It is straightforward to show that S_i satisfies

$$\dot{x}(t)=Zx(t)+Bu(t)$$

for some B. Thus, the stability of Z is equivalent to the stability of the feedback system of G and $C(\rho)$. □

Note that the requirement of stability in the preceding lemma is not equivalent to checking the eigenvalues of the closed-loop system at each ρ, this would be the case if $\rho(t)$ changes very slowly. Much more is required and an obvious sufficient choice is to pick the control gains to quadratically stabilize [15] the closed-loop system. It should also be noted that constant state-space matrices for the controller A_{C11}, A_{C21}, B_{C1}, B_{C2}, and C_{C1} can be used if stability in the presence of ρ is upheld. Only scheduling the internal model parameters $A_{H_i}(\rho)$ and $C_{H_i}(\rho)$ is required for complete regulation. The additional scheduling of $A_{C11}(\rho)$, $A_{C21}(\rho)$, $B_{C1}(\rho)$, $B_{C2}(\rho)$, and $C_{C1}(\rho)$ will improve the range of ρ for which stability holds. Moreover, the convergence rate to obtain complete regulation can be improved with the additional scheduling since more freedom is introduced into the LMI condition for quadratic stability in Definition 4.

There still remains a lot of freedom in the controller parameterization given in (7). In fact, this is really an over-parameterization and by picking the control gains in a smart manner one can arrive at two separate yet standard control problems. The following theorem explains this point.

Theorem 2
Consider the system given by (1) and the input disturbance that satisfy

$$\dot{x}_i=A_{H_i}(\rho)x_i, \quad d_i=C_{H_i}x_i,$$

where $A_{H_i}(\rho)=\mathrm{diag}(\underbrace{A_d(\rho), A_d(\rho), \ldots, A_d(\rho)}_{n_u \text{ times}})$, $A_d(\rho)$ satisfies (2), and

$$\rho(t)\in\left\{\sum_{k=1}^{N_\rho}\alpha_i(t)\zeta_i : \alpha_i(t)\geqslant 0, \sum_{k=1}^{N_\rho}\alpha_i(t)=1\right\}.$$

Suppose that there exists $P_1>0$, $P_2>0$, M_i, N_i such that

$$P_1\mathscr{A}_i - M_i\bar{C} + \mathscr{A}_i^{\mathrm{T}}P_1 - \bar{C}^{\mathrm{T}}M_i^{\mathrm{T}}<0, \tag{11}$$

$$P_2A_G^{\mathrm{T}} + N_i^{\mathrm{T}}B_G^{\mathrm{T}} + AP_2 + B_GN_i<0 \quad \forall\, i, \tag{12}$$

where

$$\mathscr{A}_i=\begin{bmatrix} A_G & B_G C_{H_i} \\ 0 & A_{H_i}(\zeta_i) \end{bmatrix}, \quad \bar{C}=[C_G \quad D_G C_{H_i}]$$

then the scheduled controller given by

$$C(\rho)=\left[\begin{array}{cc|c} A_G+B_G C_{C1}(\rho)-B_{C1}(\rho)(D_G C_{C1}(\rho)+C_G) & 0 & B_{C1}(\rho) \\ -B_{C2}(\rho)(C_G+D_G C_{C1}(\rho)) & A_{H_i}(\rho) & B_{C2}(\rho) \\ \hline C_{C1}(\rho) & C_{H_i} & 0 \end{array}\right], \tag{13}$$

where

$$\begin{bmatrix} B_{C1}(\rho) \\ -B_{C2}(\rho) \end{bmatrix}=\sum_{k=1}^{N_\rho} \alpha_i(t) P_1^{-1} M_i, \quad C_{C1}(\rho)=\sum_{k=1}^{N_\rho} \alpha_i(t) N_i P_2^{-1}$$

will quadratically stabilize the closed-loop system and regulate the input disturbance, i.e. $\lim_{t\to\infty}\|y(t)\|=0$.

Proof
From Lemma 2, if we can stabilize the upper 3×3 block of (10) (denoted as Z), then all signals will decay to the origin. Since we are considering only the asymptotic regulation of disturbances and the stability of the closed-loop system a sufficient choice is $D_C(\rho)=0$. Next, apply the following constant coordinate transformation:

$$T=\begin{bmatrix} I & -I & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

into Z, and choose $A_{C11}(\rho)=A_G+B_G C_{C1}(\rho)-B_{C1}(\rho)D_G C_{C1}(\rho)-B_{C1}(\rho)C_G$ and $A_{C21}(\rho)=-B_{C2}(\rho)(C_G+D_G C_{C1}(\rho))$ to obtain

$$TZT^{-1}=\begin{bmatrix} A_G-B_{C1}(\rho)C_G & (B_G-B_{C1}(\rho)D_G)C_{H_i} & 0 \\ B_{C2}(\rho)C_G & A_{H_i}(\rho)+B_{C2}(\rho)D_G C_{H_i} & 0 \\ B_{C1}(\rho)C_G & B_{C1}(\rho)D_G C_{H_i} & A_G+B_G C_{C1}(\rho) \end{bmatrix},$$

where it is now obvious that we can take C_{C1} to be a constant matrix if desired. From here we can apply Lemma 1 to conclude that if the block diagonals are quadratically stable, then so is the system. If (11) and (12) are satisfied with $P_1>0$ and $P_2>0$, then each sub-system is quadratically stable. □

Remark 1
The interpretation is that the controller is placing 'time-varying' invariant zeros in the closed-loop system that block the 'time-varying' eigenvalues of the disturbance model. It should be noted that the classical time-invariant analysis using invariant zeros and eigenvalues cannot be used to make this conclusion.

Remark 2
For a fixed frequency, this is the same controller as in [13], however, the derivation given here considers making the disturbance unobservable in the system output and does not use the dual system.

3.2.2. Output disturbances. Next, we consider applying the controller in Lemma 2 to a plant with output additive disturbances. In this case, when the frequency is rapidly varying, it will be shown

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that the controller that regulated input disturbances will not be able to regulate output disturbances for all plants.

Theorem 3
Consider the state-space realization for the controller $C(\rho)$ given by

$$C(\rho)=\left[\begin{array}{cc|c} A_{C11}(\rho) & 0 & B_{C1}(\rho) \\ A_{C21}(\rho) & A_{H_o}(\rho) & B_{C2}(\rho) \\ \hline C_{C1}(\rho) & C_{H_o}(\rho) & 0 \end{array}\right] \tag{14}$$

the output disturbances $d_o(t)$ satisfying

$$\dot{x}_o = A_{H_o}(\rho)x_o, \tag{15}$$

$$d_o = C_{H_o}x_o \tag{16}$$

and the state-space realization for the plant given by (1).

Suppose that the gains $A_{C11}(\rho)$, $A_{C21}(\rho)$, $B_{C1}(\rho)$, $B_{C2}(\rho)$, and $C_{C1}(\rho)$ are chosen such that the closed-loop system of G and $C(\rho)$ is stable in the presence of the time-varying parameter vector ρ. Then there exists a plant satisfying the conditions of Theorem 1 for each $\omega\in[\underline{\omega}_i, \overline{\omega}_i]\forall i$, a set of initial conditions, and a $\rho(t)\in\mathscr{F}_\rho$ such that $\lim_{t\to\infty}\|y(t)\|\neq 0$, where $y=Gu+d_o$ and $u=C(\rho)y$.

Proof
We need to show that there exists a plant satisfying the given conditions, a set of initial conditions, a controller, and a $\rho(t)\in\mathscr{F}_\rho$, such that $\lim_{t\to\infty}\|y(t)\|\neq 0$.

For example, $A_G=-1$, $B_G=1$, $C_G=1$, $D_G=0$ satisfies the given conditions. Let us assume that $d_o(t)$ is a single time-varying sinusoid (this implies that $\rho(t)=\omega(t)$).

In this case the output is written as

$$\dot{x}_G(t) = -x_G(t)+u(t), \tag{17}$$

$$y(t) = x_G(t)+d_o(t), \tag{18}$$

$$d_o(t)=A\sin(\alpha(t)), \quad \alpha(t)=\int_0^t \omega(\tau)\,\mathrm{d}\tau. \tag{19}$$

This can be written as

$$y(t)=\mathrm{e}^{-t}x(0)+\int_0^t \mathrm{e}^{\tau-t}u(\tau)\,\mathrm{d}\tau+d_o(t)$$

and for regulation it is required that eventually (as $t\to\infty$)

$$\int_0^t \mathrm{e}^{\tau-t}u(\tau)\,\mathrm{d}\tau=-d_o(t).$$

Taking derivatives and denote u^* as the steady-state control law gives

$$u^*(t)-\int_0^t \mathrm{e}^{\tau-t}u^*(\tau)\,\mathrm{d}\tau=-\dot{d}_o(t)=-\omega(t)A\cos(\alpha(t))$$

and substituting gives

$$u^*(t)+d_o(t)=-\dot{d}_o(t)=-\omega(t)A\cos(\alpha(t)),$$

$$u^*(t)=-\omega(t)A\cos(\alpha(t))-A\sin(\alpha(t)).$$

However, a controller that quadratically stabilizes the system for $\omega \in [1, 5]$ and satisfies the theorem is given by

$$C(\rho)=\left[\begin{array}{ccc|c} -12 & 0 & 0 & 1 \\ 1 & 0 & \omega & -1 \\ 1 & -\omega & 0 & -1 \\ \hline -10 & 0 & 1 & 0 \end{array}\right].$$

Now, define the coordinate transformation given by $z = Tx$ with

$$T=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin(\alpha(t)) & \cos(\alpha(t)) \\ 0 & -\cos(\alpha(t)) & \sin(\alpha(t)) \end{bmatrix}$$

then in the new coordinates the controller becomes

$$C(\rho)=\left[\begin{array}{ccc|c} -12 & 0 & 0 & 1 \\ \sin(\alpha(t))+\cos(\alpha(t)) & 0 & 0 & \cos(\alpha(t))-\sin(\alpha(t)) \\ \sin(\alpha(t))-\cos(\alpha(t)) & 0 & 0 & \cos(\alpha(t))+\sin(\alpha(t)) \\ \hline -10 & \cos(\alpha(t)) & \sin(\alpha(t)) & 0 \end{array}\right]$$

so that $u(t) = -10x_{k1} + x_{k2}\cos(\alpha(t)) + x_{k3}\sin(\alpha(t))$. The control signal error $u_e(t) := u(t) - u^*(t)$ is given by

$$u_e(t) = -10x_{k1} + (x_{k2} + \omega(t)A)\cos(\alpha(t)) + (x_{k3} + A)\sin(\alpha(t)).$$

Now, suppose that $\omega(t) = \underline{\omega}$ is constant for $0 \leqslant t < T_1$, pick T_1 such that $\alpha(T_1) = 2\pi N_1$ (for some natural number N_1), and we pick the initial conditions to be $x_{k1}(0) = 0$, $x_{k2}(0) = -\omega(t)A$, and $x_{k3}(0) = -A$, and $x(0) = 0$ so that $y(t) = 0$, $0 \leqslant t < T_1$. Thus $u_e(t) = 0$ for $0 \leqslant t < T_1$. Now, suppose that $\omega(T_1) = \overline{\omega}$ then since the controller states are integrated and cannot change instantaneously we obtain

$$|u_e(T)| = |(\underline{\omega} - \overline{\omega})||A|\cos(\alpha(T_1)) = |(\underline{\omega} - \overline{\omega})||A| > 0.$$

This error will decrease to zero if $\omega(t)$ is held constant, since the system is exponentially stable.

Next, choose an ε such that $\varepsilon < |(\underline{\omega} - \overline{\omega})||A|$ and let $\omega(t) = \overline{\omega}$ for $T_1 \leqslant t < T_2$, where T_2 is chosen such that $\alpha(T_2) = 2\pi N_2$, $|10x_{k1}| < \varepsilon/2$, and $|x_{k2} + \overline{\omega}A| < \varepsilon/2$ hold. Let $\omega(T_2) = \underline{\omega}$. This can be done since the closed-loop system is stable and when $\omega(t)$ is a constant, the results from standard servocompensator theory hold, which imply that $u_e(t) \to 0$ exp. fast. Then, we obtain

$$\begin{aligned} |u_e(T_2) + 10x_{k1}(T_2)| &= |(x_{k2}(T_2) + \omega(T_2)A)\cos(\alpha(T_2))|, \\ |u_e(T_2)| + 10|x_{k1}(T_2)| \geqslant |u_e(T_2) + 10x_{k1}(T_2)| &= |x_{k2}(T_2) + \omega(T_2)A|, \\ |u_e(T_2)| &\geqslant |x_{k2}(T_2) + \omega(T_2)A| - |\varepsilon/2| \\ &\geqslant |-\overline{\omega}A + \underline{\omega}A \pm \varepsilon/2| - |\varepsilon/2| \\ &\geqslant |\underline{\omega} - \overline{\omega}||A| - \varepsilon > 0. \end{aligned}$$

Continuing on this process defines a profile for $\omega(t) = \rho(t) \in \mathscr{F}_\rho$ that will not result in regulation. □

Remark 3

Under this structure the controller produces continuous signals. As the integral action of the plant a discontinuous signal is required to cancel a quickly changing disturbance on the output.

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Remark 4
This theorem clearly proves that input and output disturbances need separate treatment for the varying frequency case and the proof indicates why the same phenomenon does not appear in the constant frequency case.

Following Remark 4, output disturbances require separate treatment from input disturbances. Since the control signal must propagate through the plant before reaching the output the controls signal lags behind quickly changing disturbances. This is due to the fact that general linear time-varying systems do not commute with each other even in the SISO case. If, however, the disturbance model H_o commutes with the feedback system, then regulation is assured.

Before we proceed, we need to define more regarding the disturbance model H_o if we are to deal with commuting properties. Thus, we use the following notation:

$$H_o: \quad \dot{x}_o(t) = A_{H_o}(\rho)x_o(t) + B_{H_o}u(t),$$

$$y(t) = C_{H_o}x_o(t) + D_{H_o}u(t),$$

where $A_{H_o}(\rho)$ and C_{H_o} are defined in (4). For design purposes we may choose $B_{H_o} = C_{H_o}^{\mathrm{T}}$ and $D_{H_o} = 0$.

Lemma 3
Suppose that $H_o S_o = S_o H_o$ then the scheduled controller is given by

$$C = \left[\begin{array}{cc|c} A_{C11}(\rho) & A_{C12}(\rho) & B_{C1}(\rho) \\ 0 & A_{H_o}(\rho) & B_{H_o} \\ \hline C_{C1}(\rho) & C_{C2}(\rho) & 0 \end{array}\right], \tag{20}$$

where the gains $A_{C11}(\rho)$, $A_{C12}(\rho)$, $B_{C1}(\rho)$, $C_{C1}(\rho)$, and $C_{C2}(\rho)$ are chosen such that the feedback system of G and $C(\rho)$ is stable in the presence of the time-varying scheduling parameter $\rho \in \mathscr{F}_\rho$ is sufficient to asymptotically regulate the output of the plant ($\lim_{t\to\infty} \|y(t)\|$) with realization given by (1) in the presence of output disturbances that satisfy (2) in each channel.

Proof
Let $R_o := (I - D_G D_C(\rho))$ then

$$HS_o = \left[\begin{array}{cccc|c} A_{H_o} & B_{H_o}R_o^{-1}C_G & B_{H_o}R_o^{-1}D_G C_{C1} & B_{H_o}R_o^{-1}D_G C_{C2} & B_{H_o}R_o^{-1} \\ 0 & \tilde{Z}_{11} & \tilde{Z}_{12} & \tilde{Z}_{13} & B_G D_C R_o^{-1} \\ 0 & \tilde{Z}_{21} & \tilde{Z}_{22} & \tilde{Z}_{23} & B_{C1}R_o^{-1} \\ 0 & \tilde{Z}_{31} & \tilde{Z}_{32} & \tilde{Z}_{33} & B_{C2}R_o^{-1} \\ \hline C_{H_o} & D_{H_o}R_o^{-1}C_G & D_{H_o}R_o^{-1}D_G C_{C1} & D_{H_o}R_o^{-1}D_G C_{C2} & D_{H_o}R_o^{-1} \end{array}\right],$$

where

$$\tilde{Z}_{11} = A_G + B_G D_C R_o^{-1} C_G, \quad \tilde{Z}_{12} = B_G C_{C1} + B_G D_C R_o^{-1} D_G C_{C1},$$

$$\tilde{Z}_{13} = B_G C_{C2} + B_G D_C R_o^{-1} D_G C_{C2}, \quad \tilde{Z}_{21} = B_{C1} R_o^{-1} C_G,$$

$$\tilde{Z}_{22} = A_{C11} + B_{C1} R_o^{-1} D_G C_{C1}, \quad \tilde{Z}_{23} = A_{C12} + B_{C1} R_o^{-1} D_G C_{C2},$$

$$\tilde{Z}_{31} = B_{C2} R_o^{-1} C_G, \quad \tilde{Z}_{32} = A_{C21} + B_{C2} R_o^{-1} D_G C_{C1}, \quad \tilde{Z}_{33} = A_{C22} + B_{C2} R_o^{-1} D_G C_{C2},$$

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and if we apply the coordinate transformation

$$T=\begin{bmatrix} I & 0 & 0 & -I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and pick $B_{C2}=B_{H_o}$, $A_{C21}=0$, and $A_{C22}=A_{H_o}$, then we obtain

$$HS_o=\left[\begin{array}{cccc|c} A_{H_o} & 0 & 0 & 0 & 0 \\ 0 & \tilde{Z}_{11} & \tilde{Z}_{12} & \tilde{Z}_{13} & B_G D_C R_o^{-1} \\ 0 & \tilde{Z}_{21} & \tilde{Z}_{22} & \tilde{Z}_{23} & B_{C1} R_o^{-1} \\ 0 & \tilde{Z}_{31} & \tilde{Z}_{32} & \tilde{Z}_{33} & B_{C2} R_o^{-1} \\ \hline C_{H_o} & D_{H_o} R_o^{-1} C_G & D_{H_o} R_o^{-1} D_G C_{C1} & C_{H_o}+D_{H_o} R_o^{-1} D_G C_{C2} & D_{H_o} R_o^{-1} \end{array}\right]. \quad (21)$$

The disturbance states are uncontrollable and hence the input–output mapping does not depend on them.

It is straightforward to show that S_o satisfies

$$\dot{x}(t)=\tilde{Z}x(t)+\tilde{B}u(t)$$

for some $\tilde{B}$. Thus, the stability of Z is equivalent to the stability of the feedback system of G and $C(\rho)$. □

Remark 5

This result agrees with constant frequency case since $H_o=Ih_o$ implies that $H_oS_o=S_oH_o$ when the frequency is constant.

In the case if the output disturbance model H_o commutes with the feedback system, then the following theorem can be used to design the gains of the controller.

Theorem 4

Consider the system given in (1) and the output disturbance that satisfy

$$\dot{x}_o=A_{H_o}(\rho)x_o, \qquad d_o=C_{H_o}x_o,$$

where $A_{H_o}(\rho)=\text{diag}(\underbrace{A_d(\rho), A_d(\rho), \ldots, A_d(\rho)}_{n_y \text{ times}})$, $A_d(\rho)$ satisfies (2), and

$$\rho(t)\in\left\{\sum_{k=1}^{N_\rho}\alpha_i(t)\zeta_i : \alpha_i(t)\geqslant 0, \sum_{k=1}^{N_\rho}\alpha_i(t)=1\right\}.$$

Suppose that there exists $\tilde{P}_1>0$, $\tilde{P}_2>0$, $\tilde{M}_i$, $\tilde{N}_i$ such that

$$\tilde{\mathscr{A}}_i\tilde{P}_1-\tilde{\mathscr{B}}\tilde{M}_i+\tilde{P}_1\tilde{\mathscr{A}}_i^{\mathrm{T}}-\tilde{M}_i^{\mathrm{T}}\tilde{\mathscr{B}}^{\mathrm{T}}<0 \quad (22)$$

$$\tilde{P}_2A_G+\tilde{N}_iC_G+A_G^{\mathrm{T}}\tilde{P}_2+C_G^{\mathrm{T}}\tilde{N}_i^{\mathrm{T}}<0 \quad \forall\, i, \quad (23)$$

where

$$\tilde{\mathscr{A}}_i=\begin{bmatrix} A_G & 0 \\ -B_{H_o}C_G & A_{H_o}(\zeta_i)\end{bmatrix}, \quad \tilde{\mathscr{B}}^{\mathrm{T}}=[B_G \quad -B_{H_o}]$$

then the scheduled controller given by

$$C(\rho)=\left[\begin{array}{cc|c} A_G+B_{C1}C_G-(B_{C1}D_G+B_G)C_{C1} & -(B_{C1}D_G+B_G)C_{C2} & B_{C1}(\rho) \\ 0 & A_{H_o}(\rho) & B_{H_o}(\rho) \\ \hline C_{C1}(\rho) & C_{C2}(\rho) & 0 \end{array}\right], \tag{24}$$

where

$$[C_{C1}(\rho)\ \ C_{C2}(\rho)]=\sum_{k=1}^{N_\rho}\alpha_i(t)M_i\tilde{P}_1^{-1}, \quad B_{C1}(\rho)=\sum_{k=1}^{N_\rho}\alpha_i(t)\tilde{P}_2^{-1}N_i$$

will quadratically stabilize the closed-loop system and regulate the output disturbance if $H_oS_o=S_oH_o$.

Proof

From Lemma 3, if we can stabilize the lower 3×3 block of (21) (denoted as $\tilde{Z}$)

$$\tilde{Z}=\begin{bmatrix} A_G+B_GD_CR_o^{-1}C_G & B_GC_{C1}+B_GD_CR_o^{-1}D_GC_{C1} & B_GC_{C2}+B_GD_CR_o^{-1}D_GC_{C2} \\ B_{C1}R_o^{-1}C_G & A_{C11}+B_{C1}R_o^{-1}D_GC_{C1} & A_{C12}+B_{C1}R_o^{-1}D_GC_{C2} \\ B_{H_o}R_o^{-1}C_G & B_{H_o}R_o^{-1}D_GC_{C1} & A_{H_o}+B_{H_o}R_o^{-1}D_GC_{C2} \end{bmatrix}$$

then all signals will decay to the origin. Since we are only considering the asymptotic regulation of disturbances and stability of the closed-loop system a sufficient choice is $D_C(\rho)=0$. If we choose $A_{C11}=A_G+B_{C1}C_G-(B_{C1}D_G+B_G)C_{C1}$ and $A_{C12}=-(B_{C1}D_G+B_G)C_{C2}$, then we obtain

$$\tilde{Z}=\begin{bmatrix} A_G & B_GC_{C1} & B_GC_{C2} \\ B_{C1}C_G & A_G+B_{C1}C_G-B_GC_{C1} & -B_GC_{C2} \\ B_{H_o}C_G & B_{H_o}D_GC_{C1} & A_{H_o}+B_{H_o}D_GC_{C2} \end{bmatrix}.$$

Next, apply the following constant coordinate transformation:

$$T=\begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

to obtain

$$T\tilde{Z}T^{-1}=\begin{bmatrix} A_G+B_{C1}C_G & 0 & 0 \\ B_{C1}C_G & A_G-B_GC_{C1} & -B_GC_{C2} \\ B_{H_o}C_G & B_{H_o}(D_GC_{C1}-C_G) & A_{H_o}+B_{H_o}D_GC_{C2} \end{bmatrix}.$$

Thus, by Lemma 1, we can make each diagaonal block quadratically stable to quadratically stabilize the whole system. If $\tilde{P}_1>$ and $\tilde{P}_2>0$ satisfy (22) and (23), then the system is quadratically stable. □

Remark 6

For a fixed frequency, this is the same controller as in [13], however, the derivation given here considers making the disturbance uncontrollable in the system output.

The assumption $H_oS_o=S_oH_o$, indicating that the disturbance model commutes with the closed-loop system, is very restrictive. Instead of restricting the class of disturbance models, we can try to find sufficiency by restricting the class of plants that we are considering. Suppose that G is stable and stably (causally) invertible. Then we may form an equivalence between input and output disturbances by the following relation:

$$y=G(u+d_i)=Gu+d_o$$

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if $d_i = G^{-1} d_o$. This gives the following realization for d_i:

$$\dot{x}_i(t) = \begin{bmatrix} A_G - B_G D_G^{-1} C_G & -B_G D_G^{-1} C_{H_o} \\ 0 & A_{H_o}(\rho) \end{bmatrix} x_i(t), \quad d_i(t) = [D_G^{-1} C_G \quad 0] x_i(t),$$

where (A_G, B_G, C_G, D_G) are the state-space matrices for G. This new system is still a PLPV system with the same parameter set. With this approach, we arrive at the next result.

Corollary 2
Suppose that G is stable and stably (causally) invertible, and the scheduled controller $C(\rho)$ satisfies Theorem 2 with

$$\dot{x}_i = \begin{bmatrix} A_G - B_G D_G^{-1} C_G & -B_G D_G^{-1} C_{H_o} \\ 0 & A_{H_o}(\rho) \end{bmatrix} x_i(t), \quad d_i(t) = [D_G^{-1} C_G \ \ 0] x_i(t)$$

then $C(\rho)$ regulates all output disturbances satisfying

$$\dot{x}_o = A_{H_o}(\rho) x_i(t), \quad d_o(t) = C_{H_o} x_o(t).$$

Proof
Since $C(\rho)$ regulates d_i, we obtain $\lim_{t\to\infty} \|S_i d_i\| = 0$. However

$$\begin{aligned} G S_i d_i &= G(I - C(\rho)G)^{-1} d_i \\ &= G(I - C(\rho)G)^{-1} G^{-1} d_o \\ &= G(G - GC(\rho)G)^{-1} d_o \\ &= (I - GC(\rho))^{-1} d_o \\ &= S_o d_o \end{aligned}$$

holds. Therefore

$$\lim_{t\to\infty} \|G S_i d_i(t)\| = \lim_{t\to\infty} \|S_o d_o(t)\| = 0.$$

Remark 7
In some special cases, when $G^{-1} d_o$ is still a PLPV system in ρ, the inversion of G can be non-causal. An example of this is shown in Section 4.

Remark 8
This corollary indicates that if an output additive disturbance is present with a time-varying frequency, performance will be greatly increased if an inverse or approximate inverse of the plant is used for the control design.

4. SIMULATION

To demonstrate the differences between time-varying and constant frequencies we will consider the plant given by

$$G = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right] \tag{25}$$

and a controller that quadratically stabilizes the system and achieves input regulation for $1\leqslant\omega(t)\leqslant 5$ given by

$$C(\omega)=\left[\begin{array}{ccc|c} -12 & 0 & 0 & 1 \\ 1 & 0 & \omega(t) & -1 \\ 1 & -\omega(t) & 0 & -1 \\ \hline -10 & 0 & 1 & 0 \end{array}\right]. \tag{26}$$

Sim. 1 In the first simulation, the disturbance $d_i(t)=\sin(\alpha(t))$ will be applied to the input of the system and the controller in (26) is applied to the system. This controller satisfies Theorem 2 with B_{C1}, B_{C2}, C_{C1} as constants. This simulation will demonstrate the validity of Theorem 2 by showing that regulation of rapidly time-varying periodic disturbances applied at the input of the system is possible.

Sim. 2 In the second simulation, the disturbance $d_o(t)=\sin(\alpha(t))$ will be applied to the output of the system and the controller in (26) is applied to the system. This controller satisfies Theorem 2 with C_{C1}, C_{C2}, and B_{C1} as constants. This simulation will demonstrate the validity of Theorem 3 by showing that all controller that reject input disturbances will not necessarily reject output disturbances when the frequency is rapidly varying.

Sim. 3 In the third simulation, we apply Corollary 2 with a non-causal inverse for G. The non-causal inverse can be done in this case since a PLPV system results in the same parameter set. This simulation demonstrates how output regulation is possible when the system is invertible as proved in Corollary 2.

The plant $G=1/(s+1)$ can be non-causally, stably inverted via $G^{-1}=s+1$. This changes the disturbance model according to Corollary 2 as follows:

$$d_i(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}+1\right)\sin(\alpha(t))=\omega(t)\cos(\alpha(t))+\sin(\alpha(t)).$$

This implies that the controller for Sim. 3 should be

$$C(\omega)=\left[\begin{array}{ccc|c} -12 & 0 & 0 & 1 \\ 1 & 0 & \omega(t) & -1 \\ 1 & -\omega(t) & 0 & -1 \\ \hline -10 & 1 & \omega(t) & 0 \end{array}\right]. \tag{27}$$

Notice that due to the non-causal inverse $C_{H_i}(\rho)$ is a function of ρ.

Figure 2 shows the result of the simulations. The top plot shows the instantaneous frequency $\omega(t)$, where it can be seen that the frequency is switching between 1 and 5 rad/s. In the plot, the second from the top, the output of Sim. 1 is shown, denoted as $y_{\mathrm{sim}\,1}(t)$. Notice that after the output reaches zero, the output stays at zero even when the frequency is varying quickly. This is in agreement with Theorem 2, which states the conditions for regulating input disturbances with a rapidly time-varying frequency. In the third plot from the top, the result of Sim. 2 is shown, denoted by $y_{\mathrm{sim}\,2}(t)$. In this case, regulation of output disturbances is not achieved as proved in Theorem 3. Finally, the bottom plot shows the output of Sim. 3. This simulation shows how regulation of output disturbances is possible if the plant is invertible, in agreement with Corollary 2.

The control signals for the three different simulations are shown in Figure 3. Notice that the difference between the second and third simulations is that the control signal in the third simulation changes abruptly and the control signal in the second simulation slowly recovers from the abrupt change in disturbance frequency which causes the output of the plant to grow and then recover. The effect of these signal upon the output of the plant can be clearly seen in Figure 2.

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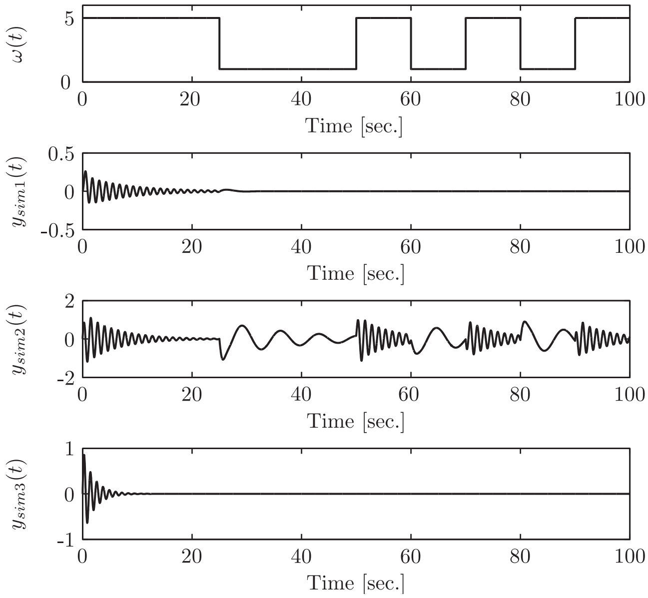

Figure 2. Three simulations of the affect that input and output disturbances with time-varying frequency have upon the plant when connected in feedback. $\omega(t)$ is the frequency of the disturbance, $y_{\text{sim}1}(t)$ is the output of Sim. 1 that validates Theorem 2, $y_{\text{sim}2}(t)$ is the output of Sim. 2 that validates Theorem 3, and $y_{\text{sim}3}(t)$ is the output of Sim. 3 that supports the claims of Corollary 2.

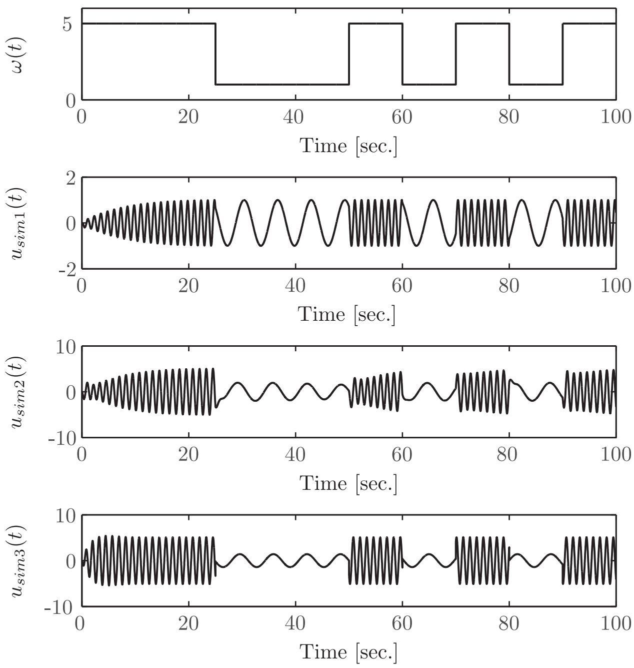

Figure 3. Control signals for the 3 different simulations. $\omega(t)$ is the frequency of the disturbance, $u_{\text{sim}1}(t)$ is the control signal of Sim. 1 that validates Theorem 2, $u_{\text{sim}2}(t)$ is the control signal of Sim. 2 that validates Theorem 3, and $u_{\text{sim}3}(t)$ is the control signal of Sim. 3 that supports the claims of Corollary 2.

5. CONCLUSIONS AND FUTURE RESEARCH

This paper considered scheduling controllers on the basis of disturbance frequency when the frequencies of the periodic disturbance are varying rapidly. It was shown that complete regulation

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of rapidly varying input disturbances is possible provided stabilizing feedback and observer gains can be found and the disturbance frequencies are contained in a compact set. In contrast, it was shown that complete regulation of rapidly varying output disturbances is not guaranteed with the same conditions. Complete regulation of output disturbances is guaranteed when the plant is invertible or the time-varying disturbance model commutes with the feedback system. Thus, scheduling on the basis of frequency is not sufficient for output additive disturbances when the frequency is rapidly varying. Simulations were presented to support the claims and demonstrate the difference that input and output disturbances have upon the plant.

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