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Subspace Identification Using Dynamic Invariance in Shifted Time-Domain Data

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Abstract— **A novel subspace identification method is presented which uses a shift-invariant property of the output data to estimate system dynamics. It is shown that the algorithm may be used with correlation function estimates in addition to input-output data. The algorithm is compared to other subspace methods in a simulation study based on an existing benchmark problem. The results show that the proposed method used with correlation function data achieves consistent system estimates in the presence of highly-colored noise.**

I. Introduction

Subspace identification methods derive a set of state-space system matrices from an estimation of the range of some alternative matrix containing the system dynamics. Most familiar subspace methods can be stated in the framework of the unifying theorem of [8], which interprets subspace identification methods in terms of the range of a weighted extended observability matrix. Once the range space of the extended observability matrix is estimated, the state-to-output behavior is typically determined using the shift-invariant structure of the extended observability matrix, and the input-to-state behavior is estimated in a weighted least-squares problem. Popular methods interpreted in this framework include Robust N4SID [7], MOESP [10], CVA [4], and ORT [2]. Thorough overviews of these methods can be found in [9] and [2]. Alternatively, in [5], a subspace identification algorithm was formulated that estimates the system Markov parameters explicitly by projecting the input and output data onto a subspace containing the system impulse response.

In this paper, we propose a method of estimating the system dynamics not from the shift-invariant structure of the extended observability matrix, but by exploiting the propagation of the system dynamics in the shifted input-output data itself. A least-squares problem is presented that will solve for the system dynamics given an estimate of an extended observability matrix with respect to some arbitrary state basis. This solution reduces to an algorithm previously studied in [6] if a specific choice of the extended observability matrix is made based on the singular-value decomposition. The method is extended to correlation function estimates and shown to provide consistent results regardless of the spectrum of the noise signal. A simulation study based on a benchmark problem in [2] illustrates the performance of the proposed subspace identification methods in comparison with the subspace methods implemented in the Matlab System Identification Toolbox.

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II. Subspace Identification Methods

Consider a linear, time-invariant, discrete-time system

$$y(t) = \sum_{k=0}^{\infty} G(k)u(t-k) + v(t). \tag{1}$$

The system Markov parameters $G(k) \in \mathbb{R}^{n_y \times n_u}$ define the relationship between the input signal $u(t) \in \mathbb{R}^{n_u}$ and the output signal $y(t) \in \mathbb{R}^{n_y}$, which contains an additive noise signal $v(t) \in \mathbb{R}^{n_y}$. We assume that the input and output signals are quasi-stationary and that the noise $v(t)$ is stationary.

Such a system has an infinite number of state-space representations of the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) + v(t) \end{aligned} \tag{2}$$

given in terms of constant matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, and $D \in \mathbb{R}^{n_y \times n_u}$. The state-space matrices are related to the system Markov parameters by

$$G(k) = \begin{cases} 0 & k < 0 \\ D & k = 0 \\ CA^{k-1}B & k > 0 \end{cases}. \tag{3}$$

We assume that all state-space representations are controllable, observable, and minimal (see [2] for relevant definitions).

The identification problem considered is to estimate (i) the system order n and (ii) state-space matrices A, B, C, and D from measured data generated by the above system. The problem of estimating a realization of the process that generates $v(t)$ is not addressed.

Consider the system described by (1) and (2), and form a column vector $y_{0|i-1}$ of output data from time $t = 0$ to $t = i - 1$

$$y_{0|i-1} = \mathrm{vec}(\begin{bmatrix} y(0) & y(1) & y(2) & \cdots & y(i-1) \end{bmatrix}).$$

This output-data vector can be expressed in terms of the initial state vector $x_0 = x(0)$, a vector of future input sequences also from $t = 0$ to $t = i - 1$

$$u_{0|i-1} = \mathrm{vec}(\begin{bmatrix} u(0) & u(1) & u(2) & \cdots & u(i-1) \end{bmatrix}),$$

and a vector of noise data from time $t = 0$ until time $t = i-1$

$$v_{0|i-1} = \mathrm{vec}(\begin{bmatrix} v(0) & v(1) & v(2) & \cdots & v(i-1) \end{bmatrix}).$$

These four vectors are related to one another by an $in_y \times n$ extended observability matrix

$$\Gamma = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots & (CA^{i-1})^T \end{bmatrix}^T. \quad (4)$$

and an $in_y \times in_u$ lower-triangular block-Toeplitz matrix of Markov parameters

$$T_{0|i-1} = \begin{bmatrix} G(0) & & & \\ G(1) & G(0) & & \\ \vdots & \vdots & \ddots & \\ G(i-1) & G(i-2) & \cdots & G(0) \end{bmatrix} \quad (5)$$

in which the first element of the index (0 above) represents the farthest right Markov parameter on the bottom row, and the second element ($i-1$ above) represents the farthest left Markov parameter on the bottom row. The output vector may then be expressed as

$$y_{0|i-1} = \Gamma x_0 + T_{0|i-1} u_{0|i-1} + v_{0|i-1}. \quad (6)$$

Our goal is to isolate the system dynamics contained in the extended observability matrix Γ by estimating a basis for its range. To accomplish this, we extend the data vectors column-wise to form full-row-rank matrices of data that contain the full range of Γ in (6). Such a basis can be found by first extending $y_{0|i-1}$ by l columns, forming an $in_y \times l$ block-Hankel matrix of output data

$$Y_{0|i-1} = \begin{bmatrix} y(0) & y(1) & \cdots & y(l-1) \\ y(1) & y(2) & \cdots & y(l) \\ \vdots & \vdots & & \vdots \\ y(i-1) & y(i) & \cdots & y(l+i-2) \end{bmatrix}.$$

Similarly expanding the input vector $u_{0|i-1}$ results in a $in_u \times l$ block-Hankel matrix

$$U_{0|i-1} = \begin{bmatrix} u(0) & u(1) & \cdots & u(l) \\ u(1) & u(2) & \cdots & u(l+1) \\ \vdots & \vdots & & \vdots \\ u(i-1) & u(i) & \cdots & u(l+i-2) \end{bmatrix}, \quad (7)$$

and expanding $v_{0|i-1}$ results in a block-Hankel matrix

$$V_{0|i-1} = \begin{bmatrix} v(0) & v(1) & \cdots & v(l-1) \\ v(1) & v(2) & \cdots & v(l) \\ \vdots & \vdots & & \vdots \\ v(m-1) & v(m) & \cdots & v(l+m-2) \end{bmatrix}.$$

Finally, we expand $x(0)$ incrementally as

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(l) \end{bmatrix}$$

resulting in the data-matrix equation

$$Y_{0|i-1} = \Gamma X + T_{0|i-1} U_{0|i-1} + V_{0|i-1}. \quad (8)$$

A. Advancing the Output

Now suppose the output vector is advanced by one time sample to form

$$y_{1|i} = \mathrm{vec}(\begin{bmatrix} y(1) & y(2) & y(3) & \cdots & y(i) \end{bmatrix}).$$

Similar to (6), $y_{1|i}$ may be expressed as $y_{1|i} = \Gamma A x_0 + T_{0|i} u_{0|i} + v_{1|i}$. Note that only the input term incremented by one and that T no longer has the initial parameter $G(0)$ appearing in the top-left corner.

Expanding $y_{1|i}$ as was done with $y_{0|i-1}$ in (8) results in

$$Y_{1|i} = \begin{bmatrix} y(1) & y(2) & \cdots & y(l) \\ y(2) & y(3) & \cdots & y(l+1) \\ \vdots & \vdots & & \vdots \\ y(i) & y(i-1) & \cdots & y(l+i-1) \end{bmatrix},$$

and applying the same expansion to $u_{0|i}$ and $v_{0|i}$ leads to the shifted data-matrix equation

$$Y_{1|i} = \Gamma A X + T_{0|i} U_{0|i} + V_{1|i}. \quad (9)$$

B. Removing the Effects of Future Input

To estimate the range of Γ, we isolate the propagation of the states X by projecting the output Y onto the null space of the future input U. Define the $l \times l$ projection matrix

$$\Pi_{0|i} = I - U_{0|i}^T (U_{0|i} U_{0|i}^T)^\dagger U_{0|i} \quad (10)$$

for which $U_{0|i}\Pi_{0|i} = 0$ and $(\cdot)^\dagger$ represents the Moore-Penrose pseudoinverse.

Lemma 1: Let $T_{0|i-k}$, $U_{0|i-k}$, and $\Pi_{0|i}$ be defined as in (5), (7), and (10), respectively. For $k \geq 0$,

$$T_{0|i-k} U_{0|i-k} \Pi_{0|i} = 0$$

Proof: For $k = 0$, the proof is trivially shown by substitution. For $k < 0$, notice in (6) that the input matrix $U_{0|i-1}$ may be extended beyond $t = i-1$ so long as columns of zeros are added to the right of $T_{0|i-1}$. Let $T_{-p|i}$ be the block-Toeplitz matrix of Markov parameters with p columns of zeros added to the right, so that $T_{-1|i}$ is $T_{0|i}$ with one additional column of zeros and so on. (This corresponds to the definition of T and the Markov parameters given in (3)). Thus for any $p \geq 0$,

$$T_{-p|i-k} U_{0|i-k+p} = T_{0|i-k} U_{0|i-k}.$$

The desired result is found by letting $p = k$ and multiplying the above on the right by $\Pi_{0|i}$. ■

Hence the same projection matrix can be used to eliminate the effects of the input on the output matrices $Y_{0|i-1}$ and $Y_{1|i}$. Multiplying (8) and (9) on the right by (10) results in

$$Y_{0|i-1}\Pi_{0|i} = \Gamma X \Pi_{0|i} + V_{0|i-1}\Pi_{0|i} \quad (11)$$

and

$$Y_{1|i}\Pi_{0|i} = \Gamma A X \Pi_{0|i} + V_{1|i}\Pi_{0|i}, \quad (12)$$

respectively, so that the matrix products $Y_{0|i-1}\Pi_{0|i}$ and $Y_{1|i}\Pi_{0|i}$ now effectively contain free-response data with additive noise.

Some additional restrictions must be put on the projection matrix $\Pi_{0|i}$ to preserve the rank of Γ in the above equation.

Lemma 2: The rank of Γ in (11) and (12) will be preserved only if

$$l \geq (i+1)n_u + n. \tag{13}$$

Proof: The null space of $U_{0|i}$ must have dimension of n or higher for the rank of Γ to be preserved, that is $\dim(\text{null}(U_{0|i})) = \text{rank}(\Pi_{0|i}) \geq n$. Since $\text{rank}(U_{0|i}) \leq \min((i+1)n_u, l)$, and recalling that $\text{rank}(U_{0|i}) + \dim(\text{null}(U_f)) = l$, a necessary condition to preserve the rank of Γ is (13). ■

Thus l must be chosen that $U_{0|i}$ is sufficiently "fat" and i must be chosen so that $Y_{0|i-1}$ and $Y_{1|i}$ are sufficiently "tall" to span the range of Γ. A thorough discussion of requirements for the input matrix to preserve the rank and span of Γ can be found in [11]. Although it strictly applies to the input matrix not extended by an additional row ($U_{0,i-1}$), the requirements on sufficiency of excitation are straightforward to extend to $U_{0|i}$ used above.

C. Estimating the Extended Observability Matrix and Identifying the System Order

In the deterministic case ($v(t) = 0$), the order of the system n may be found by simply examining the rank of $Y_{0|i-1}\Pi_{0|i}$, and an extended observability matrix Γ with respect to some arbitrary state basis may be found from any factorization

$$\Gamma(X\Pi_{0|i}) = Y_{0|i-1}\Pi_{0|i}$$

in which Γ has n columns. In the nondeterministic case, $V_{0|i-1}$ will have full rank, causing $Y_{0|i-1}\Pi_{0|i}$ to have full rank, and instead a rank-n estimate of $\Gamma X \Pi_{0|i-1}$ must be constructed from $Y_{0|i-1}\Pi_{0|i}$. Inspired by the method originally developed by [3] for approximating the system Hankel matrix from Markov parameter estimates, a reasonable goal is to find a rank-n matrix R that minimizes

$$e = \min_{\text{rank}(R)=n} \left|\left|R - Y_{0|i-1}\Pi_{0|i}\right|\right|_2. \tag{14}$$

If $Y_{0|i-1}\Pi_{0|i}$ has the singular-value decomposition

$$Y_{0|i-1}\Pi_{0|i} = \begin{bmatrix} U_n & U_s \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_s \end{bmatrix} \begin{bmatrix} V_n^T \\ V_s^T \end{bmatrix}, \tag{15}$$

in which $\Sigma_n = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ contains the first n singular values, the matrix that minimizes (14) is [1]

$$R = U_n \Sigma_n V_n^T. \tag{16}$$

Additionally, $e = \sigma_{n+1}$. Thus, if n is *unknown*, we can determine a likely value for n by examining the singular values of $Y_{0|i-1}\Pi_{0|i}$ and searching for a significant drop-off. The place of the singular value immediately prior to this drop-off is taken to be n.

An estimate of Γ — denoted $\hat{\Gamma}$ — and an estimate of the free-response states — denoted $\hat{X}\Pi_{0|i}$ — may then be taken from some factorization of R instead:

$$\hat{\Gamma}(\hat{X}\Pi_{0|i}) = R.$$

A natural choice is to use

$$\hat{\Gamma} = U_n \Sigma_n^{1/2} \tag{17}$$

for which

$$\hat{X}\Pi_{0|i} = \Sigma_n^{1/2} V_n^T = \hat{\Gamma}^\dagger Y_{0|i-1}\Pi_{0|i},$$

although any appropriately-dimensioned factorization is valid.

D. Estimating the System Dynamics — Deterministic Case

The most common means of estimating the system dynamics is by solving for A from the shift-invariant structure of the extended observability matrix $\hat{\Gamma}$ [10]. In this section, we demonstrate a method of solving for the system dynamics exactly in the deterministic case that will later be shown to be the solution to a least-squares problem in the non-deterministic case. The first results immediately follow from (11) and (12).

Theorem 1: Let the output signal $y(t)$ be a purely deterministic signal generated by $u(t)$ and some initial condition $x(0)$, that is, $v(t) = 0$ for all t. Let i be such that $Y_{0|i-1}$ and $Y_{1|i}$ have at least n rows, or $i \geq n/n_u$. Assume that condition (13) is met and that $U_{0|i}$ has full row rank. Let

$$Y_n = \text{first } n \text{ rows of } Y_{0|i-1} \qquad \overline{Y}_n = \text{first } n \text{ rows of } Y_{0|i}$$

Then

$$A_c = \overline{Y}_n \Pi_{0|i} \left(Y_n \Pi_{0|i}\right)^\dagger$$

where $A_c = TAT^{-1}$ with A in (2) with respect to some state basis of x and T a valid similarity transformation.

Proof: Let Γ_n bet the first n rows of Γ in (11) and (12). Then $Y_n\Pi_{0|i} = \Gamma_n X \Pi_{0|i}$, $\overline{Y}_n \Pi_{0|i} = \Gamma_n A X \Pi_{0|i}$. X and $\Pi_{0|i}$ must have at least rank n based on the conditions put on i and the input matrix $U_{0|i}$. While $\Pi_{0|i}$ is likely not invertible, $Y_n\Pi_{0|i}$ and $X\Pi_{0|i}$ will have full row rank and thus a right pseudoinverse. Then $A_c = \Gamma_n A X \Pi_{0|i}(X\Pi_{0|i})^\dagger \Gamma_n^\dagger = \Gamma_n A \Gamma_n^\dagger$ ■

In the single-input-single-output case, $i = n$, and A_c will be in controller-canonical form, although this is not strictly true for the multivariable case.

In general, the above will give undesirable results when applied to the nondeterministic case. We can, however, estimate the dynamics in an arbitrary but more internally-balanced state basis.

Theorem 2: Given the same assumptions in Theorem 1, assume the extended observability matrix Γ for the system (2) is known. Then for any $i \geq n/n_u$ that satisfies (13),

$$A = \Gamma^\dagger Y_{1|i}\Pi_{0|i}(\Gamma^\dagger Y_{0|i-1}\Pi_{0|i})^\dagger \tag{18}$$

in which A is with respect to the same state basis as Γ.

Proof: The proof is a straightforward extension to Theorem 1. Note that because Γ has full column rank, its left inverse will have full row rank, the term $\Gamma^\dagger Y_{0|i-1}\Pi_{0|i}$ will have full row rank, and thus the right inverse of $\Gamma^\dagger Y_{0|i-1}\Pi_{0|i}$ will exist. ■

If unknown beforehand, the extended observability matrix can be found from any factorization

$$\Gamma(X\Pi_{0|i}) = Y_{0|i-1}\Pi_{0|i}$$

in which Γ has n columns and full column rank and $X\Pi_{0|i}$ has n rows and full row rank. With C taken from the top n_y rows of Γ, the matrices B, D, and an initial condition x_0 may be solved for via a least-squares problem [10].

E. Estimating the System Dynamics – Nondeterministic Case

In the nondeterministic case, only an estimate of the extended observability matrix $\hat{\Gamma}$ will be available, and the free-response states $X\Pi_{0|i}$ will be corrupted by noise. The preceding means of solving for A, however, may still be used to provide a least-squares estimate $\hat{A}$.

We propose the following problem: given an estimate of the extended observability matrix $\hat{\Gamma}$ and an estimate of the free-response states $\hat{X}\Pi_{0|i}$, find the estimate $\hat{A}$ that best estimates the propagation of the system dynamics in the output data. We define this as the quantity

$$\begin{aligned} &\min_{\text{rank }(\hat{A})=n} \left|\left| \hat{A}\hat{X}\Pi_{0|i} - \hat{\Gamma}^{\dagger}Y_{1|i}\Pi_{0|i} \right|\right|_2 \\ &= \min_{\text{rank }(\hat{A})=n} \left|\left| \hat{A}\hat{\Gamma}^{\dagger}Y_{0|i-1}\Pi_{0|i} - \hat{\Gamma}^{\dagger}Y_{1|i}\Pi_{0|i} \right|\right|_2. \end{aligned} \tag{19}$$

The solution of the above is given by

$$\hat{A} = \hat{\Gamma}^{\dagger}Y_{1|i}\Pi_{0|i}\left(\hat{\Gamma}^{\dagger}Y_{0|i-1}\Pi_{0|i}\right)^{\dagger} \tag{20}$$

which is equivalent to (18) in the deterministic case.

With the $\hat{\Gamma}$ taken from (17), this expression reduces to

$$\hat{A} = \Sigma_n^{-1/2}U_n^T Y_{1|i}\Pi_{0|i}V_n\Sigma_n^{-1/2}. \tag{21}$$

Thus the above estimate of A, originally presented in [6], is a solution to the least-squares problem (19), which solves for A exactly in the deterministic case. With $\hat{A}$ given above and $\hat{C}$ taken from the first n_y rows of $\hat{\Gamma}$; $\hat{B}$, $\hat{D}$, and $\hat{x}_0$ may be solved for via a least-squares problem.

Expanding the expression within the norm of (19) to

$$\hat{A}\hat{\Gamma}^{\dagger}(\Gamma X - \Gamma A X)\Pi_{0|i} + \hat{\Gamma}^{\dagger}(V_{0|i-1} - V_{1|i})\Pi_{0|i},$$

we see that even when the noise signal $v(t)$ is perfectly white, the term $\hat{\Gamma}^{\dagger}(V_{0|i-1} - V_{1|i})\Pi_{0|i}$ will not be orthogonal to the regressor $\hat{\Gamma}^{\dagger}(\Gamma X - \Gamma A X)\Pi_{0|i}$, since the multiplication of the rank-reducing $\hat{\Gamma}^{\dagger}$ on the left will effectively "rotate" the noise to be in-line with the system dynamics. The unfortunate consequence of this is that (21) is not guaranteed to provide consistent estimates, with consistency defined as the condition that the eigenvalues of $\hat{A}$ converge to those of A as the number of data matrix columns $l \to \infty$.

To illustrate the difficulties this may cause, let (A_v, B_v, C_v, D_v) form a state-space representation of the process that generates the noise signal $v(t)$. The system (1) expressed in innovations form is then

$$\begin{bmatrix} x(t+1) \\ x_v(t+1) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_v \end{bmatrix}\begin{bmatrix} x(t) \\ x_v(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ K_v \end{bmatrix}e_v(t)$$

$$y(t) = \begin{bmatrix} C & C_v \end{bmatrix}\begin{bmatrix} x(t) \\ x_v(t) \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix}e_v(t)$$

where $x_v(t)$ is the state of the noise-generating process and $e_v(t)$ is white noise. Note that the states of the noise process $x_v(t)$ are uncontrollable. If n is chosen to be the correct order of the system, the poles of the estimated extended observability matrix will be biased by the poles of the noise-generating process, as shall be seen in the example given in Section V. In fact, as the signal-to-noise ratio goes to 0 the estimation problem becomes that of a stochastic realization problem. If $\hat{A}$ is estimated from the extended observability matrix (in the framework of [8]), and n is chosen artificially large, the eigenvalues of $\hat{A}$ will contain estimates of the eigenvalues of A and A_v.

It should be emphasized here that these issues are not common to all subspace identification methods. A notable difference between these methods and the solution presented here is that (20) will not identify the poles of the noise-generating process, although the poles that are identified will still be biased. Also, the ORT method of [2] will provide consistent estimates even in the presence of colored noise.

F. Obtaining Consistent Estimates

To obtain consistent estimates, we must "whiten" the effects of the noise terms so that they become orthogonal to $AX\Pi_{0|i}$ and $X\Pi_{0|i}$. Prefiltering the noise signal would require knowledge not only of the process that generates $v(t)$, but of the system dynamics in Γ that are exactly what we are trying to identify, so this is often not practical. Another means of doing so is to weight the projected output terms by weighting matrices W_1 and W_2:

$$W_1Y_{0|i-1}\Pi_{0|i}W_2 = W_1\Gamma X\Pi_{0|i}W_2 + W_1V_{0|i-1}\Pi_{0|i}W_2$$

$$W_1Y_{1|i}\Pi_{0|i}W_2 = W_1\Gamma AX\Pi_{0|i}W_2 + W_1V_{1|i}\Pi_{0|i}W_2$$

in which W_1 and W_2 are chosen such that $\lim_{N\to\infty} W_1V_*W_2 = 0$ and the rank of Γ is preserved. Thus as $N \to \infty$,

$$\begin{aligned} W_1Y_{0|i-1}\Pi_{0|i}W_2 &= W_1\Gamma X\Pi_{0|i}W_2 \\ W_1Y_{1|i}\Pi_{0|i}W_2 &= W_1\Gamma AX\Pi_{0|i}W_2. \end{aligned} \tag{22}$$

A consequence of (22) is that the weighting matrices W_1 and W_2 must somehow be chosen to satisfy the criteria $W_1V\Pi_{0|i}W_2 \to 0$, or they must at least sufficiently whiten the noise so that it does not bias the estimation of the range of Γ. If the nondeterministic components of $W_1Y_{0|i-1}\Pi_{0|i}W_2$ remain colored, however, the realization generated with Γ will naturally be biased by the poles of the process that generated the signal $v(t)$.

With this in mind, we propose a straightforward means of extending subspace identification methods to datasets beyond general input-output data that guarantee consistent estimates regardless of the noise spectrum.

III. Extension of Subspace Methods to Correlation Data

The estimation framework developed in the previous section may be directly extended to alternative forms of data, most notably correlation functions. The cross-correlation of the output $y(t)$ and the input $u(t)$ and the autocorrelation of the input $u(t)$

$$R_{yu}(\tau) = E[y(t+\tau)u^T(t)], \quad R_u(\tau) = E[u(t+\tau)u^T(t)]$$

are related by the same Markov-parameter convolution in (1):

$$R_{yu}(\tau) = \sum_{k=0}^{\infty} G(k) R_u(k-\tau)$$

Using correlation function estimates derived from N samples of data

$$\hat{R}_u^N(\tau) = \sum_{t=0}^{N-\tau-1} u(t+\tau)u^T(t)$$
$$\hat{R}_{yu}^N(\tau) = \sum_{t=0}^{N-\tau-1} y(t+\tau)u^T(t)$$

the effect of noise is reduced due to the fact that the output noise is uncorrelated with the input, or

$$\lim_{N\to\infty} \hat{R}_{vu}^N(\tau) = E[v(t+\tau)u^T(t)] = 0.$$

Thus, the use of correlation functions instead of raw data has the advantage that *consistency of the system estimates is a straightforward consequence of the consistency of the correlation function estimates*. This allows for the consistent identification of input-output behavior of a system even in the presence of colored noise.

Additionally, correlation function estimates have the attractive property that identification methods may be performed using matrices a fraction of the size of those that would be required with raw data. Signal analyzing hardware is also often capable of providing correlation function estimates computed in real-time that incorporate a larger data set than would be possible to acquire directly. For example, hardware that can compute correlation function estimates from 50,000 data points may only be able to store 5,000 data points at a time. In many situations, acquiring correlation function estimates is simply more practical.

Implementing the correlation function estimates with subspace methods is straightforward and consists of substituting the the input data with the estimate $R_u^N(\tau)$ and the output data with the estimate $R_{yu}^N(\tau)$ for some suitable range $\tau \in [\tau_{\min}, \tau_{\max}]$ ($\tau_{\min}$ may be < 0). The range over which the function is used must be chosen carefully, as the new input signal $R_u(\tau)$ will have a brief maximum at $\tau = 0$ then likely lose excitation quickly. Let $R_{0|i-1}^{yu}$ be the block-Hankel matrix filled with cross-correlation function estimates $R_{yu}^N(\tau)$, and let $R_{0|i}^u$ be the block-Hankel matrix filled with auto-correlation function estimates $R_u^N(\tau)$, in which the subscript indices agree with the definitions of $Y_{0|i-1}$ and $U_{0|i}$. Then, as $N \to \infty$,

$$R_{0|i-1}^{yu}\Pi_{0|i}^R = \Gamma X \Pi_{0|i}^R$$
$$R_{1|i}^{yu}\Pi_{0|i}^R = \Gamma A X \Pi_{0|i}^R.$$

where

$$\Pi_{0|i}^R = I - (R_{0|i}^u)^T \left(R_{0|i}^u (R_{0|i}^u)^T \right)^{\dagger} R_{0|i}^u.$$

Thus the products $R_{0|i-1}^{yu}\Pi_{0|i}^R$ and $R_{1|i}^{yu}\Pi_{0|i}^R$ contain the product of the extended observability matrix Γ and free response states $X\Pi_{0|i}^R$, which in the second case have been propagated through the system dynamics present in A. The terms $R_{0|i-1}^{yu}\Pi_{0|i}^R$ and $R_{1|i}^{yu}\Pi_{0|i}^R$ may then be used in place of $Y_{0|i-1}\Pi_{0|i}$ and $Y_{1|i}\Pi_{0|i}$ with the guarantee of consistency even when weighting matrices are omitted.

IV. Implementation Issues

Instead of projecting the output data onto the null space of the input data, more accurate results may be obtained by dividing the input data into "past" and "future" segments, with the past segment taken from time $t = -s$ to time $t = -1$. Because the deterministic component of the input-free data $Y_{0|i-1}\Pi_{0|i}$ and $Y_{1|i}\Pi_{0|i}$ is a linear combination of the deterministic components of the past input and output data, a projection onto the space of past data along the null space of the future input data will generally result in better estimates and in certain cases can be shown to be the best possible estimate of the states $X\Pi_{0|i}$[7]. The indices s and i are often called the "past horizon" and "future horizon," respectively. Applied to the algorithm presented here, the result is six data matrices instead of three: the past and future output matrices $Y_{-s|-1}$ and $Y_{0|i-1}$, the time-advanced past and future output matrices $Y_{-s+1|0}$ and $Y_{1|i}$, and the past and future input matrices $U_{-s|-1}$ and $U_{0|i}$.

Rather than implementing these projections by explicitly forming projection matrices, it can be shown that the projection matrices can be found within the LQ-decompositions [2]

$$\begin{bmatrix} U_{0|i} \\ U_{-s|-1} \\ Y_{-s|-1} \\ Y_{0|i-1} \end{bmatrix} = \begin{bmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \\ L_{31} & L_{32} & L_{33} & \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \\ Q_4^T \end{bmatrix}$$

and

$$\begin{bmatrix} U_{0|i} \\ U_{-s|-1} \\ Y_{-s+1|0} \\ Y_{1|i} \end{bmatrix} = \begin{bmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \\ L'_{31} & L'_{32} & L'_{33} & \\ L'_{41} & L'_{42} & L'_{43} & L'_{44} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \\ Q_3'^T \\ Q_4'^T \end{bmatrix}$$

The algorithm may then be performed with $W_1(L_{42}Q_2^T + L_{43}Q_3^T)W_2$ in place of $W_1 Y_{0|i-1}\Pi_{0|i}W_2$ and $W_1(L'_{42}Q_2'^T + L'_{43}Q_3'^T)W_2$ in place of $W_1 Y_{1|i}\Pi_{0|i}W_2$. To the best of the authors' knowledge, this is the only subspace identification method that requires two separate decompositions when implemented in the LQ framework.

V. Simulation Example

The algorithm was applied to signals generated by a simulated system used for several examples in [2]. The system to be identified, G, is excited by an input u that is generated by white noise e_u with variance $\sigma_u^2 = 1$ filtered through F. The output signal y is corrupted by additive noise v, the result of white noise e_v with variance σ_v^2 filtered through the noise model H. The models used for G, H, and

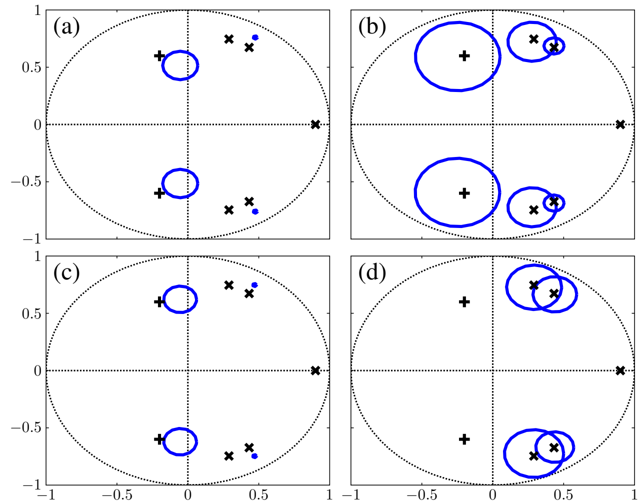

Fig. 1. Pole locations of 1000 estimates from (a) N4SID with CVA weighting, $n = 5$; (b) N4SID with CVA weighting, $n = 7$; (c) output-shifting estimate with raw data; (d) output-shifting estimate with correlation function data. The locations of the system poles are marked by 'x'; the locations of the noise-process poles are marked by '+.' The circles represent the locations of the poles estimated, centered about the average estimate. The radius of the circles is equivalent to 2 standard deviations of the pole locations. $N = 1000$, and $\sigma_v^2 = 0.002$

F are (given in terms of a the forward time-shift operator q):

$$G(q) = (0.0275q^{-4} + 0.0551q^{-5})/(1 - 2.3443q^{-1} + 3.081q^{-2} - 2.5274q^{-3} + 1.2415q^{-4} - 0.3686q^{-5}),$$

$$H(q) = \frac{1 - 0.2q^{-1} - 0.48q^{-2}}{1 + 0.4q^{-1} + 0.4q^{-2}},$$

$$F(q) = \frac{0.0513}{1 - 0.9q^{-1}}.$$

Note that the signal-generation model F and the system G share a stable, real pole. The relationship between the input, output, and noise is $y(t) = G(q)u(t) + v(t)$ with $u(t) = F(q)e_u(t)$ and $v(t) = H(q)e_v(t)$.

The simulation was run 1000 times and system estimates were generated with (a) N4SID with CVA weighting [8] and a system order of 5, (b) N4SID with CVA weighting and a system order of 7, (c) the algorithm presented in this paper with a system order of 5, and (d) the same algorithm with correlation function estimates used in place of raw data. The pole locations of the resulting estimates are represented in Figure 1. The implementations of the N4SID variants (a) and (b) were taken from Matlab's System Identification Toolbox. The toolbox automatically selects past and future horizons, and the horizons of (c) and (d) were selected to match the horizons of (a). The correlation function estimates were generated over the domain $\tau \in [-2s, 3s]$ where s is the past horizon selected by the Matlab toolbox. Figure 2 shows the singular values of the matrix products $Y_{0|i-1}\Pi_{0|i}$ and $R^{yu}_{0|i-1}\Pi^R_{0|i}$.

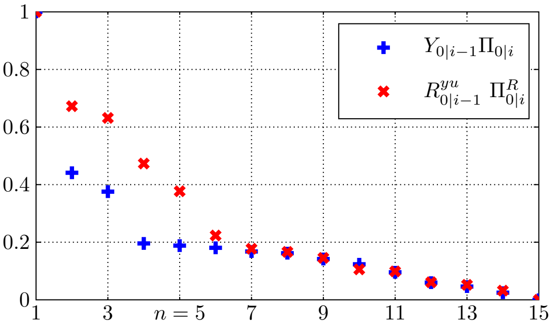

Fig. 2. The first 15 singular values of $Y_{0|i-1}\Pi_{0|i}$ and $R^{yu}_{0|i-1}\Pi^R_{0|i}$. The natural log of the singular values was taken and the values were scaled from 1 to 0 for plotting purposes.

VI. Conclusion

A new algorithm that identifies state-space realizations of systems from input-output data was presented. This algorithm uses the invariance of system dynamics in shifted time-domain data to estimate the state-dynamics matrix A, while the matrix C mapping the system state to the output is taken from an estimate of the extended observability matrix. The matrices B and D are solved for by a least-squares method. The algorithm was shown to be the solution to a least-squares problem and to reduce to a previously-presented algorithm when given a specific estimate of the extended observability matrix. Simulation results confirm that the algorithm generates estimates with the same bias as estimates generated from other subspace methods and that the algorithm generates consistent estimates when used with correlation function data.

References

[1] G.H. Golub and C.F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, third edition, 1996.

[2] T. Katayama. *Subspace Methods for System Identification*. Communications and Control Engineering. Springer-Verlag, 2005.

[3] S. Y. Kung. A new identification and model redaction algorithm via singular value decomposition. In *12th Asilomar Conference on Circuits, Systems and Computers*, pages 705–714, 1978.

[4] W. E. Larimore. Canonical variate analysis in identification, filtering, and adaptive control. In *Decision and Control, 1990., Proceedings of the 29th IEEE Conference on*, pages 596–604 vol.2, 1990.

[5] Ivan Markovsky, Jan C. Willems, Paolo Rapisarda, and Bart L.M. De Moor. Algorithms for deterministic balanced subspace identification. *Automatica*, 41(5):755–766, May 2005.

[6] D.N. Miller and R.A. de Callafon. Subspace identification from classical realization methods. In *15th IFAC Symposium on System Identification*, Saint-Malo, France, July 2009.

[7] P. Van Overschee and B. De Moor. N4SID: subspace algorithms for the identification of combined deterministic-stochastic systems. *Automatica*, 30(1):75–93, 1994.

[8] P. Van Overschee and B. De Moor. A unifying theorem for three subspace system identification algorithms. volume 2, pages 1645–1649 vol.2, 1994.

[9] P. Van Overschee and B. De Moor. *Subspace Identification for Linear Systems: Theory, Implementatin, Applications*. Kluwer Academic Publishers, 1996.

[10] Michel Verhaegen. Identification of the deterministic part of MIMO state space models given in innovations form from input-output data. *Automatica*, 30(1):61–74, 1994.

[11] Jan C. Willems, Paolo Rapisarda, Ivan Markovsky, and Bart L.M. De Moor. A note on persistency of excitation. *Systems & Control Letters*, 54(4):325–329, April 2005.