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Analysis and Design Methodologies for Robust Aeroservoelastic Structures

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Aeroservoelastic models provide a means for analyzing stability and performance, online monitoring of aerodynamic instabilities, as well as for designing active structural control systems. Of concern in the analysis and design of aeroservoelastic systems is reducing the conservatism with which stability and performance is guaranteed without significantly increasing the computational effort. Recent advances in control theory have lead to generalizations that allow system properties, specified in the frequency domain, to be analyzed via Linear Matrix Inequalities over specified finite frequency range. This paper presents robust analysis conditions which incorporate a particular class of frequency dependent multipliers that can be used to specify performance requirements. The robust analysis results are illustrated by an example that evaluates robust flutter margins for ASE systems.

I. Introduction

Modern aerosrtuctural modeling, identification and control design emphasize a progression toward light-weight airframe structures resulting in active, flexible and multifunctional wing structures. A reduction in structural weight increases the likelihood of dynamic instabilities[24] and requires extensive and costly flight-testing to safely explore the boundary of the flight envelope. In case aerodynamic instabilities are experienced during flight-testing active structural servo control strategies can be used to stabilize and expand the boundary of the flight envelope.[2]

Aeroservoelastic (ASE) models provide a means for designing active structural control systems in addition to allowing the possibility of online flight monitoring of aerodynamic instabilities. Stability margins for ASE systems are a measure of distance from a reference flight condition to a condition that induces flutter. Aeroservoelastic models can be formulated by considering a nominal (linear) model with unknown but bounded perturbations. The nominal ASE model captures the inherent dynamic interaction between structural elasticity and aerodynamic loads and contains the effects of actuator bandwidth, major structural modes and sensor locations. The unknown but bounded model perturbation is used to account for uncertainties such as variations in flight conditions, unmodeled (high frequency) elastic modes and actuator nonlinearity. Nominal flight behavior and perturbations can be estimated by identification of possibly nonlinear dynamics within a feedback control framework for an ASE system[17] or derived from first principles modeling.[1]

Computational methods for analysis of ASE models with associated uncertainty fits within robust control framework by considering a linear fractional transformation (LFT) model set. Robust stability margins are a measure of distance from a reference flight condition to a condition that could potentially induce flutter taking into consideration all possible modeling errors. The problem of determining robust flutter margins fits within the μ-analysis framework.[5,16] Of concern in the analysis and design of aeroservoelastic systems is reducing the conservatism with which stability and performance is guaranteed without significantly increasing the computational effort. Recent advances in control theory have lead to generalizations that allow system

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properties, specified in the frequency domain, to be analyzed via Linear Matrix Inequalities (LMIs) over specified finite frequency range.

In this paper we present conditions for robust analysis that are expressed as convex inequalities equivalent to checking a frequency domain inequality over finite interval.[10] These convex inequalities are generalizations of the Kalman-Yakubovich-Popov (KYP) Lemma, which can be used to compute upper bounds for the structured singular μ-analysis. The robust analysis conditions incorporate a particular class of frequency dependent multipliers that can be used to specify performance requirements over finite frequency intervals. An example in evaluating robust flutter margins is illustrated for a pitch plunge ASE system with modeling uncertainty in structural stiffness and damping parameters.

II. Aeroservoelastic Modeling for Stability Analysis

Consider the low-order pitch plunge system illustrated in Figure 1. The system is modeled as a rigid airfoil attached to a support structure composed of springs and cams that allow pitch and plunge motion. The airfoil also has a trailing edge control surface (flap) that adds an extra degree of freedom for the system. The equations of motion describing the pitch and plunge motion during aeroelastic response are derived from force and moment equations[12,16] and written in matrix form,

$$\begin{bmatrix} m & mx_\alpha b \\ mx_\alpha b & I_\alpha \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\alpha} \end{bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_\alpha \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} k_y & 0 \\ 0 & k_\alpha \end{bmatrix} \begin{bmatrix} y \\ \alpha \end{bmatrix} = \begin{bmatrix} -L \\ M \end{bmatrix}, \tag{1}$$

where y is plunge deflection and α is pitch angle. The other variables include the airfoil mass m, distance to center of mass x_α, moment of inertia I_α, chord length b, structural damping coefficients c_y, c_α, and spring constants k_y, k_α. The above system description can also be extended to generalized second order equations of motion relating structural dynamics and unsteady aerodynamics, however this discussion is beyond the intended scope of this paper and the interested reader is referred to.[16]

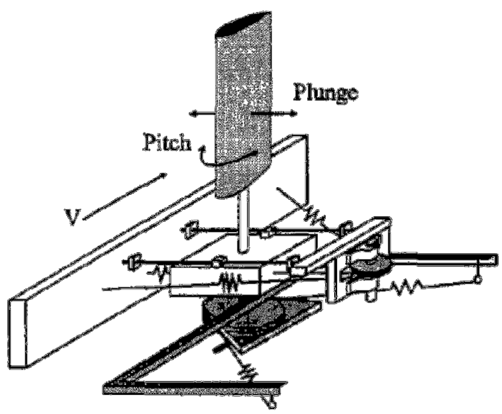

Figure 1. Isometric view of the pitch-plunge mechanism for studying aeroservoelastic systems.

The aeroelastic system (1) becomes an ASE system by including active servo control via the control surface. The control surface deflection angle β affect the lift L and moment M through the relation

$$L = 2\tilde{q}bc_{l_\alpha}\left(\alpha + 1.1\frac{b}{U}\dot{\alpha} + \frac{1}{U}\dot{y}\right) + \tilde{q}b^2c_{l_\beta}\beta, \tag{2}$$

$$M = 2\tilde{q}bc_{m_y}\left(\alpha + 1.1\frac{b}{U}\dot{\alpha} + \frac{1}{U}\dot{y}\right) + \tilde{q}b^2c_{m_\beta}\beta, \tag{3}$$

where U is the nominal airspeed, c_{l_α}, c_{m_α} are lift and moment coefficients for pitch angle and c_{l_β}, c_{m_β} are lift and moment coefficients for control surface angle. The airspeed U directly corresponds to a dynamic pressure $\tilde{q} \in \mathbb{R}$, that represents a flight condition. The system equations relate rigid body and control surface displacement and velocities through rectangular matrices of the vibration and control modes. Combining (1)-(3) the continuous-time state-space matrices for the equations of motion are given by

$$\begin{bmatrix} \dot{y} \\ \dot{\alpha} \\ \ddot{y} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} y \\ \alpha \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{M}^{-1}\mathbf{F} \end{bmatrix} \beta, \tag{4}$$

where

$$\mathbf{M} = \begin{bmatrix} m & mx_\alpha b \\ mx_\alpha b & I_\alpha \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_y & \tilde{q}U^2 b c_{l_\alpha} \\ 0 & -\tilde{q}U^2 b^2 c_{m_\alpha} + k_\alpha \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_y + \tilde{q}Ubc_{l_\alpha} & 1.1\tilde{q}Ub^2 c_{l_\alpha} \\ \tilde{q}Ub^2 c_{m_\alpha} & c_\alpha - 1.1\tilde{q}Ub^3 c_{m_\alpha} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} -U\tilde{q}bc_{l_\beta} \\ U^2\tilde{q}b^2 c_{m_\beta} \end{bmatrix}.$$

Traditional aeroservoelastic stability analysis defines the nominal stability (flutter) margin as the largest perturbation in the dynamic pressure such that the nominal aeroservoelastic model remains stable. Nominal models however are subject to errors due to the accuracy of model parameters, neglected unmodeled dynamics, and nonlinear effects. A robust aeroservoelastic model can be generated by associating uncertainty operators Δ with the nominal model and include the parametrization along with a perturbation in dynamic pressure. The robust flutter margin is the largest perturbation in dynamic pressure such that all possible feedback connections with nominal model and perturbation $\Delta \in \boldsymbol{\Delta}$ are stable. Modeling uncertainties, both parametric and dynamic in nature, can be incorporated into the aeroservoelastic stability analysis within the μ-analysis framework to be discussed in Section III.

The choice of uncertainty structure plays an important role in determining realistic stability (flutter) boundaries, however this discussion is beyond the scope of this paper. The aim here is to investigate the tools presented in the control systems literature for application in aeroservoelastic stability analysis, and in particular, model based stability analysis methods that include model uncertainty.

The aeroservoelastic model for nominal stability (flutter) analysis in the μ-analysis framework can be formulated using the model (4) specified at some nominal dynamic pressure $\tilde{q}_0$ with additional input and output signals to introduce perturbations to the dynamic pressure. Thus, given a nominal flight condition $\tilde{q}_0$, consider perturbations to dynamic pressure $\delta_{\tilde{q}}$ through a feedback relationship that is given by

$$\tilde{q} = \tilde{q}_0 + W_{\tilde{q}}\delta_{\tilde{q}}, \tag{5}$$

with $\|\delta_{\tilde{q}}\|_\infty \leq 1$. The norm bound restricts the search over dynamic pressure to perturbations of $\tilde{q} = \tilde{q}_0 \pm 1$ with respect to the nominal dynamic pressure $\tilde{q}_0$. Searching over larger range of dynamic pressure simply requires introducing an appropriate weighting $W_{\tilde{q}}$ to the additive perturbation of $\tilde{q}$ in (5). Consider also modeling uncertainties as perturbations to the structural spring constant k_α and damping coefficient c_y given by

$$k_\alpha = k_{\alpha_0} + W_{k_\alpha}\delta_{k_\alpha}, \qquad c_y = c_{y_0}(I + W_{c_y}\delta_{c_y}),$$

with $\|\delta_{k_\alpha}\|_\infty \leq 1$ and $\|\delta_{c_y}\|_\infty \leq 1$. Note the difference between the uncertainty descriptions corresponds to additive and multiplicative uncertainty structures for the spring constant and damping coefficient respectively. These model uncertainties are considered along with perturbations to dynamic pressure $\delta_{\tilde{q}}$ through the feedback relationship as in Figure 2 with structured uncertainty $\Delta \in \tilde{\boldsymbol{\Delta}}$,

$$\tilde{\boldsymbol{\Delta}} = \left\{ \tilde{\Delta} : \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \delta_{\tilde{q}} I \end{bmatrix},\ \Delta \in \mathrm{diag}(\delta_{c_y}, \delta_{k_\alpha}),\ \|\Delta\|_\infty \leq 1, \|\delta_{\tilde{q}}\|_\infty \leq 1 \right\}. \tag{6}$$

The control law is given by negative feedback of the velocity measurement,

$$H = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

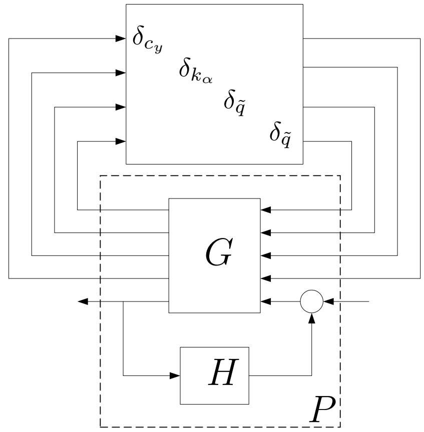

Figure 2. Pitch plunge uncertainty block diagram. The nominal aeroelastic model and controller are given by G and H respectively. The general aeroservoelastic plant model is denoted by P.

The state-space matrices of the general transfer matrix P for the uncertainty and feedback LFT configuration presented in Figure 2 is given by

$$\begin{bmatrix} \dot{y} \\ \dot{\alpha} \\ \ddot{y} \\ \ddot{\alpha} \\ \hline z_l \\ z_m \\ z_y \\ z_\alpha \\ y \\ \alpha \end{bmatrix} = \left[\begin{array}{cccc|ccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_{3l} & b_{3m} & b_{3y} & b_{3\alpha} & b_{3\beta} \\ a_{41} & a_{42} & a_{43} & a_{44} & b_{4l} & b_{4m} & b_{4y} & b_{4\alpha} & b_{4\beta} \\ \hline 0 & z_{l\alpha} & z_{l_{\dot{y}}} & z_{l_{\dot{\alpha}}} & 0 & 0 & 0 & 0 & z_{l_\beta} \\ 0 & z_{m\alpha} & z_{m_{\dot{y}}} & z_{m_{\dot{\alpha}}} & 0 & 0 & 0 & 0 & z_{m_\beta} \\ 0 & 0 & W_{c_y} & 0 & 0 & 0 & 0 & 0 & 0 \\ W_{k_\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} y \\ \alpha \\ \dot{y} \\ \dot{\alpha} \\ \hline w_l \\ w_m \\ w_y \\ w_\alpha \\ y \\ \alpha \end{bmatrix}, \qquad (7)$$

where z_l, z_m, w_l and w_m are the input-output signals of the feedback connection with the dynamic pressure perturbation and y, α correspond to pitch-plunge sensor measurements. Values for the elements within the general transfer matrix are derived from **M**, **C**, **K**, and **F**.[16]

More complex model uncertainties can be considered by augmenting the outputs of the state-space system with feedback perturbations. Additional model perturbations could include parametric uncertainty in the aerodynamic models, structural modes and model properties, as well as dynamic uncertainty overbounding responses measured from flight data.[14,19] The μ-analysis framework discussed in the following sections is a method for analyzing stability of uncertain feedback systems for which the pitch-plunge mechanism described by the model in Figure 2 is a special case.

III. Robustness Analysis

It is common practice in systems and control to specify performance and robustness of dynamical systems using the frequency domain. Frequency domain plots, such as Bode plots, Nyquist plots,[7] or more recently singular value plots,[25] convey lots of information about the system under consideration and are popular tools among the systems and control community. Various important properties in the systems and control community can be characterized by a set of inequality constraints in the frequency domain. For instance, specifications on the $\mathcal{H}_\infty$-norm (largest singular value $\overline{\sigma}$) of a transfer matrix $P(s)$, such as $\|P\|_\infty :=$

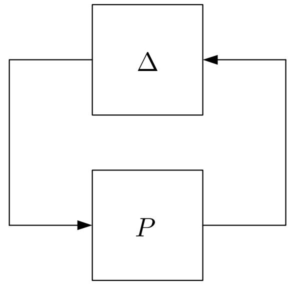

Figure 3. Standard uncertain system connection.

$\overline{\sigma}(P(j\omega)) < \beta$ for all frequencies $\omega \in \mathbb{R}$, can be specified by FDI

$$\begin{bmatrix} P(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\beta^2 I \end{bmatrix} \begin{bmatrix} P(j\omega) \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega \in \mathbb{R}. \tag{8}$$

The challenge in working with performance specifications in Frequency Domain Inequality (FDI) form is of course the infinite dimensionality of the problem, which must be specified for every frequency $\omega \in \mathbb{R}$. Indeed, various engineering design problems can be formulated as model-based specifications expressed by inequality constraints in the frequency domain. Conversion of such specifications into analytical conditions suitable to numerically tractable optimization problems remains an important area of research for dynamical system analysis and design.

Consider the standard setup for robustness and performance μ analysis i.e. the LFT feedback connection of a nominal map P and an uncertainty or perturbation Δ as depicted in Figure 3. The nominal map $P(s)$ is assumed to be a rational function of the complex variable s, being a proper and square matrix that is analytic in the closed right-half plane. The unknown uncertainty is assumed to have the following structure

$$\mathbf{\Delta} := \left\{ \text{diag}[\delta_1 I_{s_1}, \cdots, \delta_r I_{s_r}, \phi_1 I_{s_1}, \cdots, \phi_c I_{s_c}, \Delta_1, \cdots, \Delta_F] : \delta_i \in \mathbb{R}, \phi_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j} \right\}.$$

Note that the aeroservoelastic modeling technique presented in the previous section fits within this general setup, where Figure 2 is a special case of Figure 3 with structured uncertainty $\mathbf{\Delta} = \left\{ \text{diag}[\delta_{c_y}, \delta_{k_\alpha}, \delta_{\tilde{q}}] \right\}$. Let $\mathcal{T}(\Delta)$ denote the set of all block diagonal and stable rational transfer function matrices that have block structures such as $\mathbf{\Delta}$

$$\mathcal{T}(\Delta) := \{ \Delta(\cdot) \in \mathcal{RH}_\infty : \ \Delta(s_0) \in \mathbf{\Delta} \ \forall \ s_0 \in \mathbb{C}_+ \}$$

The feedback connection of (P, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{T}(\Delta)$ with $\|\Delta\|_\infty < 1$ if and only if[25]

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(P(j\omega)) \leq 1, \tag{9}$$

where $\mu_{\mathbf{\Delta}}$ denotes the structured singular value of a matrix, which is defined as

$$\mu_{\mathbf{\Delta}}(P) := \left(\inf_{\Delta \in \mathbf{\Delta}} \{ \|\Delta\| : \det(I - P\Delta) = 0 \} \right)^{-1}.$$

In case no $\Delta \in \mathbf{\Delta}$ makes $(I - P\Delta)$ singular $\mu_{\mathbf{\Delta}}(P) := 0$.

In general, the structured singular value $\mu_{\mathbf{\Delta}}$ cannot be computed in polynomial time, being a problem for which no algorithm exists (NP-hard).[23] In practice, the introduction of appropriate *scalings* or *multipliers* using duality theory is commonly used to provide computable upper bounds for $\mu_{\mathbf{\Delta}}$. For instance, define the set of scaling matrices

$$\mathbf{Z} := \left\{ \text{diag}[Z_1, \cdots, Z_{s_r + s_c}, z_1 I_{m_1}, \cdots, z_F I_{m_F}] : Z_i \in \mathbb{C}^{s_i \times s_i}, Z_i = Z_i^* > 0, z_j \in \mathbb{R}, z_j > 0 \right\}, \tag{10}$$

and

$$\mathbf{Y} := \{ \text{diag}[Y_1, \cdots, Y_{s_r}, 0, \cdots, 0] : \ Y_i = Y_i^* \in \mathbb{C}^{s_i \times s_i} \}. \tag{11}$$

Note that $\mathbf{Z}$ and $\mathbf{Y}$ commute with the matrices in $\mathbf{\Delta}$. Now define the matrix valued function

$$\Gamma_\beta(P(j\omega), Z(\omega), Y(\omega)) := \begin{bmatrix} P(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Z(\omega) & -jY(\omega) \\ jY(\omega) & -\beta^2 Z(\omega) \end{bmatrix} \begin{bmatrix} P(j\omega) \\ I \end{bmatrix} \tag{12}$$

and the optimization problem

$$\rho_{\mathbf{\Delta}}(P) := \inf_{\beta\in\mathbb{R}, Z\in\mathbf{Z}, Y\in\mathbf{Y}} \sup_{\omega\in\Omega} \{\beta : \ \Gamma_\beta(P(j\omega), Z(\omega), Y(\omega)) \prec 0\}. \tag{13}$$

It follows from duality theory[4] that

$$\sup_{\omega\in\Omega} \mu_{\mathbf{\Delta}}(P(j\omega)) \leq \rho_{\mathbf{\Delta}}(P). \tag{14}$$

The problem on the right hand side of the above inequality is, in some sense, simpler than the original problem (9). Yet it cannot be easily solved as well. The following are commonly found strategies for approaching this problem:

(i) **Constant multipliers on $\Omega = \mathbb{R}$:** When $\Omega = \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \, \forall \omega \in \mathbb{R}$, then problem (13) can be converted into an LMI using the KYP Lemma. The particular case $Z = I$, $Y = 0$ reduces to the well known BRL (Bounded-Real Lemma). This approach produces upper bounds for $\rho_{\mathbf{\Delta}}$.

(ii) **Constant multipliers on $\Omega \subset \mathbb{R}$:** When $\Omega = [\omega_1, \omega_2] \subset \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \, \forall \, |\omega| \in [\omega_1, \omega_2]$, then problem (13) can be converted into an LMI using the Generalized KYP results.[11] This approach produces upper bounds for $\rho_{\mathbf{\Delta}}$ with tight upper bounds obtained by splitting Ω in N segments $\Omega_i = [\omega_i, \omega_{i+1}]$, $i = 1, \ldots, N$ such that $\Omega = \cup_i \Omega_i$.

(iii) **Constant multipliers on $\Omega = \{w_1\}$:** For a single frequency ω_1, i.e., $\Omega = \{\omega_1\}$ problem (13) is an LMI. Lower bounds for $\rho_{\mathbf{\Delta}}$ can be obtained by solving this LMI on a finite grid $\Omega = \cup_i\{\omega_i\}$. In most cases, to achieve a reasonable approximation for $\rho_{\mathbf{\Delta}}$ a very dense grid must be used.

Recently, the authors developed a method for including frequency dependent multipliers.[10]

(iv) **Affine multipliers on $\Omega \subset \mathbb{R}$:** When $\Omega = [\omega_1, \omega_2] \subset \mathbb{R}$ and $Z(\omega)$ and $Y(\omega)$ are affine functions of the frequency variable ω. This will be discussed in detail in Section V.

IV. Robust Aeroservoelastic Analysis

Computation of robust aeroservoelastic stability (robust flutter) margins is considered in the μ-analysis framework through an augmented generalized plant (7) and uncertainty structure (6).

Lemma 1 *Let nominal plant P, derived for nominal dynamic pressure $\tilde{q}_0$, be given and define the relationship for perturbations to dynamic pressure $\delta_{\tilde{q}}$ as in Figure 2(a). Let scaling $W_{\overline{q}}$ and uncertainty structure $\mathbf{\Delta}$ be also given such that $\|\Delta\|_\infty < 1$, for all $\Delta \in \mathbf{\Delta}$. Define the scaled plant $\tilde{P}$ as*

$$\tilde{P} = P \begin{bmatrix} W_{\tilde{q}} & 0 \\ 0 & I \end{bmatrix}. \tag{15}$$

Then W_q is the robust aeroservoelastic stability (robust flutter) margin

$$\Gamma_{rob} = W_{\tilde{q}}$$

if and only if $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{P}(j\omega)) = 1$, where $\mu_{\mathbf{\Delta}}(\cdot)$ is defined in (9). Furthermore, $\tilde{q}_{rob} := \tilde{q}_0 + W_q$ is the dynamic pressure which is at the limit of stability.

For a proof see [16, chapter 8]. The robust aeroservoelastic stability (robust flutter) margin is determined by iterating over the scaling $W_{\tilde{q}}$ until the largest dynamic pressure $\tilde{q}$ given in (5) is found such that $\tilde{P}$ is robustly stable with respect to the set of uncertainties $\mathbf{\Delta}$. At each iteration a solution to the μ-analysis problem is required. As outlined in the previous section, μ_Δ can be approximated by evaluating upper bounds via ρ_Δ specified by the optimization problem (13).

Let the scaled plant (15) be described by state space matrices $[A, B, C, D]$ of appropriate dimension. The matrix valued function $\Gamma_\beta(\tilde{P}(j\omega), Z(\omega), Y(\omega))$ defined in (12) with frequency response given by

$$\tilde{P}(j\omega) = C\,(j\omega I - A)^{-1}\,B + D$$

can be written as

$$\Gamma_\beta(\tilde{P}(j\omega), Z(\omega), Y(\omega)) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z(\omega) & -jY(\omega) \\ jY(\omega) & -\beta^2 Z(\omega) \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}. \tag{16}$$

Then the robust ASE stability margin can be evaluated via the optimization problem (13) written as

$$\rho_{\mathbf{\Delta}}(\tilde{P}) = \inf_{\beta\in\mathbb{R}, Z\in\mathbf{Z}, Y\in\mathbf{Y}} \sup_{\omega\in\Omega} \left\{ \beta : \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \right\}, \tag{17}$$

where

$$\Theta(\omega) = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z(\omega) & -jY(\omega) \\ jY(\omega) & -\beta^2 Z(\omega) \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{18}$$

There are several methods available for evaluating ρ_Δ that are characterized a choice for the form of scaling matrices Z and Y as well as the frequency interval Ω that is considered. The following section outlines recent developments of the authors[10] for computing frequency affine scaling matrices.

V. Robust Analysis with Affine Scaling Matrices

In the last decades, thanks mostly to the result known as Kalman-Yakubovich-Popov (KYP) Lemma,[20] FDIs became a major tool in the analysis of dynamic systems. The KYP Lemma establishes the equivalence between the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega \in \mathbb{R}, \tag{19}$$

where matrices A, B and the Hermitian matrix Θ of appropriate finite dimensions are given, and the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \tag{20}$$

which should hold for some Hermitian matrix P. The main role of the KYP Lemma is to convert the infinite dimensional inequality (19) into the finite dimensional inequality (20) where appropriate choices for Θ represent the analysis of various system properties. The LMI formulation of the KYP Lemma (20) provides a constant multiplier relaxation for the FDI, see[21,22] over the entire frequency axis $\omega \in \mathbb{R}$, that is Θ is constant over the frequency axis. Particular choice of structure for the coefficient matrix Θ in the KYP Lemma specifies a μ-analysis over bound problem (12) using constant scaling matrices

$$\Theta = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z & -jY \\ jY & -\beta^2 Z \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{21}$$

Equivalent LMI conditions (20) characterizing FDIs (19) over the entire frequency range can be conservative in most practical applications where specifications of interest are considered only over finite frequency ranges. Including appropriate weighting functions in FDIs has demonstrated usefulness in practice for incorporating requirements over finite frequency intervals. However the process of selecting weights is non-trivial especially when considering the trade off between complexity and accuracy in capturing desired specifications.[13] Searching simultaneously for a frequency-dependent multiplier while satisfying the FDI over all frequencies has revealed to be a hard problem, for example the challenges of robust analysis via μ-analysis[4, 18] leading to the often used method of reducing the search domain by finite but sufficiently dense frequency grid. For some problems this technique is adequate, while for others it is unreliable, particularly in highly flexible aeroservoelastic systems with narrow and high peaks in the frequency domain plot since it becomes possible to miss the critical frequency and thus under-evaluate the system response.[6]

The above LMI presents a case where the scaling matrix Θ is constant. This case can be extended to incorporate a particular class of frequency-dependent Θ over finite frequency intervals. From a practical perspective, this allows for posing and checking frequency domain specifications within a certain frequency range which might be the most relevant to a specific application. Furthermore, by combining ranges one can pose frequency specifications in different ranges without augmenting the plant with frequency dependent scalings or weights. In particular we consider frequency-dependent multipliers of the form

$$\Theta(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1}\Theta_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1}\Theta_2. \tag{22}$$

Note that $\Theta(\omega)$ is not a proper rational function of ω which means that $\Theta(\omega)$ cannot be realized as a proper rational transfer function of ω. The affine function $\Theta(\omega)$ is restricted to a single frequency interval only for simplicity of the exposition and can be extended to more general piecewise affine functions.[9]

Theorem 2 *Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_i = \Theta_i^* \in \mathbb{C}^{(n+m)\times(n+m)}$ for $i = \{1, 2\}$ be given. If the LMI*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} I \\ j\omega_i I \end{bmatrix} \begin{bmatrix} F \\ E \end{bmatrix}^* + \begin{bmatrix} F \\ E \end{bmatrix} \begin{bmatrix} I \\ j\omega_i I \end{bmatrix}^* \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta_i \prec 0, \qquad i = \{1, 2\} \tag{23}$$

has feasible solution for some $F \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{m\times n}$, then the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } \omega_1 \le \omega \le \omega_2, \tag{24}$$

holds with $\Theta(\omega)$ given in (22).

Sufficiency for the pair of inequalities (23) introduces concepts from the analysis of polytopic systems[3] in treating the frequency an real uncertain parameter. To show sufficiency[8] a convex combination of (23) is formed with the use of the parameter

$$\lambda(\omega) := \frac{\omega_2 - \omega}{\omega_2 - \omega_1}, \qquad \omega_1 \le \omega \le \omega_2. \tag{25}$$

Indeed $\lambda(\omega) \in [0, 1]$ for all $\omega_1 \le \omega \le \omega_2$, which forms the basis for the frequency-dependent matrix Θ

$$\Theta(\omega) = \lambda(\omega)\Theta_1 + (1 - \lambda(\omega))\Theta_2,$$

where $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$. The convex combination is formed by taking the sum of (23) for $i = 1$ multiplied by $\lambda(\omega)$ and (23) for $i = 2$ multiplied by $(1 - \lambda(\omega))$, which results in the feasible FDI

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} I \\ j\omega I \end{bmatrix} \begin{bmatrix} F \\ E \end{bmatrix}^* + \begin{bmatrix} F \\ E \end{bmatrix} \begin{bmatrix} I \\ j\omega I \end{bmatrix}^* \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta(\omega) \prec 0, \qquad \text{for all } \omega_1 \le \omega \le \omega_2.$$

Define the matrix

$$\mathcal{N}(\omega) := \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

and multiply the above inequality by $\mathcal{N}(\omega)$ on the right and by its conjugate transpose on the left to obtain

$$\mathcal{N}(\omega)^*\Theta(\omega)\mathcal{N}(\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0.$$

Theorem 2 can be used to produce upper bounds to ρ_Δ which has as its main advantage the fact that the scaling matrices Z and Y are allowed to be affine functions of frequency ω. That is

$$\begin{aligned} Z(\omega) &= \frac{\omega_2 - \omega}{\omega_2 - \omega_1} Z_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1} Z_2, \\ Y(\omega) &= \frac{\omega_2 - \omega}{\omega_2 - \omega_1} Y_1 + \frac{\omega - \omega_1}{\omega_2 - \omega_1} Y_2. \end{aligned}$$

The scaling matrices Z_1, Z_2 and Y_1, Y_2 appear linearly in the coefficient matrices Θ_1, Θ_2 as well as in the LMI conditions of Theorem 2 maintaining a convex optimization for computing upper bounds to ρ_Δ in (13). Furthermore, allowing for frequency dependent coefficient matrices $\Theta(\omega)$ can significantly reduce conservatism particularly in applications that include searching over variables in Θ.

Necessity of the pair of LMI (23) with Θ constant, that is $\Theta_1 = \Theta_2 = \Theta$, can be established[10] using results on the generalized KYP Lemma.[11] In this case, if the FDI (19) holds for $\omega_1 \leq \omega \leq \omega_2$ then there exists a particular choice of F and E which is guaranteed to satisfy the above inequalities.

Note that Theorem 2 has the difficulty of handling infinite frequency intervals, for instance in case $\omega_2 \to \infty$. This difficulty can overcome through a change of variables on the frequency ω via the bilinear transformation. Consider the transformation of the frequency variable

$$\psi = \frac{\omega - z}{1 + \omega - z}, \tag{26}$$

which maps the semi-infinite segment of the real axis $\omega \in [z, \infty)$ onto the finite segment of the real axis $\psi \in [0, 1)$. The following is an extension of Theorem 2 where the transformed frequency variable (26) has been substituted into the LMI conditions.[10]

Theorem 3 *Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_i = \Theta_i^* \in \mathbb{C}^{(n+m)\times(n+m)}$ for $i = \{1, 2\}$ be given. If the LMI*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} (1-\psi_i)I \\ j[z + \psi_i(1-z)] \end{bmatrix} \begin{bmatrix} F & E \end{bmatrix} + \begin{bmatrix} F \\ E \end{bmatrix} \begin{bmatrix} (1-\psi_i)I & -j[z + \psi_i(1-z)] \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta_i \prec 0, \tag{27}$$

with $\psi_1 = 0$, $\psi_2 = 1$ has feasible solution for some $F \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{m\times n}$, then the FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta\left(\frac{\omega - z}{1 + \omega - z}\right) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \qquad \text{for all } z \leq \omega < \infty, \tag{28}$$

holds with $\Theta(\cdot)$ given in (22).

The above theorem can handle the case $\omega_2 \to \infty$ by making $\psi_2 \to 1$. Furthermore, for real-valued matrices $[A, B, C, D]$ with appropriate dimension, the above theorem can handle the case $\omega \in \mathbb{R}$ by setting $z = 0$.[10] Note that Theorems 2 and 3 are not completely equivalent and may produce different results for the same frequency interval since the coefficient matrix in Theorem 3 is an affine function of the transformed frequency variable ψ. The LMI conditions (27) are finite in ψ despite the infinite dimension of the FDI in ω.

VI. Evaluating Robust Aeroservoelastic Stability Margins

The robust aeroservoelastic stability margin uses μ-analysis tools to give worst case stability parameters. Safe and efficient expansion of the flight envelope can be performed using on-line implementation of the above analysis algorithm. Since computing the upper bound is a convex optimization, it does not introduce excessive computational burden. The predictive nature of model based analysis methods, such as the μ-method allow for so called flutterometer tools to be developed for tracking flutter margin during flight tests.[15]

The generalized plant is scaled such that the peak structured singular value plot is approximately, but not over, unity as shown in Figure 4. That is the system is robustly stable, with respect to the model uncertainty and scaled perturbation in dynamic pressure. The maximum lower bound for ρ_Δ evaluated on the frequency grid is $\rho_\Delta = 0.982$ at a frequency $\omega = 1.4 \times 10^{-6} Hz$. Using either Theorem 2 with $\omega \in [0, 1]$ or Theorem 3 with $\omega \in \mathbb{R}$ provides an upper bound $\rho_\Delta = 0.995$.

For this feedback control configuration however, the singular value plot approaches the robust stability limit at frequencies much lower than the predominant dynamics of the system. This is not realistic since these dynamics occur at a frequency of approximately every 8 days. Furthermore, analysis performed using conventional techniques that hold for all frequencies $\omega \in \mathbb{R}$, such as the original KYP Lemma conditions (20), give an upper bound $\rho_\Delta = 5.2$ where the stability margin would have been greatly underestimated due to the conservativeness introduced by the numerical computations. Analysis performed on a finite frequency range above $10^{-4} Hz$ provides a less conservative stability margin. The maximum lower bound for ρ_Δ evalauted on the frequency grid above $1Hz$ is $\rho_\Delta = 0.601$. Using either Theorem 2 with $\omega \in [1, 100]$ or Theorem 3 with $1 \le \omega \le \infty$ provides an upper bound $\rho_\Delta = 0.602$.

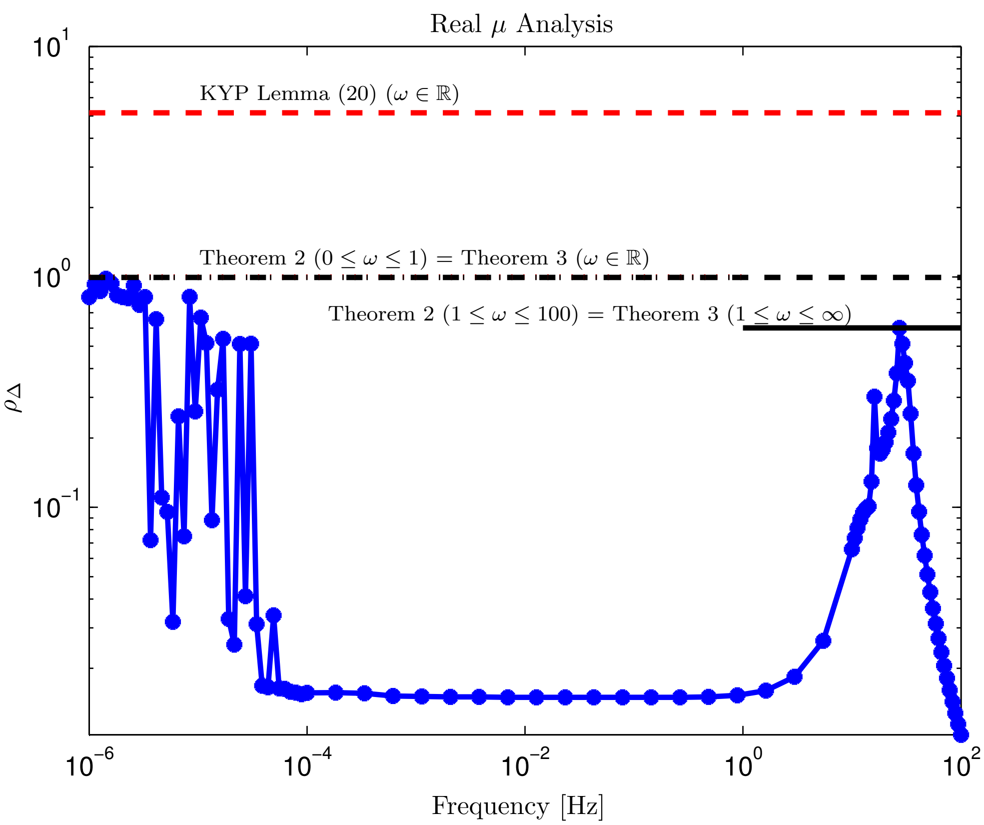

Figure 4. Robust flutter analysis with velocity feedback controller. The curved line is the greatest lower bound for ρ_Δ with dots located at the frequency grid. Other lines are labeled according to the method and frequency range used to compute them.

VII. Conclusions

A model based ASE stability analysis method is presented as a standard robust analysis problem from the robust control theory framework. A computationally efficient method is then presented for robustness analysis over finite frequency intervals. The simple pitch plunge ASE system example illustrates the usefulness of the results presented in previous sections for practical problems in which performance and robustness are considered.

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