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TECHNICAL PAPER

Frequency domain subspace identification of a tape servo system

Michel Claes · Matthew R. Graham · Raymond A. de Callafon

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Abstract As capacity demands for magnetic tape storage systems grow, servo actuator design for tracking data on high density tape media presents new modeling and control design challenges. In this paper a frequency weighted subspace identification algorithm is presented for control relevant model estimation of a tape servo actuator. Common to other subspace identification methods, the proposed algorithm is based on linear algebra techniques providing means for model order selection and model computation. The proposed subspace identification also allows for frequency dependent weightings to emphasize frequency data around the cross-over frequency to find models relevant for control design. The algorithm is applied to data obtained from a tape storage device, demonstrating model order selection and the estimation of servo actuator dynamics with control relevant model fit criteria.

1 Introduction

Due to its enormous volumetric storage density, high data rate and excellent archival capability, magnetic tape storage is the primary choice for backup applications. However, as the cost/storage-capacity ratio for hard disk drives keeps decreasing, tape drives can only remain competitive when their volumetric storage density increases as well. The latter can be accomplished by improving the track density, but results in increased sensitivity to high frequency disturbances, such as lateral tape motion. Since lateral cross-track displacements between the data track and the read/write head on the order of 10% of the track width causes read/write errors, it is clear that high performance control is crucial.

Reliable measurements of the frequency response of the servo actuator can be obtained from a tape servo system. Such measurements can be used to construct a low order model of the tape servo system on the basis of frequency domain subspace methods (De Schutter 2000; McKelvey et al. 1996; Viberg 1995). The main advantage of a subspace method over a standard curve fitting methodology is determined by the following three facts. Firstly, a subspace method is based on projection techniques that can be computed via standard linear algebra procedures and does not require a non-linear or iterative optimization, as is required in curve fitting (de Callafon et al. 1996; Ljung 1999). Secondly, a subspace method does not require a specific parametrization in terms of the polynomial coefficients of a transfer function, as a model is found directly in terms of its state space realization. This is an obvious advantage when estimating multivariable (multi-input, multi-output models), as the subspace method can be used without any modification, while a curve fitting methodology would have to parametrize the coefficients of a multivariable transfer function explicitely (Gaikwad and Rivera 1997). Finally, a subspace method can handle large order models, as state space models are computed by numerically stable linear algebra techniques. Estimating high order

M. Claes · M. R. Graham · R. A. de Callafon (✉)
Department of Mechanical and Aerospace Engineering,
University of California San Diego,
9500 Gilman Drive, La Jolla, CA 92093-0411, USA
e-mail: callafon@ucsd.edu

models in a polynomial or transfer function format requires a specific polynomial parametrization (Rolain et al. 1994; Schoukens and Pintelon 2001) to guarantee numerical stability of the parameter estimates.

Based on well-established results for subspace identification (McKelvey et al. 1996), a new procedure for computing low-order models on the basis of frequency domain data is presented in this paper. The focus of this paper is an algorithm for which the frequency response data is obtained by spectral analysis and is equidistantly spaced along the frequency axis. The computational procedure is shown to have a strong resemblance to the conventional Ho-Kalman algorithm (Ho and Kalman 1966) to compute low order models using a singular value decomposition (SVD) of a symmetric Hankel matrix. In addition, the procedure allows for both input and output weighting functions to emphasize the frequency range around the cross-over frequency for the control relevant estimation of models (Gevers 2002). The corresponding subspace identification technique is applied to experimental data obtained from a tape drive system to demonstrate the usefulness of the method in control relevant estimation.

2 Motivation and problem formulation

2.1 Motivation for control relevant estimation

Consider the measured frequency response data of a tape servo actuator given in Fig. 1. This frequency response data is obtained via spectral analysis of a measurement of the position error signal (PES) from the tape servo system, the servo actuator input and an external reference signal. More details on these signals and their underlying relations are given in Sect. 5.1. The data in Fig. 1 is presented for motivational purposes, where one can observe the characterizing double integrator in the low and middle frequency area, while the high frequency area is dominated by resonance modes and noise.

The dynamic behavior of a mechanical system is dominated by resonance modes present around the feedback control bandwidth of the servo system and modeling these resonance modes are most relevant for control design. Control relevant estimation recognizes the fact that a model inevitably will be an approximation of the frequency domain data. Motivated by stability requirements of a closed-loop system, control relevant estimation requires frequency dependent weighting filters to emphasize the frequency range around the cross-over frequency (Gevers 2002; Hjalmarsson 2005; Van Den Hof and Schrama 1995) and a model estimation procedure should support this feature.

The objective of the modeling procedure presented in this paper is to obtain a linear time invariant (LTI) model in the form of a state-space realization through frequency domain system identification techniques in which the essential dynamics of the system for feedback control design purposes is modeled. For that purpose we will employ subspace based estimation techniques along with a flexible choice of frequency domain input/output weightings to emphasize a frequency range of interest.

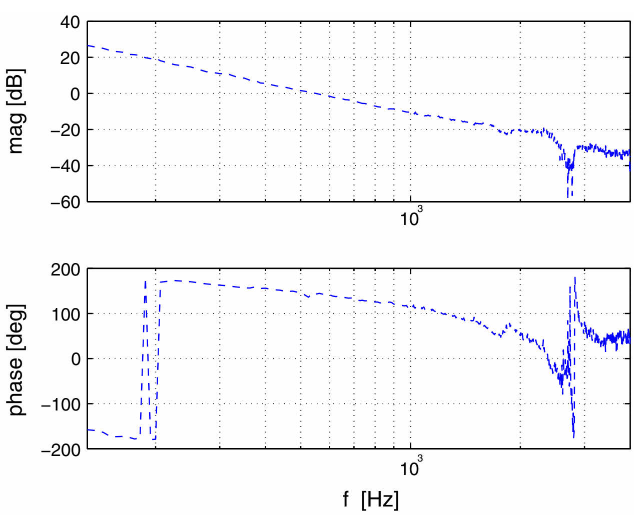

Fig. 1 Bode plot (amplitude: *top*, phase: *bottom*) of the frequency response measurements of a tape servo actuator

2.2 Notation

Consider an nth order LTI model given by a state-space representation

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Although the state-space realization is not unique, all realizations are related through similarity transformation of the state variable $x = T\overline{x}$ such that

$$\begin{aligned} \overline{x}_{t+1} &= T^{-1}AT\overline{x}_t + T^{-1}Bu_t \\ y_t &= CTx_t + Du_t \end{aligned}$$

where T is a non-singular matrix. The Markov parameters M_k are defined via series expansion of the state-space model

$$G(q) = C(qI - A)^{-1}B + D$$
$$= \sum_{k=1}^{\infty} M_k \mathrm{q}^{-k} + D : \qquad M_k = CA^{k-1}B$$

where q^{-1} is the backward time shift operator ($\mathrm{q}^{-1}x_t = x_{t-1}$). Note that Markov parameters M_k are invariant to a similarity transformation. Important matrices associated to the state-space realization are the extended observability and controllability matrices

$$\mathcal{O}^r(C,A) = \begin{bmatrix} C^T & [CA]^T & \ldots & [CA^{r-1}]^T \end{bmatrix}^T \in \mathbb{R}^{rp\times n},$$
$$\mathcal{C}^r(A,B) = \begin{bmatrix} B & AB & \ldots & A^{r-1}B \end{bmatrix} \in \mathbb{R}^{n\times rm}$$

that have full rank n for $r \geq n$ if the pairs (C, A) and (A, B) are observable and controllable respectively (Skelton 1988). Moreover, the product of $\mathcal{O}^r(C,A)$ and $\mathcal{C}^r(A,B)$ can be related to the Markov parameters M_k via the Hankel matrix

$$H^{qr}(M_k) = \mathcal{O}^q(C,A)\mathcal{C}^r(A,B) = \begin{bmatrix} M_1 & M_2 & \ldots & M_r \\ M_2 & M_3 & \ldots & M_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_q & M_{q+1} & \ldots & M_{q+r-1} \end{bmatrix} \tag{1}$$

and the rank of H^{qr} is n, provided that (C,A), (A,B) is observable and controllable, and $q,r \geq n$. Finally, the Markov parameters M_k can be related to the frequency response $G(e^{j\omega}) = C\,(e^{j\omega}I - A)^{-1}B + D$ over an equidistantly spaced frequency grid via the (inverse) discrete-time Fourier transform

$$M_k = \frac{1}{2\pi}\int_0^{2\pi} G(e^{j\omega})e^{j\omega k}\, d\omega$$
$$G(e^{j\omega}) = \sum_{k=0}^{\infty} M_k e^{-j\omega k}$$

2.3 Problem statement

Consider (noisy) equidistantly spaced frequency domain measurements $G_k = G(e^{j\omega_k}) + n_k$ for $\omega_k = \pi k/N$, $k = 0, \ldots, N$, of a nth order LTI system. Equidistantly spaced frequency measurements can be obtained from standard Fourier or spectral analysis or by linear interpolation of non-equidistantly spaced frequency domain data. Let the measurements G_k be converted to Markov parameter estimates g_k via an inverse discrete Fourier transform (IDFT) and consider the Hankel matrix $H^{qr}(g_k)$ constructed from these Markov parameter estimates as in (1). It should be noted that in general the Hankel matrix $H^{qr}(g_k)$ will have full rank $\max(q,r) \geq n$, due to noise n_k on the frequency domain measurements.

Given the (full rank) Hankel matrix $H^{qr}(g_k)$ and using $\|X\|_2$ to indicate the induced 2-norm of a matrix X:

$$\|X\|_2 = \sup_{y\neq 0}\frac{\|Xy\|_2}{\|y\|_2} = \bar{\sigma}(X)$$

where $\bar{\sigma}(X)$ is the largest singular value of X, the modeling problem considered in this paper is to find the order $\hat{n}$ and a realization $(\hat{A}, \hat{B}, \hat{C})$ from a rank $\hat{n}$ matrix $\hat{H}^{qr}$ defined by

$$\hat{H}^{qr} = \arg\min_H \|H^{qr}(g_k) - H\|_2 \tag{2}$$

It should be noted that (2) aims at finding the best rank $\hat{n}$ matrix $\hat{H}^{qr}$ of size compatible to the full rank $\max(q,r)$ Hankel matrix $H^{qr}(g_k)$. In case the frequency domain data G_k was generated by a nth order system, $\hat{n} = n$ would be a consistent estimate of the model order.

As indicated in the motivation for control relevant model estimation, the modeling procedure should facilitate the estimation of a low order model $\hat{n} < n$, emphasizing certain features in the frequency response. To accomplish this task, we will consider the weighted version of (2) given by

$$\hat{H}^{qr} = \arg\min_H \left\| \underline{\Gamma}_y^q (H^{qr}(g_k) - H)\overline{\Gamma}_u^r \right\|_2 \tag{3}$$

to find a order $\hat{n}$ and a realization $(\hat{A}, \hat{B}, \hat{C})$ of a model that approximates the measured frequency response data represented by the Markov parameter g_k in the (full rank) Hankel $H^{qr}(g_k)$. To anticipate the results mentioned in this paper, $\underline{\Gamma}_y^q$ and $\overline{\Gamma}_u^r$ in (3) are yet to be determined weighting matrices, obtained from a user specified output/input frequency weighting.

3 Frequency domain subspace identification

3.1 Model order determination

The given frequency response data G_k measured on $[0, \pi]$ can be symmetrically extended on the interval $[\pi, 2\pi]$ by taking the complex conjugate

$$G_{N+k} = G^*_{N-k} \qquad k = 1, \ldots, N-1$$

Taking the IDFT of the extended frequency response gives the estimates g_k of the Markov parameters M_k of the system

$$g_k = \frac{1}{2N}\sum_{s=0}^{2N-1} G_s e^{j2\pi ks/2N} \qquad k = 0, \ldots, 2N-1$$

Note that for a system with well-damped poles, as the number of frequency points N tend to infinity, the estimates g_k will converge to the true system Markov parameters M_k, making $H_{qr}(g_k)$ a rank n matrix. Using the fact that the complex exponential functions are orthogonal, it was shown in McKelvey (1995) that the estimated Markov parameters g_k satisfy

$$g_k = CA^{k-1}\left(\sum_{m=0}^{\infty} A^{2mN}\right)B = CA^{k-1}\left(I - A^{2N}\right)^{-1}B$$

in the noise-free case. Moreover, $\sum_{m=0}^{\infty} A^{2mN} = \left(I - A^{2N}\right)^{-1}$ holds in case $\max_i|\lambda_i(A)| < 1$. where $\lambda_i(A)$ denotes the ith eigenvalue of A. As a result, the Hankel matrix $H^{qr}(g_k)$ can be factorized as

$$H^{qr}(g_k) = \mathcal{O}^q\left(I - A^{2N}\right)^{-1}\mathcal{C}^r \tag{4}$$

under the assumption that the underlying model is discrete-time stable. With a minimal state space realization (A,B,C) it can be seen that a noise-free estimated Hankel matrix $H^{qr}(g_k)$ is also of rank n, despite the fact it was based on only a finite number N of frequency domain measurements.

As indicated before, the g_k are found via an IDFT of noisy frequency domain measurements G_k and the Hankel matrix $H^{qr}(g_k)$ will be full rank max(q,r). Using the shorthand notation $H^{qr}(g_k) = H^{qr}$, the minimization in (2) approximates H^{qr} by a matrix $\hat{H}^{qr}$ of the same size as H^{qr} with rank $\hat{n} < \max(q,r)$ by minimizing $\bar{\sigma}(H^{qr} - \hat{H}^{qr})$, where $\bar{\sigma}(H)$ denotes the maximum singular value of a matrix H. This minimization problem can be solved using a SVD, where dominant singular values of the system are separated from less important ones (Zeiger and McEwen 1974). A lower rank approximate Hankel matrix $\hat{H}^{qr}$ can be extracted by performing a SVD on H^{qr}

$$H^{qr} = \begin{bmatrix} U_{\hat{n}} & U_o \end{bmatrix} \begin{bmatrix} \Sigma_{\hat{n}} & 0 \\ 0 & \Sigma_o \end{bmatrix} \begin{bmatrix} V_{\hat{n}}^T \\ V_o^T \end{bmatrix}$$

and partitioning the result by the $\hat{n}$ largest singular values into $\Sigma_{\hat{n}}$ to construct

$$\hat{H}^{qr} = U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T \tag{5}$$

Considering the (unweighted) minimization problem in (2), it can be seen that the solution $\hat{H}^{qr}$ given in (5) satisfies $H^{qr} - \hat{H}^{qr} = U_o\Sigma_o V_o^T$ and $\bar{\sigma}\{H^{qr} - \hat{H}^{qr}\} = \sigma_{\hat{n}+1}\{H^{qr}\}$ is minimized.

Typically the McMillan degree (order) n of a system is not known and the SVD can provide insight into the order of the system as follows. In the noise-free case $\hat{n}$ can be chosen such that $\bar{\sigma}\{\Sigma_o\} = 0$ making $\hat{n} = n$. In the case of noisy frequency response measurements, plotting all non-zero singular values allows one to make a distinction between $\hat{n}$ large and $\hat{n} - \max(q,r)$ small singular values. One can determine the model order $\hat{n}$ by deciding on a level of the Hankel singular values that is significantly different from 0. The level is up to the user but should be determined by the value of the higher singular values that determine Σ_o. In addition, emphasizing frequency data around the cross-over frequency to find models relevant for control design allows one to choose a value $\hat{n} < n < \max(q,r)$ in order to find a low order approximation. Once the SVD and $\hat{n}$ have been determined, the realization of the state-space model is relatively straightforward and various model orders can be compared easily.

3.2 State-space realization from $\hat{H}^{qr}$

Following the Ho-Kalman algorithm (Ho and Kalman 1966), a state-space system representation can be derived from (1) by recognizing that the Hankel matrix H^{qr} has the same column space as the extended observability matrix $\mathcal{O}^q$. Although there are several methods for extracting a state-space realization $(\hat{A}, \hat{B}, \hat{C})$, a straightforward method uses the rank $\hat{n}$ approximation $\hat{H}^{qr}$ and a shifted version $\overline{H}^{qr}$ of the Hankel matrix H^{qr} defined by

$$\overline{H}^{qr} = \begin{bmatrix} g_2 & g_3 & \cdots & g_{r+1} \\ g_3 & g_4 & \cdots & g_{r+2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{q+1} & g_{q+2} & \cdots & g_{q+r} \end{bmatrix}$$

The matrix H^{qr} (based on noisy data) can be written in a factorization

$$\begin{aligned} H^{qr} &= H_1H_2 + E = U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T + U_o\Sigma_oV_o^T \\ &= \hat{H}^{qr} + U_o\Sigma_oV_o^T \end{aligned} \tag{6}$$

where E indicates an error matrix due to the noisy frequency domain measurements. Using $g_k = CA^{k-1}B$, it is straightforward to see that

$$\overline{H}^{qr} = H_1AH_2 + \bar{E} \tag{7}$$

where $\bar{E}$ is a shifted version of the same error matrix E. With the choice $\hat{n} \le n$, an estimate $\hat{A}$ of size $n \times n$ can be computed via a least squares (LS) solution

$$\hat{A} = H_1^{\dagger}\overline{H}^{qr}H_2^{\dagger} = \Sigma_{\hat{n}}^{-1/2}U_{\hat{n}}^T\overline{H}^{qr}V_{\hat{n}}\Sigma_{\hat{n}}^{-1/2}$$

where the non-singularity of $\Sigma_{\hat{n}}$ is guaranteed as $\hat{n} \le n$.

Finally it can be observed from (4) that the Hankel matrix H^{qr} can be written as the decomposition

$$H^{qr} = \mathcal{O}^q\left(I - A^{2N}\right)^{-1}\mathcal{C}^r + E$$

in which the extended observability and controllability Gramians of the (unknown) state space realization (A,B,C) appear. Using the same decomposition for the rank $\hat{n}$ matrix $\hat{H}^{qr} = U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T$ allows $\hat{H}^{qr}$ to be written as

$$\hat{H}^{qr} = \hat{\mathcal{O}}^q\left(I - \hat{A}^{2N}\right)^{-1}\hat{\mathcal{C}}^r + \hat{E} \tag{8}$$

where $\hat{\mathcal{O}}^q$ and $\hat{\mathcal{C}}^r$ are the extended observability and controllability Gramians

$$\hat{\mathcal{O}}^q = \begin{bmatrix}\hat{C}^T & [\hat{C}\hat{A}]^T & \dots & [\hat{C}\hat{A}^{q-1}]^T\end{bmatrix}^T$$
$$\hat{\mathcal{C}}^r = \begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \dots & \hat{A}^{r-1}\hat{B}\end{bmatrix}$$

of the to be computed state space realization $(\hat{A}, \hat{B}, \hat{C})$. The error matrix $\hat{E} = 0$ in case $\hat{H}^{qr}$ obeys a Hankel structure similar as in (1). Since $\hat{H}^{qr}$ is computed via a SVD of a Hankel matrix $H^{qr}, \bar{\sigma}\{\hat{E}\} = 0$ in case $\bar{\sigma}\{H^{qr} - \hat{H}^{qr}\} = \sigma_{\hat{n}+1}\{H^{qr}\} = 0$. As indicated before, the (unweighted) minimization problem in (2) solved by a SVD minimizes $\bar{\sigma}\{H^{qr} - \hat{H}^{qr}\}$, resulting in a small error matrix $\hat{E}$. This allows the computation of the extended observability and controllability matrices via

$$\hat{\mathcal{O}}^q = U_{\hat{n}}\Sigma_{\hat{n}}^{1/2}\left(I - \hat{A}^{2N}\right)^{1/2}$$
$$\hat{\mathcal{C}}^r = \left(I - \hat{A}^{2N}\right)^{1/2}\Sigma_{\hat{n}}^{1/2}V_{\hat{n}}^T$$

The matrices $\hat{C}$ and $\hat{B}$ can now be extracted from the first p rows and m columns of $\hat{\mathcal{O}}^q$ and $\hat{\mathcal{C}}^r$ via $\hat{C} = J_C\hat{\mathcal{O}}^q$ and $\hat{B} = \hat{\mathcal{C}}^r J_B$, where

$$J_C = \begin{bmatrix}I_p & 0_{p\times(q-1)p}\end{bmatrix}, \quad J_B = \begin{bmatrix}I_m & 0_{(r-1)m\times m}\end{bmatrix}^T \tag{9}$$

and I_i denotes the $i \times i$ identity matrix, $0_{k\times l}$ the $k \times l$ zero matrix.

4 Frequency weighted subspace identification

4.1 Frequency weighted balanced truncation

Frequency dependent weightings in a model identification algorithm allows one to place emphasis on frequency data (amplitude and phase) around the cross-over frequency of the servo system to find models relevant for control design. In subspace identification, the frequency weighted balanced truncation technique introduced by Enns (1984), and further developed in Van Overschee and De Moor (1993), can be used to achieve control-relevant weighting of the identified models and the weighted subspace formulation was given by the minimization posed in (3).

Consider the cascaded system consisting of an input weighting filter F_u, the plant G and an output weighting filter F_y. The weighting filters are user specified, and without loss of generality, should be stable and stable invertible. The input of the filter F_u is $u \in \mathbb{R}^m$, the input of the plant is given by the signal $w \in \mathbb{R}^m$, the output of the plant is $y \in \mathbb{R}^p$ and the output of the weighting filter is designated as $z \in \mathbb{R}^p$ (Fig. 2).

It is shown in Claes (2006) that an input/output frequency domain weighting in a subspace technique based on equidistantly spaced frequency domain data simply requires a weighted SVD of the matrix $\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r$. Similar as in the unweighted case, the matrix H^{qr} is the Hankel matrix built from the (noisy) estimates of the Markov parameters of the system. The matrices $\underline{\Gamma}_y^q$ and $\overline{\Gamma}_u^r$ are Toeplitz matrices that can be computed from the Markov parameters of the input/output weightings.

Let (A_y,B_y,C_y,D_y) denote the state space realization of the output-filter F_y and let (A_u,B_u,C_u,D_u) be the state space realization of the input-filter F_u, then the Toeplitz matrices $\underline{\Gamma}_y^q$ and $\overline{\Gamma}_u^r$ are given by

$$\underline{\Gamma}_y^q = \begin{bmatrix} D_y & 0 & \dots & 0 \\ C_yB_y & D_y & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_yA_y^{q-2}B_y & C_yA_y^{q-3}B_y & \dots & D_y \end{bmatrix}$$
$$\overline{\Gamma}_u^r = \begin{bmatrix} D_u & C_uB_u & \dots & C_uA_u^{r-2}B_u \\ 0 & D_u & \dots & C_uA_u^{r-3}B_u \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_u \end{bmatrix}$$

It can be seen that $\underline{\Gamma}_y^q$ is a lower triangular Toeplitz matrix constructed from the Markov parameters of the output filter and $\overline{\Gamma}_u^r$ is an upper triangular Toeplitz matrix that contains the Markov parameters of the input weighting filter. Since the weighting filters are stable and stable invertible, the matrices $\underline{\Gamma}_y^q$ and $\overline{\Gamma}_u^r$ will be invertible and have full rank. In case the state-space

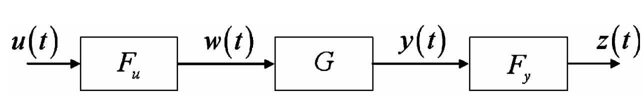

Fig. 2 Input filter–plant–output filter cascaded system

realizations of the weighting filters are unknown, the Markov parameters of the filters can be constructed from the frequency response data of the filters by IDFT.

As the matrices $\underline{\Gamma}_y^q$ and $\overline{\Gamma}_u^r$ have full rank, the rank properties of the Hankel matrix are preserved. As a result, $\operatorname{rank}\left(\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r\right) = \operatorname{rank}(H^{qr})$ and taking the SVD of $\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r$ yields a weighted low rank matrix approximation $\underline{\Gamma}_y^q \hat{H}^{qr}\overline{\Gamma}_u^r$ given by

$$\underline{\Gamma}_y^q \hat{H}^{qr}\overline{\Gamma}_u^r = U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T, \quad \text{with}$$
$$\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r = [U_{\hat{n}} \quad U_o]\begin{bmatrix}\Sigma_{\hat{n}} & 0\\ 0 & \Sigma_o\end{bmatrix}\begin{bmatrix}V_{\hat{n}}^T\\ V_o^T\end{bmatrix}$$

Using a computational procedure analogous to (6) and (7) allows the $\hat{A}$ matrix to be computed from

$$\hat{A} = \Sigma_{\hat{n}}^{-1/2}U_{\hat{n}}^T\underline{\Gamma}_y^q\overline{H}^{qr}\overline{\Gamma}_u^r V_{\hat{n}}\Sigma_{\hat{n}}^{-1/2}$$

Furthermore, writing an equivalent expression as (8) for the matrix $\underline{\Gamma}_y^q \hat{H}^{qr}\overline{\Gamma}_u^r$ yields

$$\underline{\Gamma}_y^q \hat{H}^{qr}\overline{\Gamma}_u^r = U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T = \underline{\Gamma}_y^q\hat{\mathcal{O}}^q\left(I-\hat{A}^{2N}\right)^{-1}\hat{\mathcal{C}}^r\overline{\Gamma}_u^r + \hat{E} \qquad (10)$$

As in the unweighted case, if $\hat{E}=0$ then $\hat{H}^{qr}$ obeys a Hankel structure and the extended observability and controllability matrices can be found from (10). With a small error matrix $\hat{E}$, the matrices $\hat{C}$ and $\hat{B}$ can be computed via

$$\hat{C} = J_C\left(\underline{\Gamma}_y^q\right)^{-1}U_{\hat{n}}\Sigma_{\hat{n}}^{1/2}\left(I-\hat{A}^{2N}\right)^{1/2}$$
$$\hat{B} = \left(I-\hat{A}^{2N}\right)^{1/2}\Sigma_{\hat{n}}^{1/2}V_{\hat{n}}^T\left(\overline{\Gamma}_u^r\right)^{-1}J_B$$

with the operators J_C and J_B defined in (9).

The method described here can also be used when there is only input weighting or only output weighting. Respectively the matrix $\underline{\Gamma}_y^q$ or $\overline{\Gamma}_u^r$ is then set equal to the identity matrix and the remainder of the procedure is the same as above. If in a SISO identification problem an input weighting filter is applied (with no output weighting) and then the same filter is used as output weighting (but without input weighting), the algorithm will return the same input-output model in both cases.

4.2 Summary of weighted equidistant frequency data subspace algorithm

To summarize the computational procedure for the weighted frequency domain subspace method proposed in this paper, consider complex valued frequency response measurements G_k of a plant and frequency domain points F_{y_k}, F_{u_k} respectively of an output and input weighting filter given at k = 0,..., N equidistantly spaced frequency points $\omega_k = \pi k/N$. The algorithm for finding a minimal state-space realization of the plant is then given by the following steps:

1. Extend the frequency response samples to the full unit circle:

$$\begin{cases} G_{N+k} = G_{N-k}^* \\ F_{y_{N+k}} = F_{y_{N-k}}^*, \qquad k = 1,\ldots,N-1 \\ F_{u_{N+k}} = F_{u_{N-k}}^* \end{cases}$$

2. Let g_i, g_{y_i} and g_{u_i} be defined by the $2N$-point IDFT:

$$\begin{cases} g_i = \frac{1}{2N}\sum_{k=0}^{2N-1} G_k e^{j2\pi ik/2N} \\ g_{y_i} = \frac{1}{2N}\sum_{k=0}^{2N-1} F_{y_k} e^{j2\pi ik/2N}, \qquad i = 0,\ldots,q+r-1 \\ g_{u_i} = \frac{1}{2N}\sum_{k=0}^{2N-1} F_{u_k} e^{j2\pi ik/2N} \end{cases}$$

3. Then the block Hankel matrix $H^{qr} \in \mathbb{R}^{qp\times rm}$ and shifted block Hankel matrix $\overline{H}^{qr} \in \mathbb{R}^{qp\times rm}$ are defined as:

$$H^{qr} = \begin{bmatrix} g_1 & g_2 & \cdots & g_r \\ g_2 & g_3 & \cdots & g_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ g_q & g_{q+1} & \cdots & g_{q+r-1} \end{bmatrix},$$
$$\overline{H}^{qr} = \begin{bmatrix} g_2 & g_3 & \cdots & g_{r+1} \\ g_3 & g_4 & \cdots & g_{r+2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{q+1} & g_{q+2} & \cdots & g_{q+r} \end{bmatrix}$$

and the Toeplitz matrices corresponding to the filters ($\overline{\Gamma}_u^r \in \mathbb{R}^{rm\times rm}$ and $\underline{\Gamma}_y^q \in \mathbb{R}^{qp\times qp}$):

$$\overline{\Gamma}_u^r = \begin{bmatrix} g_{u_0} & g_{u_1} & \cdots & g_{u_{r-1}} \\ 0 & g_{u_0} & \cdots & g_{u_{r-2}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & g_{u_0} \end{bmatrix}, \quad \underline{\Gamma}_y^q = \begin{bmatrix} g_{y_0} & 0 & \ldots & 0 \\ g_{y_1} & g_{y_0} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{y_{q-1}} & g_{y_{q-2}} & \cdots & g_{y_0} \end{bmatrix}$$

where $q+1 > \hat{n}, r \geq \hat{n}$ and $q + r + 1 \leq 2N$.

4. Calculate the SVD of $\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r$, determine the order $\hat{n}$ of the plant by inspecting the singular values and partition the SVD so that $\Sigma_{\hat{n}} = \operatorname{diag}\{\sigma_1,\ldots,\sigma_{\hat{n}}\}$ contains the $\hat{n}$ largest singular values:

$$\underline{\Gamma}_y^q H^{qr}\overline{\Gamma}_u^r = [U_{\hat{n}} \quad U_o]\begin{bmatrix}\Sigma_{\hat{n}} & 0\\ 0 & \Sigma_o\end{bmatrix}\begin{bmatrix}V_{\hat{n}}^T\\ V_o^T\end{bmatrix}$$

5. Determine the state-space matrices $\hat{A}, \hat{C}$ and $\hat{B}$ as:

$$\begin{aligned}\hat{A} &= \Sigma_{\hat{n}}^{-1/2} U_{\hat{n}}^T \underline{\Gamma}_y^q \overline{H}^{qr} \overline{\Gamma}_u^r V_{\hat{n}} \Sigma_{\hat{n}}^{-1/2} \\ \hat{C} &= J_C \left(\underline{\Gamma}_y^q\right)^{-1} U_{\hat{n}} \Sigma_{\hat{n}}^{1/2} \left(I - \hat{A}^{2N}\right)^{1/2} \\ \hat{B} &= \left(I - \hat{A}^{2N}\right)^{1/2} \Sigma_{\hat{n}}^{1/2} V_{\hat{n}}^T \left(\overline{\Gamma}_u^r\right)^{-1} J_B\end{aligned} \tag{11}$$

where:

$$J_C = \begin{bmatrix} I_p & 0_{p \times (q-1)p} \end{bmatrix}$$
$$J_B = \begin{bmatrix} I_m \\ 0_{(r-1)m \times m} \end{bmatrix}$$

and I_i denotes the $i \times i$ identity matrix and $0_{k \times l}$ the $k \times l$ zero matrix.

6. In case the discrete-time model has no time-delay, an additional feedthrough matrix $\hat{D} \neq 0$ has to be estimated. The weighted LS problem

$$\hat{D} = \arg \min_{D \in \mathbb{R}^{p \times m}} \times \sum_{k=0}^{N} \left\| F_{y_k} \left(G_k - D - \hat{C} \left(e^{j\omega_k} I - \hat{A} \right)^{-1} \hat{B} \right) F_{u_k} \right\|_F^2 \tag{12}$$

where $\|X\|_F^2 = \sum_k \sum_s |x_{ks}|^2$ is the Frobenius norm, can be used to compute the unique minimizing argument $\hat{D}$.

7. The resulting estimated frequency response of the input–output model on the basis of the matrices $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is given by:

$$\hat{P}(e^{j\omega}) = \hat{C}\left(e^{j\omega} I - \hat{A}\right)^{-1} \hat{B} + \hat{D} \tag{13}$$

5 Application to tape servo system

5.1 Experimental set-up

Consider the schematic diagram given in Fig. 3 of a linear tape open (LTO) tape drive where modifications were made to allow for injection of a reference signal R into the closed loop system. The PES of the read/write head from the desired position on the track is fed back as input to the servo controller C. The controller provides a control signal U as the input to the servo actuator dynamics P. When the reference signal R is sufficiently exciting, frequency domain measurements of various closed-loop transfer functions can be estimated via spectral analysis (Ljung 1999). For notational purposes we define the spectral estimates

$$Q_k = \frac{\phi_{\mathrm{PES},R}(\omega_k)}{\phi_R(\omega_k)}, \quad S_k = \frac{\phi_{U,R}(\omega)}{\phi_R(\omega)} \tag{14}$$

where $\varphi_{\mathrm{PES}}, R(\omega_k)$ and $\varphi_{U,R}(\omega)$ indicated the cross spectral estimates between PES, control signal U and reference signal R. The spectral estimate S_k denotes the frequency domain measurements of the (discrete time) error rejection or sensitivity function $S(e^{j\omega}) = (1 + P(e^{j\omega})C(e^{j\omega}))^{-1}$ that characterizes the performance of the servo system.

Taking the ratio of the closed-loop transfer function estimates in (14) yields the frequency domain measurements $G_k = Q_k/S_k$ of the open-loop (uncontrolled) servo actuator P. Motivated by stability requirements of the servo system, control relevant estimation of a model $\hat{P}$ requires frequency dependent weighting filters to emphasize the frequency range around the cross-over frequency (Gevers 2002; Hjalmarsson 2005; Van Den Hof and Schrama 1995). For a disturbance rejection performance criteria, an appropriate control relevant frequency weighting filter F can be computed by

$$\frac{1}{1+CP} - \frac{1}{1+C\hat{P}} = F[P - \hat{P}], \quad F = \frac{C}{(1+CP)(1+C\hat{P})}$$

In case $\hat{P}$ is a good model of the servo actuator dynamics P, the filter F can be approximated by the frequency domain data of the (output) filter

$$F_{y_k} = C_k S_k^2 \tag{15}$$

using the knowledge of the currently implemented servo controller C and the measurements of S_k obtained via the spectral analysis in (14). The frequency domain data F_{y_k} will be used as an output weighting in the frequency weighted subspace method.

5.2 Subspace identification results

Measurements were taken using a Siglab data acquisition system, which allows for easy post-processing in Matlab. Excitation was done with a chirp signal up to the Nyquist frequency of 4 kHz to obtain frequency domain measurements of Q_k and S_k in (14) and used to compute the frequency domain data $G_k = Q_k/S_k$ of the open-loop (uncontrolled) tape servo actuator.

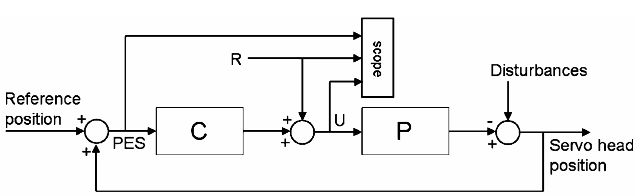

Fig. 3 Schematic of the tape servo system and relevant signals for identification purposes

Applying the control relevant filter (15) as output weighting and computation of the weighted SVD of the Hankel matrix returns the singular value plot depicted in Fig. 4.

As shown in Sect. 3.1, one of the advantages of subspace identification techniques is the selection of the model order. From the singular values of the Hankel matrix one can determine the model order such that the estimated model captures most of the relevant dynamics present in the frequency response data. One can determine the model order by drawing a line, that indicates a level of the Hankel singular values that is significantly different from 0. The level of this line is up to the user but should be determined by the value of the tail of the higher singular values in the Hankel Singular Value plot as drawn in Fig. 4.

Drawing a level line around the value of 1 in Fig. 4, indicates that 11 Hankel singular values are significantly different from 0. This results in the choice of an 11th order model that should be able to capture the main dynamic phenomena represented in the weighted frequency response. Selection of $\hat{n} = 11$ and computing a state space realization via (11) followed by a weighted LS estimation (12) yields a Bode plot of the model (13) that is given in Fig. 5.

It can be observed from Fig. 5 that the 11th order subspace model fits the data closely around the current servo bandwidth of 500 Hz, as both amplitude and phase Bode plots of the estimated model follow the measured frequency response data. A verification of the accuracy of the model for servo system design is the comparison of the measured frequency response S_k of the current sensitivity function or error rejection function $S(e^{j\omega})$ and the sensitivity function

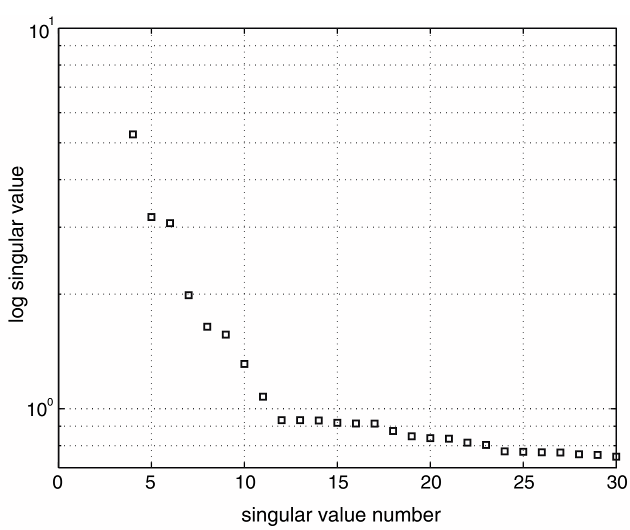

Fig. 4 Logarithmic distribution of singular values found by the singular value decomposition of the weighted Hankel matrix

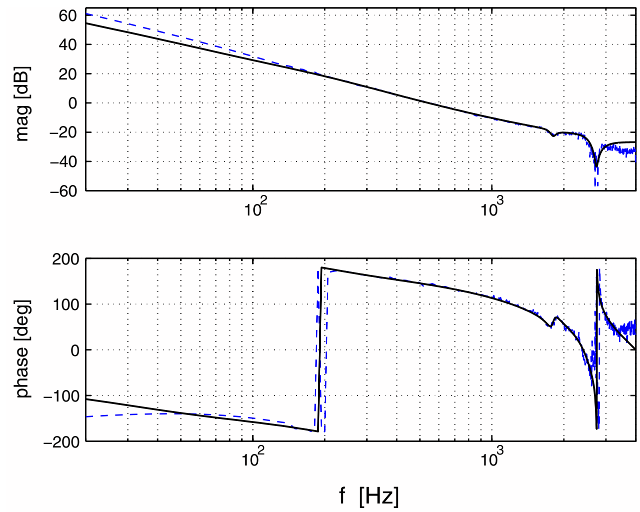

Fig. 5 Bode plot comparison of frequency domain data G_k (*dashed*) with frequency response $\hat{P}(e^{jw})$ of the estimated 11th order subspace model (solid)

$$\hat{S}(e^{j\omega_k}) = \frac{1}{(1 + \hat{P}(e^{j\omega_k})C(e^{j\omega_k}))}$$

as predicted by the estimated model $\hat{P}(e^{j\omega})$ in (13). This comparison has been depicted in Fig. 6 where it can be seen that both the measured closed-loop frequency domain data S_k and the modeled closed-loop behavior $\hat{S}(e^{j\omega_k})$ on the basis of the 11th order model captures all the main resonance modes in the closed-loop error rejection function.

6 Conclusion

Subspace system identification algorithms rely on linear algebra results for realizing consistent model estimates. A frequency weighted balanced realization technique was presented for determining lower order approximations for system dynamics. While preserving the beneficial computational properties associated with standard realization algorithms, the formulation allows for the use of user-specified frequency dependent weightings to emphasize frequency data around the cross-over frequency to find models relevant for control design.

The algorithm is applied to closed-loop frequency domain measurements obtained from a tape storage device. A SVD on a frequency weighted Hankel matrix is used to determine appropriate model order for closed-loop model estimates as well as for the servo actuator model estimates. Low-order control relevant models were realized through frequency weighted subspace identification techniques. The frequency

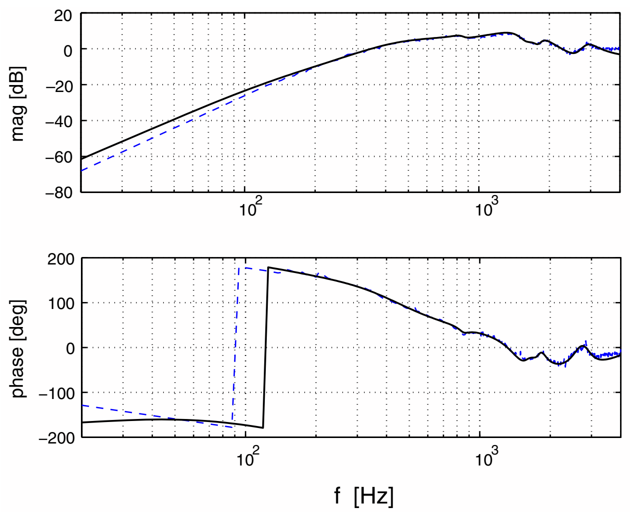

Fig. 6 Bode plot comparison of closed-loop frequency domain data S_k (*dashed*) with frequency response of the computed sensitivity function $\hat{S}(e^{j\omega_k})$ (solid)

weighted subspace identification algorithm could be used in alleviating or reducing the computational burden associated with the non-linear optimizations involved in standard frequency domain curve fitting techniques and can easily be extended to identification of multivariable models in case of dual-stage actuation of a tape servo system.

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