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Frequency Domain Conditions via Linear Matrix Inequalities

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Abstract— **This paper revisits a pair of Linear Matrix Inequalities (LMIs) that are related to checking a frequency domain inequality (FDI) over a finite interval. The first contribution is to show that the proposed pair of LMIs contain the original formulation of the Kalman-Yakubovich-Popov Lemma when the coefficient matrix is constant. The coefficient matrix can be made affine on the frequency variable at no extra computational cost. The second contribution is to show how to transform the frequency variable in order to extend the proposed results to infinite frequency intervals. In applications such as robustness analysis, allowing for frequency dependent coefficient matrices can be significant in reducing conservatism, a feature which is illustrated with a simple numerical example.**

I. Introduction

Specifications for performance and robustness of dynamical systems are commonly described in terms of frequency domain inequalities (FDIs), which due to infinite dimensionality are not directly tractable in analysis and design. The Kalman-Yakubovich-Popov (KYP) Lemma, also known as the positive real lemma, is a fundamental result in systems theory that establishes equivalence between an infinite dimensional FDI, namely

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} \prec 0 \qquad (1)$$

for all $\omega \in \mathbb{R}$, where A, B and Hermitian Θ have appropriate finite dimension, and the search for Hermitian matrix P such that the linear matrix inequality (LMI)

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}0 & P \\ P & 0\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0 \qquad (2)$$

is feasible [1], [2]. Many problems in systems and control theory can be posed in the form (1) where appropriate choices for Θ represent the analysis of various system properties. The significance of the KYP Lemma lies in the reduction of infinitely many FDIs to a single LMI. In this original form, the KYP Lemma holds for all frequencies $\omega \in \mathbb{R}$, which can be conservative in practical applications where specifications of interest are considered only over finite frequency ranges.

The finite frequency KYP Lemma was considered in [3] where equivalence was established between the FDI (1) over finite interval $\omega_1 \leq \omega \leq \omega_2$ and a LMI

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1\omega_2 Q\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Theta \prec 0,$$
$$\omega_c := (\omega_1 + \omega_2)/2, \qquad \omega_1 \leq \omega \leq \omega_2 \quad (3)$$

with Hermitian matrix variables P and Q, where Q is positive semidefinite. The matrix in (3) that characterizes finite frequency intervals provides a multiplier relaxation, see [4], for the LMI conditions presented in the original KYP Lemma. Further generalizations, features and discussion of the above results can be found in [5]. Additionally, for particular choice of coefficient matrix Θ, the finite frequency KYP Lemma specifies a μ-analysis problem over finite interval that uses constant scaling matrices, which can be shown through the losslessness of the D, G-scalings [3], [6].

In [7] the authors proved sufficiency of the pair of LMI

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}I \\ j\omega_i I\end{bmatrix} \begin{bmatrix}F \\ G\end{bmatrix}^* + \begin{bmatrix}F \\ G\end{bmatrix} \begin{bmatrix}I \\ j\omega_i I\end{bmatrix}^* \begin{bmatrix}A & B \\ I & 0\end{bmatrix}$$
$$+ \Theta \prec 0, \qquad i = \{1, 2\}, \quad (4)$$

on complex matrix variables F and G, in the context of robust analysis. The first contribution of this paper is to prove the necessity of the pair of LMI (4), that is if (1) hold for all $\omega_1 \leq \omega \leq \omega_2$, or equivalently (3) holds for some P and $Q \succ 0$, then there exists a particular choice of F and G which is guaranteed to satisfy the above inequalities.

At no extra computational cost, the class of coefficient matrices Θ can be extended to be affine on the frequency variable ω. As in [7], given Hermitian matrices Θ_1, Θ_2, if the LMI

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}I \\ j\omega_i I\end{bmatrix} \begin{bmatrix}F \\ G\end{bmatrix}^* + \begin{bmatrix}F \\ G\end{bmatrix} \begin{bmatrix}I \\ j\omega_i I\end{bmatrix}^* \begin{bmatrix}A & B \\ I & 0\end{bmatrix}$$
$$+ \Theta_i \prec 0, \qquad i = \{1, 2\} \quad (5)$$

has some feasible solution F and G, then the FDI (1) holds over the finite frequency interval $\omega_1 \leq \omega \leq \omega_2$ with

$$\Theta(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1}\Theta_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1}\Theta_2.$$

The second contribution of this paper is to show how the LMI (5) can be modified to handle infinite frequency ranges. The μ-analysis application of [7] is revisited with the developments presented in the current paper, particularly focusing on the reduction in conservativeness when evaluating upper bounds on the entire frequency interval.

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A. Notation

The following notation will be used throughout the paper. The scalar $j = \sqrt{-1}$. For a matrix $X \in \mathbb{C}^{n\times n}$: $\overline{X}$, X^* are the complex-conjugate and complex-conjugate transpose of the matrix X respectively and X^{-1}, $X_\perp$ are full rank matrices such that $XX^{-1} = I$ and $XX_\perp = 0$. He$\{X\}$ is short-hand notation for $X + X^*$. We denote by $\mathbb{HC}^n$ the space of $\mathbb{C}^{n\times n}$ Hermitian matrices.

II. Finite Frequency KYP Lemma

The following finite frequency KYP Lemma is developed from a generalized S-procedure [3]. It gives necessary and sufficient conditions for replacing FDI constraints with a pair of inequalities by incorporating a multiplier matrix that is lossless in characterizing finite frequency intervals.

Lemma 1: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and a Hermitian matrix $\Theta \in \mathbb{HC}^{n+m}$ be given. Let scalars $\omega_1, \omega_2 \in \mathbb{R}$ be also given, then the following statements are equivalent.

(i) The finite FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^* \Theta \begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0, \quad (6)$$

holds for all $\omega_1 \le \omega \le \omega_2$.

(ii) There exist matrices $P, Q \in \mathbb{HC}^n$ such that $Q \succeq 0$ and

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q & P + j\omega_c Q\\ P - j\omega_c Q & -\omega_1\omega_2 Q\end{bmatrix} \begin{bmatrix}A & B\\ I & 0\end{bmatrix} + \Theta \prec 0, \quad (7)$$

where $\omega_c := (\omega_1 + \omega_2)/2$.

Proof of the above result is omitted here since it is readily available in [3] or [5] with more discussion and applications.

Our first contribution is to show necessity of the results in [7], that is that feasibility of the FDI (6) implies feasibility of the pair of LMI (4). The following theorem can be seen as an alternative finite frequency KYP Lemma.

Theorem 2: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and a Hermitian matrix $\Theta \in \mathbb{HC}^{n+m}$ be given. Let scalars $\omega_1, \omega_2 \in \mathbb{R}$ be also given, then the following statements are equivalent.

(i) The FDI (6) holds for all $\omega_1 \le \omega \le \omega_2$.

(ii) There exist matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$ such that

$$\mathrm{He}\left\{\begin{bmatrix}F\\ G\end{bmatrix}\begin{bmatrix}I & -j\omega_i I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta \prec 0, \quad i = \{1, 2\}. \quad (8)$$

Proof: Follow the methods in [7] to show that (ii) implies (i). To show that (i) implies (ii), assume that (6) holds for all $\omega_1 \le \omega \le \omega_2$. From Lemma 1 there exists matrices $P, Q \in \mathbb{HC}^n$, $Q \succeq 0$ such that (7) holds. Define the matrix

$$X(\omega) := \begin{bmatrix}1 & -j\omega\\ j\omega & \hat\omega^2 + 2\omega\,\omega_c - \omega_c^2\end{bmatrix} \otimes Q, \quad (9)$$

where $\hat\omega = (\omega_2 - \omega_1)/2$ and $\omega_c = (\omega_1 + \omega_2)/2$. Since $Q \succeq 0$ the equivalence

$$\begin{bmatrix}1 & -j\omega\\ j\omega & \hat\omega^2 + 2\omega\,\omega_c - \omega_c^2\end{bmatrix} \otimes Q \succeq 0 \quad \Leftrightarrow \quad \hat\omega^2 - (\omega - \omega_c)^2 \ge 0,$$

can be established from using Schur complement. Note that

$$\hat\omega^2 - (\omega - \omega_c)^2 = (\omega - \omega_1)(\omega_2 - \omega) \ge 0,$$

which implies that $X(\omega)$ is positive semidefinite for all $\omega_1 \le \omega \le \omega_2$. Now add the matrix

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^* X(\omega) \begin{bmatrix}A & B\\ I & 0\end{bmatrix} \succeq 0, \qquad \omega_1 \le \omega \le \omega_2.$$

to the right hand side of (7) to obtain

$$\Theta \prec \begin{bmatrix}A & B\\ I & 0\end{bmatrix}^* \Psi \begin{bmatrix}A & B\\ I & 0\end{bmatrix},$$

where

$$\Psi := \left(\begin{bmatrix}Q & -P - j\omega_c Q\\ -P + j\omega_c Q & \omega_1\omega_2 Q\end{bmatrix} + \begin{bmatrix}Q & -j\omega Q\\ j\omega Q & (\hat\omega^2 + 2\,\omega\,\omega_c - \omega_c^2)Q\end{bmatrix}\right).$$

Note that

$$\begin{aligned}\Psi &= \begin{bmatrix}2Q & -P - j(\omega + \omega_c)Q\\ -P + j(\omega + \omega_c)Q & 2\,\omega\,\omega_c Q\end{bmatrix},\\ &= \begin{bmatrix}Q\\ -P + j\omega_c Q\end{bmatrix}\begin{bmatrix}I & -j\omega I\end{bmatrix} + \begin{bmatrix}I\\ j\omega I\end{bmatrix}\begin{bmatrix}Q & -P - j\omega_c Q\end{bmatrix}.\end{aligned}$$

This implies that

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^*\left(\begin{bmatrix}Q\\ -P + j\omega_c Q\end{bmatrix}\begin{bmatrix}I & -j\omega I\end{bmatrix} + \begin{bmatrix}I\\ j\omega I\end{bmatrix}\begin{bmatrix}Q & -P - j\omega_c Q\end{bmatrix}\right)\begin{bmatrix}A & B\\ I & 0\end{bmatrix} + \Theta \prec 0.$$

Choosing

$$\begin{bmatrix}F\\ G\end{bmatrix} = \begin{bmatrix}A & B\\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q\\ P - j\omega_c Q\end{bmatrix} \quad (10)$$

we have that

$$\mathrm{He}\left\{\begin{bmatrix}F\\ G\end{bmatrix}\begin{bmatrix}I & -j\omega I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta \prec 0 \quad (11)$$

holds for all $\omega_1 \le \omega \le \omega_2$. In particular, for $\omega = \omega_1$ and $\omega = \omega_2$ which imply that the pair of inequalities (8) are feasible, therefore that (ii) should hold. ∎

Now consider the affine function

$$\Theta(\omega) := \lambda(\omega)\Theta_1 + [1-\lambda(\omega)]\Theta_2, \qquad (12)$$

where $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ and

$$\lambda(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1}, \qquad \omega_1 \le \omega \le \omega_2, \qquad (13)$$

so that $\lambda \in [0,1]$. Note that $\Theta(\omega)$ is not a proper rational function of ω.

The above theorem presents the case where Θ is constant. This case can be extended to incorporate a particular class of frequency dependent Θ without incurring extra computational cost in solving the LMI (8). The next theorem, which has been introduced in [7], extends the class of coefficient matrices allowed in the LMI (8).

Theorem 3: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. Let ω_1, $\omega_2 \in \mathbb{R}$ be also given. If there exist matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$ such that

$$\mathrm{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -j\omega_i I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta_i \prec 0, \qquad i = \{1,2\} \qquad (14)$$

then the FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^*\Theta(\omega)\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0 \qquad (15)$$

holds for all $\omega_1 \le \omega \le \omega_2$ with $\Theta(\omega)$ as given in (12).

Proof: See [7]. ■

Results obtained for the affine function $\Theta(\omega)$ in (12) can be easily extended to the piecewise affine case by juxtaposition of the conditions to be obtained on the appropriate frequency intervals, i.e.

$$\Theta(\omega) := \begin{cases}\lambda_1(\omega)\Theta_1 + [1-\lambda_1(\omega)]\Theta_2, & \omega_1 \le \omega \le \omega_2,\\ \lambda_2(\omega)\Theta_3 + [1-\lambda_2(\omega)]\Theta_4, & \omega_3 \le \omega \le \omega_4,\\ \quad\vdots & \\ \lambda_N(\omega)\Theta_{2N-1} + [1-\lambda_N(\omega)]\Theta_{2N}, & \omega_{2N-1} \le \omega \le \omega_{2N},\end{cases} \qquad (16)$$

where N is any integer, $\lambda_i(\omega) := (\omega_{2i} - \omega)/(\omega_{2i} - \omega_{2i-1})$ and Θ_{2i}, $\Theta_{2i-1} \in \mathbb{HC}^{n+m}$ for $i = 1, \dots, N$. Theorem 3 can be extended to cope with piecewise affine functions of the form (16) by simply solving one pair of inequalities (14) for each sub-interval $\omega_{2i-1} \le \omega \le \omega_{2i}$, $i = 1, \dots, N$. The resulting conditions are omitted for brevity, however an interesting case of piecewise affine coefficient matrices will be developed in Section III-C which enables an alternative KYP Lemma that holds for the infinite frequency interval.

III. Infinite Frequency KYP Lemma

Notice that Theorem 3 has difficulties in handling unbounded frequency ranges, for instance in the case $\omega_2 \to \infty$. One could conceptually search for limits on the problem variables as ω_2 increases by solving a sequence of pairs of inequalities (14). A more elegant solution, however, is to transform the frequency variable and solve a modified problem on the transformed frequency that now has a finite limit. This is done in this section. Note that only sufficiency will be derived here, but that the results all become necessary for constant Θ, see [8] for this presentation in greater generality.

A. High-Frequency Condition

Consider the transformation of the frequency variable

$$\psi = \omega^{-1}, \qquad (17)$$

which is useful in altering the high-frequency extreme case $\omega \to \infty$ to the low-frequency case $\psi \to 0$. This transformation has been used in [9] to develop a high-frequency version of Lemma 1, namely establishing the equivalence between the FDI (6) for all $|\omega| \ge \hat{\omega}$ and the LMI

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^*\begin{bmatrix}Q & P\\ P & -\hat{\omega}^2 Q\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix} + \Theta \prec 0. \qquad (18)$$

Note that if $\hat{\omega} = 0$, then Q can be set to zero, recovering the original formulation of the KYP Lemma (2).

Using the transformation (17) one can establish

$$(j\omega I - A)^{-1}B = -(I + j\psi A)^{-1}j\psi B. \qquad (19)$$

The next theorem is a version of Theorem 3 that handles a FDI in the transformed frequency variable ψ.

Theorem 4: Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. Let ω_1, $\omega_2 \in \mathbb{R}$ be also given. If there exist matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI

$$\mathrm{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}j\psi_i I & I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\} + \Theta_i \prec 0, \qquad \psi_i = \omega_i^{-1}, \qquad i = \{1,2\} \qquad (20)$$

have feasible solutions then the finite FDI

$$\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix}^{-1}\Theta(\psi)\begin{bmatrix}(j\omega I - A)^{-1}B\\ I\end{bmatrix} \prec 0, \qquad \psi(\omega) = \omega^{-1},$$

holds for all $\omega_1 \le \omega \le \omega_2$ with $\Theta(\cdot)$ as given in (12).

Proof: Following the methods in [7], assume the pair of inequalities (20) have feasible solutions. The sum of (20) for $i = 1$ multiplied by

$$\lambda(\psi) = (\psi_2 - \psi)/(\psi_2 - \psi_1) \in [0,1] \qquad (21)$$

and of (20) for $i=2$ multiplied by $[1-\lambda(\psi)]$ implies that

$$\text{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}j\psi I & I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\}+\Theta(\psi)\prec 0,$$

is feasible for all $\psi_1 \le \psi \le \psi_2$ with $\Theta(\psi)$ given as in (12). Note the frequency dependent matrix

$$N_\perp(\psi)=\begin{bmatrix}-(I+j\psi A)^{-1}j\psi B\\ I\end{bmatrix}$$

which exists for all $\psi \in \mathbb{R}$ due to the assumption that A has no eigenvalues on the imaginary axis. Multiply the above inequality by $\mathcal{N}_\perp(\psi)$ on the right and by its transpose conjugate on the left to obtain

$$\mathcal{N}_\perp^*(\psi)\Theta(\psi)\mathcal{N}_\perp(\psi)= \begin{bmatrix}(j\omega I-A)^{-1}B\\ I\end{bmatrix}^*\Theta(\psi)\begin{bmatrix}(j\omega I-A)^{-1}B\\ I\end{bmatrix}\prec 0,$$

after using (19), which is the desired result. Note that $\Theta(\psi)$ is an affine function of the form (12) for the transformed frequency variable $\psi=\omega^{-1}$ ■

The above theorem can handle the case $\omega_2\to\infty$ without further ado by making $\psi_2\to 0$. Note that Theorem 3 and Theorem 4 are not completely equivalent, in that they may produce different results for the very same frequency range. This is a direct result from the fact that the coefficient matrix produced in Theorem 4 is an affine functions of $\psi=\omega^{-1}$ and not of ω.

One is often interested in checking feasibility of a FDI in the semi-infinite range $0\le\omega<\infty$. Theorem 4 treats the case $\omega_2\to\infty$ by using the transformation $\psi=\omega^{-1}$, which has a singularity at $\omega=0$. Therefore, it cannot be used to check such inequalities. In the next section we introduce another frequency transformation that achieves this goal.

B. Handling Semi-Infinite Frequency Intervals

In order to handle semi-infinite FDIs we will use the bilinear transformation [10]. Consider the special case of the bilinear transformation

$$\psi=\frac{\omega-z}{1+\omega-z}. \tag{22}$$

It maps the semi-infinite segment of the real axis $[z,\infty)$ onto the finite segment of the real axis $[0,1)$. Note that the inverse transformation is given by

$$\omega=\frac{\psi}{1-\psi}+z,$$

so that one can establish that

$$(j\omega I-A)^{-1}B=(j\psi\mathcal{E}-\mathcal{A})^{-1}(\mathcal{B}-j\psi\mathcal{H}), \tag{23}$$

where

$$\begin{aligned}\mathcal{A}&:=A-jzI, & \mathcal{B}&:=B,\\ \mathcal{E}&:=I-zI-jA, & \mathcal{H}&:=-jB.\end{aligned} \tag{24}$$

This motivates the presentation of an extension of Theorem 3 for a class of regular descriptor systems.

Theorem 5: Let matrix $\mathcal{A}\in\mathbb{C}^{n\times n}$ and $\mathcal{E}\in\mathbb{C}^{n\times n}$ be given such that the $\det(j\psi\mathcal{E}-\mathcal{A})\neq 0$ for all $\psi\in\mathbb{R}$. Let matrices $\mathcal{B}\in\mathbb{C}^{n\times m}$, $\mathcal{H}\in\mathbb{C}^{n\times m}$ and $\Theta_1,\Theta_2\in\mathbb{HC}^{n+m}$ and $\psi_1,\ \psi_2\in\mathbb{R}$ be also given. If there exist matrices $F\in\mathbb{C}^{n\times n}$ and $G\in\mathbb{C}^{m\times n}$ such that

$$\text{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -j\psi_i I\end{bmatrix}\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{E} & \mathcal{H}\end{bmatrix}\right\}+\Theta_i\prec 0,\quad i=\{1,2\} \tag{25}$$

then the FDI

$$\begin{bmatrix}\mathcal{M}(j\psi)\\ I\end{bmatrix}^*\Theta(\psi)\begin{bmatrix}\mathcal{M}(j\psi)\\ I\end{bmatrix}\prec 0,$$
$$\mathcal{M}(j\psi):=(j\psi\mathcal{E}-\mathcal{A})^{-1}(\mathcal{B}-j\psi\mathcal{H}), \tag{26}$$

holds for all $\psi_1\le\psi\le\psi_2$ with $\Theta(\cdot)$ as given in (12).

Proof: Assume that the pair of inequalities (25) have feasible solutions. The sum of (25) for $i=1$ multiplied by $\lambda(\psi)$ given in (21) and of (25) for $i=2$ multiplied by $[1-\lambda(\psi)]$ implies that

$$\text{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -j\psi I\end{bmatrix}\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{E} & \mathcal{H}\end{bmatrix}\right\}+\Theta(\psi)\prec 0,$$

is feasible for all $\psi_1\le\psi\le\psi_2$ with $\Theta(\psi)$ given as in (12). Define the frequency dependent matrix

$$\mathcal{N}_\perp(\psi)=\begin{bmatrix}(j\psi\mathcal{E}-\mathcal{A})^{-1}(\mathcal{B}-j\psi\mathcal{H})\\ I\end{bmatrix}$$

which exists for all $\psi\in\mathbb{R}$ due to the assumption that the matrix $(j\psi\mathcal{E}-\mathcal{A})$ is nonsingular. Multiply the above inequality by $\mathcal{N}_\perp(\psi)$ on the right and by its transpose conjugate on the left to obtain

$$\mathcal{N}_\perp^*(\psi)\Theta(\psi)\mathcal{N}_\perp(\psi)=\begin{bmatrix}\mathcal{M}(j\psi)\\ I\end{bmatrix}^*\Theta(\psi)\begin{bmatrix}\mathcal{M}(j\psi)\\ I\end{bmatrix}\prec 0,$$

where $\mathcal{M}(j\psi)=(j\psi\mathcal{E}-\mathcal{A})^{-1}(\mathcal{B}-j\psi\mathcal{H})$, which is the FDI (26). ■

The above result can be used to provide a check for a FDI involving the system (23) on the transformed frequency variable ψ defined in (22).

Corollary 6: Let matrices $A\in\mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B\in\mathbb{C}^{n\times m}$ and $\Theta_1,\Theta_2\in\mathbb{HC}^{n+m}$ and $z\in\mathbb{R}$ be given. If there exist matrices $F\in\mathbb{C}^{n\times n}$, $G\in\mathbb{C}^{m\times n}$ such that

$$\text{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -j\psi_i I\end{bmatrix}\begin{bmatrix}A-jzI & B\\ I-zI-jA & -jB\end{bmatrix}\right\}+\Theta_i\prec 0, \tag{27}$$

where $\psi_1=0$ and $\psi_2=1$, then the following FDI holds

$$\begin{bmatrix}(j\omega I-A)^{-1}B\\ I\end{bmatrix}^*\Theta(\psi)\begin{bmatrix}(j\omega I-A)^{-1}B\\ I\end{bmatrix}\prec 0,$$
$$\psi(\omega)=\frac{\omega-z}{1+\omega-z}, \tag{28}$$

for all $z \leq \omega < \infty$ with $\Theta(\cdot)$ as given in (12).

Proof: Use Theorem 5 with matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$ and $\mathcal{H}$ given by (24) with the frequency transformation (22) and apply limits as $\omega \to z$ and $\omega \to \infty$. ∎

The high-frequency LMIs (27) are finite even when $z \to 0$, or equivalently, $\omega \to 0$. In the next section we show how these results can be used to construct a condition for checking a FDI that holds in the entire frequency range, i.e. $\omega \in \mathbb{R}$. This can be used as an alternative formulation for the KYP Lemma.

C. An Alternative KYP Lemma

The following interesting result holds when A and B are real matrices.

Theorem 7: Let matrices $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI

$$\mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (1-\psi_i)I & -j\psi_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \Theta_i \prec 0, \quad (29)$$

are feasible where $\psi_1 = 0$ and $\psi_2 = 1$, then the following FDI holds

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad (30)$$

where

$$\Theta(\omega) = \begin{cases} \dfrac{1}{1+\omega}\Theta_1 + \dfrac{\omega}{1+\omega}\Theta_2, & \omega \geq 0, \\ \dfrac{1}{1-\omega}\overline{\Theta}_1 - \dfrac{\omega}{1-\omega}\overline{\Theta}_2, & \omega < 0. \end{cases} \qquad (31)$$

for all $\omega \in \mathbb{R}$.

Proof: First note that (29) is precisely condition (27) in Corollary 6 for $z = 0$, which, if feasible for some F and G, implies that the FDI (30) holds for all $\omega \geq 0$.

To prove that the FDI (30) holds in $\omega < 0$ assume that (29) is feasible for some F and G and take the conjugate of (29) to produce

$$\mathrm{He}\left\{ \begin{bmatrix} \overline{F} \\ \overline{G} \end{bmatrix} \begin{bmatrix} (1-\psi_i)I & j\psi_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} + \overline{\Theta_i} < 0, \qquad i = \{1, 2\}.$$

Note that

$$\begin{bmatrix} (1-\psi_i)I & j\psi_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & -j\psi_i I \end{bmatrix} \begin{bmatrix} A & B \\ -(I+jA) & -jB \end{bmatrix}$$

so that by Theorem 5 we have

$$\begin{bmatrix} \mathcal{M}(j\psi) \\ I \end{bmatrix}^* \Theta(\psi) \begin{bmatrix} \mathcal{M}(j\psi) \\ I \end{bmatrix} \prec 0$$

for all $\psi \in [0, 1)$ where

$$\mathcal{M}(j\psi) = [(\psi A - A) - j\psi I]^{-1}(B - \psi B).$$

Now define the pair of frequency transformations

$$\omega = \frac{-\psi}{1-\psi}, \qquad \psi = \frac{-\omega}{1-\omega}.$$

If $\psi \in [0, 1)$ then $\omega \in (-\infty, 0]$. Substituting the above in

$$\mathcal{M}(j\psi) = (j\omega I - A)^{-1} B,$$

we conclude that the FDI (30) is feasible for all $\omega < 0$ with (31). ∎

Note that $\Theta(\omega)$ defined in (31) may be discontinuous at $\omega = 0$. However, the above proof implies that the FDI (30) is satisfied at $\omega = 0$ for both $\Theta(0) = \Theta_1$ or $\Theta(0) = \overline{\Theta}_1$

This alternative formulation for the KYP Lemma over all frequencies will be useful in reducing conservativeness in the computation of upper bounds for the structured singular value in μ-analysis in the next section.

IV. Application to μ-Analysis

This section revisits earlier work of the authors [7] that focused on the analysis of robustness via the structured singular value μ. The structured singular value of a matrix is defined as

$$\mu_{\mathbf{\Delta}}(M) := \left(\inf_{\Delta \in \mathbf{\Delta}} \{ \|\Delta\| : \det(I - M\Delta) = 0 \} \right)^{-1}$$

where the uncertainty is given block-diagonal structure

$$\mathbf{\Delta} := \{ \mathrm{diag}[\phi_1 I_{s_1}, \cdots, \phi_r I_{s_r}, \delta_1 I_{s_1}, \cdots, \delta_c I_{s_c}, \Delta_1, \cdots, \Delta_F] : \phi_i \in \mathbb{R}, \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \}.$$

In practice, the introduction of appropriate *scalings* through duality theory is used to provide computable upper bounds for $\mu_{\mathbf{\Delta}}$. For instance, define the set of scaling matrices that commute with Δ

$$\mathbf{Z} := \{ \mathrm{diag}[Z_1, \cdots, Z_{s_r+s_c}, z_1 I_{m_1}, \cdots, z_F I_{m_F}] : Z_i \in \mathbb{C}^{s_i \times s_i}, Z_i = Z_i^* > 0, z_j \in \mathbb{R}, z_j > 0 \},$$

$$\mathbf{Y} := \{ \mathrm{diag}[Y_1, \cdots, Y_{s_r}, 0, \cdots, 0] : Y_i = Y_i^* \in \mathbb{C}^{s_i \times s_i} \},$$

and the optimization problem

$$\rho_{\mathbf{\Delta}}(M) := \inf_{\beta \in \mathbb{R}, Z \in \mathbf{Z}, Y \in \mathbf{Y}} \sup_{\omega \in \Omega} \{ \beta : \ \Phi_\beta(M(j\omega), Z(\omega), Y(\omega)) < 0 \}, \qquad (32)$$

where the matrix valued function Φ_β is given by

$$\Phi_\beta(M, Z, Y) := M^* Z M - j(M^* Y + Y M) - \beta^2 Z.$$

It follows from duality theory [11] that

$$\sup_{\omega \in \Omega} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq \rho_{\mathbf{\Delta}}.$$

The main focus in the context of this paper comes from recognizing that the coefficient matrix Θ appears linearly in the LMI conditions. This leads to the following corollary for evaluating upper bounds on $\mu_{\mathbf{\Delta}}$ by computing scaling matrices Z and Y that are affine on the frequency variable.

Corollary 8: Let $M(j\omega) = C(j\omega I - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$ are given. If there exist matrices $Z_1, Z_2 \in \mathbf{Z}$, $Y_1, Y_2 \in \mathbf{Y}$, $F \in \mathbb{C}^{n\times n}$, and $G \in \mathbb{C}^{m\times n}$ such that

$$\Theta_i + \mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} -I & j\omega_i I \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \right\} \prec 0, \quad i = \{1, 2\}$$

where

$$\Theta_i = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -jY_i \\ jY_i & -\beta^2 Z_i \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix},$$

has feasible solutions then $\rho_{\mathbf{\Delta}}(M(j\omega), \Omega) < \beta$ for all $|\omega| \in \Omega = [\omega_1, \omega_2]$.

Proof: Follows from Theorem 3 and the results of (29) for real matrices A and B. Note that the parametrization for $\Theta_i \in \mathbb{HC}^{n+m}$, $i = \{1, 2\}$ reduces the inequality (14) to the feasibility of $\rho_{\mathbf{\Delta}}(M(j\omega), \Omega) < \beta$ given in (32), where Ω is all $\omega_1 \leq |\omega| \leq \omega_2$. ∎

The combination of Corollary 8 with Theorem 7 for infinite frequency ranges, is illustrated for the same example used in [7]. These results are compared with the original formulation of the KYP Lemma for computing upper bounds of μ. Additionally the finite frequency results proposed in this paper are compared with the finite frequency KYP Lemma [5]. All comparisons are shown in Figure 1.

For this example, the proposed alternative conditions for the KYP Lemma significantly reduce the conservatism in computing the μ upper bound when evaluated over the entire imaginary axis as well as over semi-infinite and finite frequency ranges.

V. Conclusions

The contributions are this paper are two fold: first, we show that the sufficient LMI test for frequency domain inequalities in finite frequency ranges introduced in [7] are also necessary in the classical case the matrix coefficient Θ is constant; second, we show how transformations of the frequency variable can lead to several useful extensions, including an alternative formulation of the standard KYP Lemma that holds over all frequencies. These new results are used to compute less conservative upperbounds for the structured singular value in robust analysis.

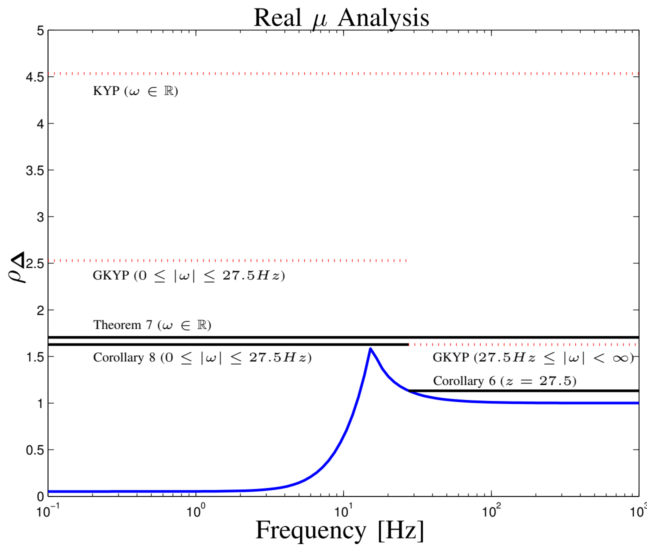

Fig. 1. Computation of upper bounds for μ. The blue solid line is the greatest lower bound for $\rho_{\mathbf{\Delta}}$. Other lines are labeled according to the method and frequency range used to compute them.

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