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A Comparison of Fixed Point Designs And Time-varying Observers for Scheduling Repetitive Controllers

C.E. Kinney and R.A. de Callafon

Abstract— **This paper presents two distinct methods of designing scheduled repetitive controllers for the case where the frequency of the disturbance is slowly varying and when it is rapidly varying. In both cases the plant is subjected to random and periodic disturbances. The first method uses set points to design a family of controllers and then interpolates between them online. The conditions for designing sub-optimal gains are given. The second method utilizes results from time-varying Riccati equations to design a time-varying observer. It is shown via simulation that the time-varying Riccati method is superior but at a greater computational cost.**

I. Introduction

Repetitive control [1], [2], [3], [4], [5] has been successfully applied to many systems where the frequency of the disturbance is know and constant. These control system have been shown to be a particular realization of the internal model principle [6] and the connection between the internal model principle and repetitive controllers is discussed in [7]. In many systems the frequency of the disturbance is time-varying and measurable, for example the tachometer of a cooling fan.

In practice, to deal with the time-varying frequency of the disturbance it is often feasible to design a family of controllers at different operating points and switch or interpolate between them as the frequency of the disturbance varies. However, there are circumstances where this methodology does not work. The problem lies in the fact that the controller is time-varying and a time-invariant design method was used. This can lead to instabilities in the closed loop system.

Several adaptive techniques that have arisen from internal model-based techniques require that the disturbance slowly varying with time and in the analysis of the closed loop stability the adaptation and control are coupled. For example, Brown and Zhang [8] have implemented an adaptive internal-model based controller that is able to track and reject a periodic disturbance. Their analysis is based upon averaging and singular perturbation and requires the assumption that the disturbance is slowly time-varying.

The main point of this paper is to show that the results from time-varying Riccati equations can be used to design and analyze scheduled repetitive controllers. Further, the design of the interpolated controller is described in detail for the case when random disturbances are present in addition to the periodic disturbance. This issue has not be fully address in past work since there are unobservable/uncontrollable issues with the internal model. Most of the previous work has concentrated upon designing stable controllers and do not consider random disturbances.

C.E. Kinney is a graduate student in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, U.S.A cekinney@ucsd.edu

R.A. de Callafon is an Associate Professor in the Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0411, U.S.A callafon@ucsd.edu

II. Problem Description

This section provides the mathematical description of the system and internal model that will be used for the remainder of the paper. Additionally, the controller and cost function are introduced.

The dynamics of the system or *plant* will be modeled with the state space model

$$\begin{array}{ll} x_p(k+1) & = A_p x_p(k) + B_u u(k) + B_w w(k) \\ y_p(k) & = C_p x_p(k) + D_{yw} w(k) + d(k) \\ z(k) & = C_z x_p(K) + D_{zu} u(k) \end{array} , \quad (1)$$

where $x_p(k) \in \mathbb{R}^{n_p}$ are the plant states, $u(k) \in \mathbb{R}^{n_u}$ is the control signal, $w(k) \in \mathbb{R}^{n_w}$ is white noise, $d(k) \in \mathbb{R}^{n_y}$ is the time-varying periodic disturbance, $y_p(k) \in \mathbb{R}^{n_y}$ is the measurable output of the plant, and $z(k) \in \mathbb{R}^{n_z}$ is the performance channel. It will also be assumed throughout the paper that $D_{yw} D_{yw}^T > 0$.

For the purposes of the control design, a model of the disturbance or *internal model* will be used and is given by

$$\begin{array}{ll} x_m(k+1) & = \mathcal{A}_m(\omega(k)) x_m(k) + B_m u_m(k) \\ y_m(k) & = C_m x_m(k) \end{array} ,$$

where $x_m(k) \in \mathbb{R}^{n_m}$ are the internal model states, $u_m(k) \in \mathbb{R}^{n_y}$ is the input to the internal model, $y_m(k) \in \mathbb{R}^{n_{ym}}$ is the output of the internal model, and for each k, $\omega(k)$ is measurable or known.

The purpose of the time-varying internal model is to create a controller that will cancel the time-varying disturbance. This methodology is based upon the internal model principle [6] that states that the controller must contain the dynamics of the disturbance to completely cancel it. Since the disturbance is time-varying it is natural to try an internal model that is time-varying as well.

The overall goal is to eliminate unwanted disturbances that appears as a time-varying periodic signal and random noise due to measurement and state errors.

Using this reasoning, the objective of the control design becomes rejecting the time-varying disturbance $d(k)$ in the presence of random noise while minimizing (or over bounding) the cost function

$$J = \lim_{N\to\infty} \mathbf{E}\left\{\frac{1}{N}\sum_{k=0}^{N} z(k)^T z(k)\right\}. \quad (2)$$

III. Internal Model

In order for the frequency cancelation to work properly it is important to construct an internal model that captures the relevant properties of the unknown signal. In our case, a model of a periodic signal with a time-varying frequency is needed.

It was shown in [9] that a continuous time model that represents the periodic signal $y(t) = \cos(\omega t + \phi)$ with a time-varying frequency is given by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_d(t)^2 & \dot{\omega}_d(t)\frac{1}{\omega_d(t)}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$
$$y(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},$$

where $\omega_d(t) = \frac{d}{dt}(\omega t + \phi)$ and another realization is

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} 0 & \omega_d(t) \\ -\omega_d(t) & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$
$$y(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}. \quad (3)$$

Notice that one must be carefull when choosing the realization, since the normal LTI realizations are not always the same as the time-varying realizations.

Applying a zero-order hold, with sample time Δt, to (3) gives

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} = \begin{bmatrix} \cos(\omega\Delta t) & \sin(\omega\Delta t) \\ -\sin(\omega\Delta t) & \cos(\omega\Delta t)\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$$
$$y(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} \quad (4)$$

where $\omega := \omega_d(t)|_{t=k}$ is held constant over the samping interval. This model will be used as the building blocks of an internal model. The part that remains is chose how the input should affect the states of the internal model. This choice is important since the internal model will be inside the feedback loop and thus will change the dynamics of the closed loop system.

Since the realization (4) does not have any zeros (it doesn't have an input), we will choose an internal model that has an input but does not have any zeros. One such realization is given by

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix} = \begin{bmatrix} \cos(\omega\Delta t) & \sin(\omega\Delta t) \\ -\sin(\omega\Delta t) & \cos(\omega\Delta t)\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u(k)$$
$$y(k) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$$

and this realization will be used for the remainder of the paper as a subsystem for the internal model. The subsystems need to be connected to accommodate disturbances that may contain more than one sinusoid.

IV. Scheduled Internal Model Control Algoritms

A. Internal Model-Based Control

Since the goal of the internal model principle is to place closed loop transmission zeros at the location of the poles of the periodic disturbance, the following definition of IMB is presented.

Definition 1 (IMB Controller): A strictly proper controller with a minimal state space realization $(A_c, B_c, C_c, 0)$ is IMB if the eigenvalues of A_c contain the eigenvalues of the internal model.

Here we consider only strictly proper controllers since we are in discrete time and require at least one step time delay between the input and output for calculations. This definition guarantees that the sensitivity function will have transmission zeros [10] at the proper location. The following proposition clarifies this point.

Proposition 1: Consider the state space system given by (A, B, C, D) and a controller with a state space realization $(A_c, B_c, C_c, 0)$. If the controller is IMB then the output sensitivity function $S(q)$ will have transmission zeros located at the eigenvalues of the internal model.

Proof: The output sensitivity function is defined as

$$S(q) := (I - P(q)C(q))^{-1},$$

where $P(q) := \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ is the plant and $C(q) := \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0\end{array}\right]$ is the controller. The state space realization of (I-P(q)C(q)) is given by

$$(I - P(q)C(q)) = \left[\begin{array}{cc|c} A_c & B_cC & B_cD \\ 0 & A & B \\ \hline -C_c & 0 & I\end{array}\right].$$

The inverse of the above is

$$(I - P(q)C(q))^{-1} = \left[\begin{array}{cc|c} A_c + B_cDC_c & B_cC & B_cD \\ BC_c & A & B \\ \hline C_c & 0 & I\end{array}\right]$$

and the transmission zeros are determined by the rank($R(\lambda)$), where

$$R(\lambda) = \begin{bmatrix} A_c + B_cDC_c - \lambda I & B_cC & B_cD \\ BC_c & A - \lambda I & B \\ C_c & 0 & I\end{bmatrix}$$

and $\lambda \in \mathbb{C}$. Multiplying on the right with a full rank matrix gives

$$\begin{bmatrix} A_c + B_cDC_c - \lambda I & B_cC & B_cD \\ BC_c & A - \lambda I & B \\ C_c & 0 & I\end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C_c & 0 & I\end{bmatrix} = \begin{bmatrix} A_c - \lambda I & B_cC & B_cD \\ 0 & A - \lambda I & B \\ 0 & 0 & I\end{bmatrix}.$$

Therefore the normal rank of $R(\lambda)$ drops when λ is an eigenvalue of A_c, and since A_c is IMB there are transmission zeros at the location of the eigenvalues of the internal model. ■

The output sensitivity function is investigated since, for our problem, it represents the path from periodic disturbances to the output of the system and placing transmission zeros will asymptotically reject these disturbances.

B. Interpolation of LTI Controllers

The idea of this approach is to interpolate between LTI controllers designed at different operating conditions. For instance, one controller might be tuned to reject disturbances at 100 Hz and another 105 Hz and if the disturbance is at 102 Hz then interpolate. The idea behind this method is to create a family of controllers and interpolate between them as the disturbance varies (hoping that the interpolation error is small). This method is justified by a well known continuous time result (see [11] and others) about slowly varying LPV systems. The result implies that stability, in the slowly varying case, can be determined by the collection of eigenvalues of the entire family of closed loop systems. From this perspective, the family or collection of LTI controllers will be designed as a function of the frequency of the disturbance. Also, a pair of Riccati equations can be solved each time step to eliminate the interpolation error, but this is much more computationally expensive.

It was shown in [12] that the LTI controller given by

$$C(q) = \left[\begin{array}{cc|c} A_p - L_pC_p - B_uK & 0 & L_p \\ L_mC_p & A_m & -L_m \\ \hline -K & C_m & 0 \end{array}\right], \quad (5)$$

is an internal model-based controller and has an interpretation of a learning controller when the appropriate internal model is chosen. This controllers is able to cancel periodic disturbances that have a fixed, known frequency.

In this paper, we use this LTI controller as a starting point to design an adaptive internal model-based controller that can track and cancel time-varying periodic disturbances and simultaneously cope with the random disturbances in the system. To accomplish this task, the internal model is parameterized as a function of ω. This implies that the observer gains in (5) necessarily will change as a function of ω to preserve stability of the fixed point designs.

Following the same methodology as [13] the controller is parameterized with respect to ω this gives

$$C(q,\omega) = \left[\begin{array}{cc|c} A_p - L_p(\omega)C_p - B_uK(\omega) & 0 & L_p(\omega) \\ L_m(\omega)C_p & A_m(\omega) & -L_m(\omega) \\ \hline -K(\omega) & C_m & 0 \end{array}\right], \quad (6)$$

where the controller gains must be designed so that the closed loop system is stable and has desirable properties.

With the controller given by (6) and the system given by (1) the closed loop system can be written as (after a similarity transformation)

$$\begin{bmatrix} \bar{x}_p(k+1) \\ \bar{x}_c^{(2)}(k+1) \\ \bar{x}_c^{(1)}(k+1) \end{bmatrix} = \begin{bmatrix} A_p - L_p(\omega)C_p & B_uC_m & 0 \\ -L_m(\omega)C_p & A_m(\omega) & 0 \\ L_p(\omega)C_m & 0 & A_p - B_uK(\omega) \end{bmatrix} \begin{bmatrix} \bar{x}_p(k) \\ \bar{x}_c^{(2)}(k) \\ \bar{x}_c^{(1)}(k) \end{bmatrix} + \begin{bmatrix} B_w - L_p(\omega)D_{yw} \\ -L_m(\omega)D_{yw} \\ L_p(\omega)D_{yw} \end{bmatrix} w(k) + \begin{bmatrix} -L_p(\omega) \\ -L_m(\omega) \\ L_p(\omega) \end{bmatrix} d(k)$$

$$z(k) = \begin{bmatrix} C_z & D_{zu}C_m & C_z - D_{zu}K(\omega) \end{bmatrix} \begin{bmatrix} \bar{x}_p(k) \\ \bar{x}_c^{(2)}(k) \\ \bar{x}_c^{(1)}(k) \end{bmatrix}$$

where the controller states have been partitioned accordingly. Notice that this is an LTI system for each fixed ω. Creating a grid over ω (not necessarily evenly spaced) with N_ω points creates N_ω LTI control problems that each can be solved with standard results from LQG control theory.

For each fixed ω, as long as the closed loop system is stable, the effect that $d(k)$ has upon the output (in the limit) is zero since the controller is IMB by definition 1 and by proposition 1 there are zeros at in the closed loop system from $d(k) \rightarrow y(k)$ located at the poles of the disturbance. For this reason, the affect of $d(k)$ can be neglected in the design of the gains.

The optimal IMB control problem is to find the optimal gains K^*, L_p^*, and L_m^* for the controller given by (6) such that the cost (2) is reduced. For exponentially stable systems it is known that this cost is equivalent to

$$J = \lim_{k\to\infty} \mathbb{E}\left\{z(k)^T z(k)\right\}. \quad (7)$$

For notational purposes it is convenient to define the following

$$L := \begin{bmatrix} L_p \\ L_m \end{bmatrix} \quad \mathcal{A} := \begin{bmatrix} A_p & B_uC_m \\ 0 & A_m \end{bmatrix} \quad \mathcal{B}_w := \begin{bmatrix} B_w \\ 0 \end{bmatrix}$$
$$\mathcal{C}_p := \begin{bmatrix} C_p & 0 \end{bmatrix} \qquad \mathcal{C}_m := \begin{bmatrix} C_m & 0 \end{bmatrix} \quad (8)$$

The optimal IMB controller is obtained by direct minimization of the cost function. The following theorem clarifies the design of the optimal gains for the internal model-based problem.

Theorem 1 (Sub-optimal IMB Controller): Consider the plant given in (1) and the controller defined in (5) where the matrices A_m and C_m are a given. If (A_p, B_u) is controllable, (A_p, C_z) is observable, and $\exists$ matrices $P_1 \in (S)^{n_p+n_m}$, $M \in \mathbb{R}^{(n_p+n_m)\times n_y}$, and $W \in \mathbb{R}^{(n_p+n_m)\times n_w}$ such that

$$\begin{bmatrix} P_1 & \mathcal{A}^TP_1 - \mathcal{C}_p^TM^T \\ (\cdot)^T & P_1 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} C_z^T \\ (D_{zu}C_m)^T \end{bmatrix} \begin{bmatrix} C_z & D_{zu}C_m \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (9)$$

and

$$\begin{bmatrix} W & (P_1\mathcal{B}_w - MD_{yw})^T \\ P_1\mathcal{B}_w - MD_{yw} & P_1 \end{bmatrix} \geq 0$$
$$tr\{W\} \leq \gamma$$

hold.

Then the control gains given by

$$K^* = (\mathcal{B}_u^T P_c \mathcal{B}_u + \mathcal{D}_{zu}^T \mathcal{D}_{zu})^{-1} \mathcal{B}_u^T P_c \mathcal{A} \quad (10)$$

where P_c satisfies

$$P_c = A_p^T P_c A_p - A_p^T P_c B_u (B_u^T P_c B_u + D_{zu}^T D_{zu})^{-1} B_u^T P_c A_p + C_z^T C_z$$

and

$$L^* = P_1^{-1} M, \quad (11)$$

over-bound the cost function defined in (7). Moreover, the over-bound can be written as

$$\begin{aligned} J &\leq tr\{(\mathcal{B}_w - LD_{yw})^T P_1 (\mathcal{B}_w - LD_{yw})\} \\ &\quad + tr\{\bar{P}_2 L_p (\mathcal{C}_m P_1 \mathcal{C}_m^T + D_{yw} D_{yw}^T) L_p^T\} \\ &\leq \gamma + tr\{\bar{P}_2 L_p (\mathcal{C}_m P_1 \mathcal{C}_m^T + D_{yw} D_{yw}^T) L_p^T\}. \end{aligned}$$

Proof: Omitted for space. ■

This theorem is very similar to the separation principle for standard LQG controllers. The difference here is due to the internal model that creates an uncontrollable pair $(\mathcal{A}, \mathcal{B}_w)$ and therefore an upperbound must be found. In the dual problem, there is an observability problem that can be handled similarly or by changing the cost function. In the current problem, we can modify $\mathcal{B}_w$ so that $(\mathcal{A}, \mathcal{B}_w)$ is controllable and the solution will approximate the solution that we are looking for. Either way, we have a method for designing the controller gains to reduce the criteria that is relevant for our problem.

Corollary 1: Let $\mathcal{A}$ be defined the same as in (IV-B), $\bar{\mathcal{B}}_w = \mathcal{B}_w + \begin{bmatrix} 0 \\ \Delta \end{bmatrix}$ with Δ a matrix with the appropriate dimensions, $(\mathcal{A}, \mathcal{C}_p)$ be observable, and $D_{yw} D_{yw}^T > 0$. If $(\mathcal{A}, \bar{\mathcal{B}}_w)$ is controllable and $\bar{\mathcal{B}}_w D_{yw}^T = 0$ then

$$P = \mathcal{A} P \mathcal{A}^T - \mathcal{A} P \mathcal{C}_p^T (\mathcal{C}_p P \mathcal{C}_p^T + D_{yw} D_{yw}^T)^{-1} \mathcal{C}_p P \mathcal{A}^T + \bar{\mathcal{B}}_w \bar{\mathcal{B}}_w^T \quad (12)$$

will have a unique positive definite solution P that satisfies (9).

Proof: The existence and uniques of the solution is obtained since the systems completely controllable, completely observable, and $D_{yw} D_{yw}^T > 0$.

Define $L := \mathcal{A} P \mathcal{C}_p^T (\mathcal{C}_p P \mathcal{C}_p^T + D_{yw} D_{yw}^T)^{-1}$, then (12) can be written as

$$\begin{aligned} P &= \mathcal{A} P \mathcal{A}^T + L(\mathcal{C}_p P \mathcal{C}_p + D_{yw} D_{yw}^T) L^T \\ &\quad + \bar{\mathcal{B}}_w \bar{\mathcal{B}}_w^T - L \mathcal{C}_p P \mathcal{A}^T - \mathcal{A} P \mathcal{C}_p^T L^T \\ &= (\mathcal{A} - L\mathcal{C}_p) P (\mathcal{A} - L\mathcal{C}_p)^T + (\mathcal{B}_w - LD_{yw})(\mathcal{B}_w - LD_{yw})^T \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & \Delta\Delta^T \end{bmatrix} \end{aligned}$$

that is bounded by

$$P \geq (\mathcal{A} - L\mathcal{C}_p) P (\mathcal{A} - L\mathcal{C}_p)^T + (\mathcal{B}_w - LD_{yw})(\mathcal{B}_w - LD_{yw})^T$$

which can be shown via duality, the change of variables $M = P_1 L$, and Schur complement to satisfy (9). ■

This corollary shows that it is possible to solve for a stabilizing controller that will over bound the desired cost with standard Riccati equation solvers. The reason that the exact optimal solution cannot be found is due to a controllability issue with the internal model which is unstable. In the following section, we will use this sub-optimal problem to solve for the observer gains online using a recursive Riccati equation. The result is a controller with a time-varying observer and a time-invariant state feedback gain. The time-varying nature of the observer mimics the time-varying nature of the disturbance so that the controller is able to track and reject the periodic disturbances.

C. Time-varying Observer

In this section we use some results on the time-varying Riccati filter to solve for a time-varying controller that rejects periodic disturbances in the presence of random noise. Recall, that the control design is composed of two subproblems: an observer design and a state feedback design. To embed the internal model into the controller the observer design is done for the series connection of the internal model and plant. For stability of the closed loop system, the state feedback gain is determined for the plant.

To change from a family of LTI controllers to a time-varying controller only the observer subproblem needs to be considered. As it turns out, the solution of the time-varying Kalman filter is a causal function of time. This is in contrast to the solution of the time-varying state feedback gain and is the crucial element that allows for this method to work while the dual problem will not. Here, the dual problem is finding a time-varying state feedback gain and a time-invariant observer.

The series connection of the time-varying internal model and the time invariant plant is given by

$$\begin{aligned} \begin{bmatrix} x_p(k+1) \\ x_m(k+1) \end{bmatrix} &= \begin{bmatrix} A_p & B_u C_m \\ 0 & A_m(k) \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_m(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ B_m \end{bmatrix} u_m(k) + \begin{bmatrix} B_w \\ 0 \end{bmatrix} w(k) \\ y_p(k) &= \begin{bmatrix} C_p & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_m(k) \end{bmatrix} + D_{yw} w(k) \\ z(k) &= \begin{bmatrix} C_z & 0 \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_m(k) \end{bmatrix} + D_{zu} u(k) \end{aligned} \quad (13)$$

which can be expressed as

$$\begin{aligned} x(k+1) &= \mathcal{A}(k) x(k) + \mathcal{B}_w w(k) + \mathcal{B}_m u_m(k) \\ y_p(k) &= \mathcal{C}_p x(k) + D_{yw} w(k) \\ z(k) &= \mathcal{C}_z x(k) + D_{zu} u(k). \end{aligned} \quad (14)$$

Since we are considering a time-varying system, the cost function that is minimized is slightly different because the time-varying system may not converge. Given all of the inputs and outputs up to time k, denoted by the set $Y_{k-1} = \{u_m(0), y_p(0), u_m(1), y_p(1), \ldots, u_m(k-1), y_p(k-1)\}$, the objective is to find the best predictor of the states $\hat{x}_k$ in the L_2 sense and it is assumed that the initial condition $x(0)$ is a gaussian random variable with known mean $\bar{x}(0)$ and covariance $\mathbb{X}(0)$. The cost J_e therefore can be written as

$$J_e = \mathbb{E}\{\|x(k) - \hat{x}(k)\|^2 | Y_{k-1}\}.$$

The Kalman predictor for this system is

$$\hat{x}(k+1) = (\mathcal{A}(k) - \mathcal{L}(k)\mathcal{C}_p)\,\hat{x}(k) + \mathcal{L}_m u_m(k) + \mathcal{L}(k) y_p(k) \quad (15)$$

and the error system is

$$\tilde{x}(k+1) = (\mathcal{A}(k) - \mathcal{L}(k)\mathcal{C}_p)\,\tilde{x}(k) + (\mathcal{B}_w - \mathcal{L}(k) D_{yw})\, w. \quad (16)$$

Recall that $(\mathcal{A}, \mathcal{B}_w)$ is not controllable and therefore $\mathcal{B}_w$ needs to be augmented with Δ in agreement with corollary 1 for each time step k. When this is done, we arrive at the following Ricatti difference equation for the Kalman predictor gain

$$\begin{aligned} P_{k+1} &= \mathcal{A}(k) P_k \mathcal{A}(k)^T \\ &\quad - \mathcal{A}(k) P_k \mathcal{C}_p^T (\mathcal{C}_p P_k \mathcal{C}_p^T + D_{yw} D_{yw}^T)^{-1} \mathcal{C}_p P_k \mathcal{A}(k)^T \\ &\quad + \bar{\mathcal{B}}_w \bar{\mathcal{B}}_w^T \\ P_0 &= \mathcal{A}(0) \mathbb{X}(0) \mathcal{A}(0)^T + \bar{\mathcal{B}}_w \bar{\mathcal{B}}_w^T \end{aligned} \quad (17)$$

and the predictor gain is

$$\mathcal{L}(k) = \mathcal{A}(k) P_k \mathcal{C}_p^T (\mathcal{C}_p P_k \mathcal{C}_p^T + D_{yw} D_{yw}^T)^{-1}. \quad (18)$$

Theorem 2 (Stability of time-varying IMC): Consider the plant given in (1) and the controller defined in (5) where the matrices $A_m(k)$ and C_m are given for each k. Suppose that the Kalman predictor in (17) and $A_p - B_u K$ are uniformly exponentially stable then the closed loop system is uniformly exponentially stable.

Proof: Follows from definition of exponential stability and the properties of norms. ■

This theorem shows that this adaptive algorithm is stable and standard results about recursive Riccati equations can be used to determine when both systems will be exponentially stable. For example, results from Floquet's Theory [14] or periodic Riccati equations [15] can be used to determine the stability of periodic systems, i.e., systems such that

$$A(k) = A(k+T).$$

More generally, from Theorem 5.3 in [16], if the conditions of the following theorem are met then the system will be exponentially stable.

Theorem 3: Let $(\mathcal{A}(k), \mathcal{C}_p)$ be uniformly detectable and $(\mathcal{A}(k), \bar{\mathcal{B}}_w)$ be uniformly stabilizable, as defined in [16], then $(\mathcal{A}(k) - \mathcal{L}(k)\mathcal{C}_p)$ is exponentially stable with $\mathcal{L}(k)$ defined in (18).

This theorem concludes that the stability of the system reduces to determining the uniform detectability and stabilizability of the system.

Notice that if $\omega(k)$ is constant for all k then as $k \to \infty$ the two observers (above and in Section IV-B will produce the same results provided that the interpolation error is negligible. The main purpose in this paper is to compare these two methods when the disturbance is time-varying, and it is expected that the time-varying observer will outperform the interpolation method at a greater computational cost.

V. Simulation Results

In this section the two internal model based algorithms discussed in this paper are compared. The first algorithm is based upon a fixed point design and interpolates between a family of LTI controllers, it will be denoted the interpolation method. The second algorithm is based upon the time-varying Kalman filter and updates the filter gains at each time step, it will be denoted the time-varying method. The first algorithm is simple and easy to implement, but not guaranteed to be stable during the interpolation. The second algorithm is guaranteed to be exponentially stable under mild conditions, but the computational cost is greater than the first. This is the tradeoff for improved performance.

The frequency response of the plant we are simulating is shown in Figure 1. Notice that the plant has a couple of lightly damped resonance modes, is open-loop stable, and has a couple of lightly damped zeros.

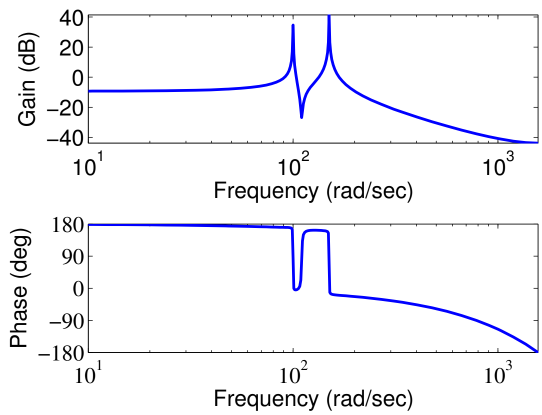

Fig. 1. Frequency response of the plant.

Consider the case where the interpolation error is zero, the Riccati equations are solved at each time k, and the disturbance is a single sinusoid and is given by

$$d(k) = \sin(\alpha(k))$$

where $\omega(k) := (\alpha(k) - \alpha(k-1))$ and Δ_t is the sampling time. Notice that this definition is necessary since we are dealing with a time-varying sinusoid and it agrees with the time-invariant case since if $\alpha(k) = \omega\Delta_t k + \phi$ then $\omega(k) = \omega$.

For the first simulation, the frequency will be slowly modulated. Setting $\omega(k) = 100 + 50\sin(0.5\Delta_t k)$, $\alpha(k) = \Delta t \sum_{i=0}^{k} \omega(k)$, results in Figure 2. Here, both the time-varying method and the interpolation method perform almost identically since the frequency is varying slowly in comparison to the closed loop dynamics.

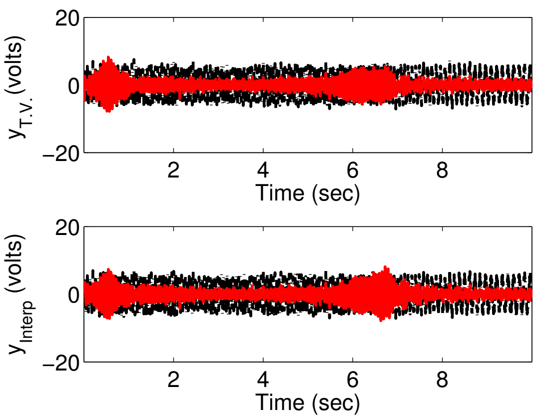

Fig. 2. Comparison of controllers with slowly varying frequency.

Next, consider the situation where $\omega(k) = 100 + 50\sin(100\Delta_t k)$. As it can be seen in Figure 3, the interpolation method results in an unstable closed loop system while the time-varying method does not.

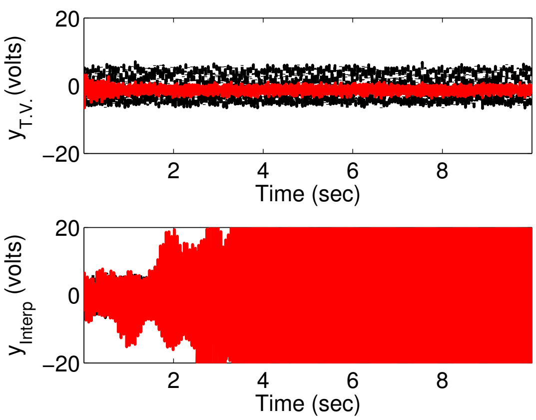

Fig. 3. Comparison of controllers when the frequency of the disturbance is rapidly varying.

Finally, if the extreme case where $\omega(k)$ alternates between 50 and 150 at each time step is shown in Figure 4. Again, the time-varying method is able to stabilize the closed loop system while the interpolation method does not.

VI. Conclusions

In this paper, we discuss the design of scheduled repetitive controllers. The design of sub-optimal gains were presented for the case where random disturbances are present in addition to periodic disturbances. This method uses set points to design a family of controllers. This design method is limited to the case where the frequency of the disturbance is slowly-varying. For the case where the frequency may be varying quickly, an alternative design method was presented that relies upon known results from time-varying Riccati equations. Both methods were simulated and the simulation shows that the time-varying Riccati equations outperform the set point designs when the frequency varies rapidly.

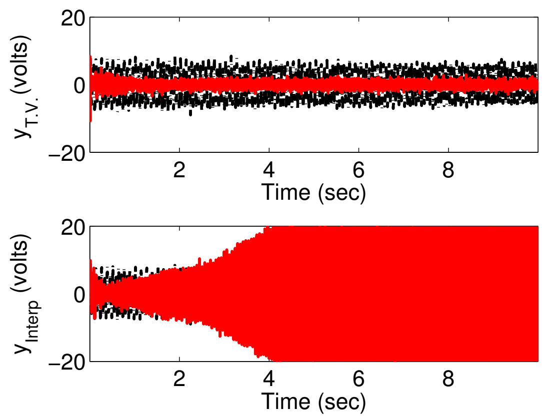

Fig. 4. Comparison of controllers when the frequency of the disturbance is switching.

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