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A Linear Matrix Inequality for Robust Stability Analysis with Frequency-Dependent Multipliers

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Abstract— **In this paper we introduce a Linear Matrix Inequality (LMI) condition for robust stability analysis. The condition is expressed as a pair of convex inequalities that provides an upper bound for the structured singular that can be used to verify stability and performance robustness. This robust analysis test incorporates a particular class of frequency-dependent multipliers and can be limited to finite frequency intervals, features which can significantly reduce conservatism as compared to existing conditions with similar complexity. The results are illustrated with a simple numerical example that illustrates the improvement of the proposed LMI condition.**

I. Introduction

The structured singular value, often referred as μ and introduced in [1], has been a popular analytical tool for analyzing stability and performance robustness of linear systems with parametric and dynamic uncertainties. Despite the fact that computing μ, or even finding tight bounds for μ, has proved an extremely hard problem, the use of the μ terminology and methodology in robust control is widespread.

A number of algorithms has been developed to compute the maximum of the structured singular value over a specific frequency interval. Many popular methods make use of the concept of frequency-dependent multipliers [2]. Searching simultaneously for a frequency-dependent multiplier and the maximum μ over all frequencies has also revealed to be a hard problem. An often used device is the reduction of the search domain by using finite yet sufficiently dense frequency grid. Search algorithms have been proposed for determining the maximum μ over frequency interval [3], [4] avoiding unnecessarily dense grids. Other approaches have considered the frequency itself as uncertain parameter augmented to the original system [5] eliminating difficulties associated with gridding methods. For improved accuracy finite frequency intervals can be considered via linear fractional transformation mapping of the real frequency parameter [6].

The KYP (Kalman-Yakubovich-Popov) Lemma establishes equivalence between frequency domain inequalities on the system transfer function and a certain Linear Matrix Inequality (LMI). Generalization of the KYP Lemma allows for treatment of finite frequency ranges [7]. This result is closely related with standard μ-analysis [8], and can be proved from results on the losslessness of scalings for mixed-μ [9]. Recent results show the connection of these methods and propose generalizations in the context of powerful relaxation techniques [10], [11] for which robust analysis tests that approach exactness can be derived in a systematic way at the expense of increasing the large size of the problem to be solved and number of optimization variables.

The results presented in this paper provide new robust stability conditions with a complexity that is comparable to the generalized KYP lemma [7]. The tests are expressed as a pair of LMI that, if solvable, provide an upper bound to μ over some specified and possibly finite frequency interval along with a particular frequency-dependent multiplier that is used to prove robust stability. We borrow ideas from robust analysis of uncertain polytopic systems [12] in the treatment of frequency as a real uncertain parameter to formulate the results in this paper.

The paper is organized as follows: Section III provides a review of robust analysis using the structured singular value μ. Subsequently, Section IV defines the affine frequency dependent multipliers and presents the main results for robust analysis. The proof of the main result is provided in Section V. To illustrate the reduction in conservatism of the computed upper bounds, the proposed method is illustrated by a numerical example in Section VI.

II. Notation

The following notation will be used throughout the paper. The scalar $j = \sqrt{-1}$. For a matrix $X \in \mathbb{C}^{n\times n}$, X^{-1}, $\overline{X}$ and X^* are the inverse, complex-conjugate and complex-conjugate transpose of the matrix X respectively. $\mathrm{He}\{X\}$ is short-hand notation for $X + X^*$.

III. Robust Analysis

Consider the standard setup for robustness and performance analysis as the linear fractional transformation (LFT) feedback connection of a nominal map M and an uncertainty or perturbation Δ, depicted in Figure 1.

The nominal map $M(s)$ is assumed to be a rational function of the complex variable s, being a proper and square

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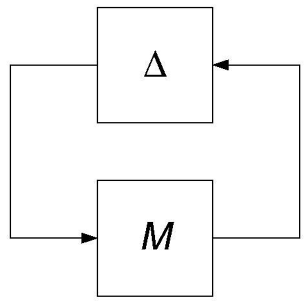

Fig. 1. Standard uncertain system connection.

matrix that is analytic in the closed right-half plane. The unknown uncertainty is restricted to have the structure

$$\mathbf{\Delta} := \{\text{diag}[\phi_1 I_{s_1}, \cdots, \phi_r I_{s_r}, \delta_1 I_{s_1}, \cdots, \delta_c I_{s_c}, \Delta_1, \cdots, \Delta_F] : \phi_i \in \mathbb{R}, \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{p_j \times p_j}\}. \quad (1)$$

allowing a combination scalar real-valued and complex valued parameter perturbations and complex (unstructured) uncertainty blocks. By choosing the number, size, and dynamic nature of the elements of Δ, a variety of uncertainty structures can be translated into this standard form (see for instance [13]).

Let $\mathcal{M}(\Delta)$ denote the set of all block diagonal and stable rational transfer function matrices that have block structures such as $\mathbf{\Delta}$

$$\mathcal{M}(\Delta) := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \ \Delta(s_0) \in \mathbf{\Delta} \ \forall \ s_0 \in \mathbb{C}_+\} \quad (2)$$

The feedback connection of (M, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{M}(\Delta)$ with $\|\Delta\|_\infty < \beta^{-1}$ if and only if [13]

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(M(j\omega)) \le \beta, \quad (3)$$

where $\mu_{\mathbf{\Delta}}$ denotes the structured singular value of a matrix, which is defined as

$$\mu_{\mathbf{\Delta}}(M) := \left(\inf_{\Delta \in \mathbf{\Delta}} \{\|\Delta\| : \det(I - M\Delta) = 0\} \right)^{-1}.$$

In case no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular $\mu_{\mathbf{\Delta}}(M) := 0$.

In general the structured singular value $\mu_{\mathbf{\Delta}}$ cannot be computed in reasonable (polynomial) time, being a problem for which no polynomial-time algorithm can ever be found (NP-hard) [14]. In practice, the introduction of appropriate *scalings* or *multipliers* through duality theory is commonly used to provide computable upper bounds for $\mu_{\mathbf{\Delta}}$ in polynomial time.

For instance, define the set of scaling matrices

$$\mathbf{Z} := \{\text{diag}[Z_1, \cdots, Z_{s_r+s_c}, z_1 I_{p_1}, \cdots, z_F I_{p_F}] : Z_i \in \mathbb{C}^{s_i \times s_i}, Z_i = Z_i^* > 0, z_j \in \mathbb{R}, z_j > 0\}, \quad (4)$$

and

$$\mathbf{Y} := \{\text{diag}[Y_1, \cdots, Y_{s_r}, 0, \cdots, 0] : Y_i = Y_i^* \in \mathbb{C}^{s_i \times s_i}\} \quad (5)$$

where it can be noted that $\mathbf{Z}$ and $\mathbf{Y}$ commute with the matrices in $\mathbf{\Delta}$. Now define the matrix valued function

$$\Phi_\beta(M, Z, Y) := M^* Z M - j\,(M^* Y - Y M) - \beta^2 Z, \quad (6)$$

and the optimization problem

$$\rho_{\mathbf{\Delta}}(M) := \inf_{\beta \in \mathbb{R}, Z \in \mathbf{Z}, Y \in \mathbf{Y}} \sup_{\omega \in \Omega} \{\beta : \ \Phi_\beta(M(j\omega), Z(\omega), Y(\omega)) < 0\}. \quad (7)$$

It follows from duality theory [15] that

$$\sup_{\omega \in \Omega} \mu_{\mathbf{\Delta}}(M(j\omega)) \le \rho_{\mathbf{\Delta}}. \quad (8)$$

The problem on the right hand side of the above inequality is, in some sense, simpler than the original problem (3). Yet it cannot be easily solved as well and the following strategies are commonly known approaches to tackle this problem:

a) Constant multipliers on $\Omega = \mathbb{R}$: When $\Omega = \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \, \forall \, \omega \in \mathbb{R}$, then problem (7) can be converted into an LMI (Linear Matrix Inequality) using the KYP (Kalman-Yakubovich-Popov) Lemma. LMIs are convex problems and can be solved in polynomial time. The particular case $Z = I$, $Y = 0$ reduces to the well known BRL (Bounded-Real Lemma). This approach produces upper bounds for $\rho_{\mathbf{\Delta}}$.

b) Constant multipliers on $\Omega \subset \mathbb{R}$: When $\Omega = [\omega_1, \omega_2] \subset \mathbb{R}$ and Z and Y are assumed to be constant, i.e., $Z(\omega) = Z$ and $Y(\omega) = Y \, \forall \, |\omega| \in [\omega_1, \omega_2]$, then problem (7) can be converted into an LMI (Linear Matrix Inequality) using the Generalized KYP Lemma [7]. This approach produces upper bounds for $\rho_{\mathbf{\Delta}}$. Tight upper bounds are obtained by splitting Ω in N segments $\Omega_i = [\omega_i, \omega_{i+1}]$, $i = 1, \ldots, N$ such that $\Omega = \cup_i \Omega_i$.

c) Constant multipliers on $\Omega = \{w_1\}$: For a single frequency ω_1, i.e., $\Omega = \{\omega_1\}$ problem (7) is an LMI. Lower bounds for $\rho_{\mathbf{\Delta}}$ can be obtained by solving this LMI on a finite grid $\Omega = \cup_i \{\omega_i\}$. In most cases, to achieve a reasonable approximation for $\rho_{\mathbf{\Delta}}$ a very dense grid must be used.

This paper introduces a procedure to produce upper bounds to $\rho_{\mathbf{\Delta}}$ which has as its main advantage the fact that Z and Y are allowed to be some specific affine functions of ω on a set $\Omega = [\omega_1, \omega_2] \subset \mathbb{R}$. The result, as illustrated by examples, produces upper bounds for $\rho_{\mathbf{\Delta}}$ that can be significantly less conservative than the ones obtained by the methods discussed above while been relatively cheap to compute [16].

IV. The Main Result

The basic idea behind the results of this paper is to treat the frequency ω as an uncertain parameter. We borrow ideas from the robustness analysis of uncertain systems with polytopic uncertainty [12] to develop robustness analysis conditions with frequency-dependent multipliers. These tests allow the value of ω to be constrained in a segment of the real line, i. e., $|\omega| \in \Omega = [\omega_1, \omega_2] \subset \mathbb{R}^+$, where ω_1 can be possibly zero and ω_2 infinity.

A. Frequency-dependent Multipliers

The main contribution of our results is the possibility of incorporating frequency-dependent multipliers. In particular, we consider frequency-dependent multipliers of the form

$$\begin{aligned} Z(\omega) &= \begin{cases} \overline{Z_a} - \omega \overline{Z_b}, & \omega < 0, \\ Z_a + \omega Z_b, & \omega > 0, \end{cases} \\ Y(\omega) &= \begin{cases} \omega \overline{Y_b} - \overline{Y_a}, & \omega < 0, \\ Y_a + \omega Y_b, & \omega > 0. \end{cases} \end{aligned} \tag{9}$$

Note that this definition implies

$$Z(-\omega) = \overline{Z(\omega)}, \quad Y(-\omega) = -\overline{Y(\omega)}, \quad \omega \neq 0. \tag{10}$$

The multipliers Z and Y are affine functions of ω for positive and negative real values of ω and it is not defined at $\omega = 0$. We will extend this definition to handle this case later. Also of interest is the fact that a spectral factorization of $Z(\omega) \in \mathbf{Z}$ will not in general produce a rational function of $j\omega$, which means that these multipliers cannot be realized as rational transfer functions of $j\omega$.

We now introduce alternative parametrizations for these multipliers based on the affine functions

$$\begin{aligned} Z_\xi(\xi) &:= \xi Z_1 + (1-\xi) Z_2, \quad Z_1, Z_2 \in \mathbf{Z}, \\ Y_\xi(\xi) &:= \xi Y_1 + (1-\xi) Y_2, \quad Y_1, Y_2 \in \mathbf{Y}. \end{aligned}$$

This parametrization will prove better suited to our future developments. Note that

$$Z_\xi(\xi) \in \mathbf{Z}, \qquad Y_\xi(\xi) \in \mathbf{Y}, \qquad \forall\, \xi \in [0,1],$$

and that for all $|\omega| \in \Omega = [\omega_1, \omega_2]$

$$\frac{\omega_2 - |\omega|}{\omega_2 - \omega_1} \in [0,1].$$

Finally define

$$Z_\Omega(\omega) := Z_\xi\left(\frac{\omega_2 - |\omega|}{\omega_2 - \omega_1}\right), \quad Y_\Omega(\omega) := Y_\xi\left(\frac{\omega_2 - |\omega|}{\omega_2 - \omega_1}\right),$$

so that Z and Y can be rewritten in the form

$$\begin{aligned} Z(\omega) &= \begin{cases} \overline{Z_\Omega(-\omega)}, & \omega < 0, \\ Z_\Omega(\omega), & \omega > 0, \end{cases} \\ Y(\omega) &= \begin{cases} -\overline{Y_\Omega(-\omega)}, & \omega < 0, \\ Y_\Omega(\omega), & \omega > 0. \end{cases} \end{aligned} \tag{11}$$

The two main advantages of this alternative parametrization are: a) the constraint $Z(\omega) \in \mathbf{Z}$ and $Y(\omega) \in \mathbf{Y}$ can be enforced by imposing Z_1, $Z_2 \in \mathbf{Z}$ and Y_1, $Y_2 \in \mathbf{Y}$; b) linearity of Z_ξ and Y_ξ with respect to $\xi = (\omega_2 - |\omega|)/(\omega_2 - \omega_1)$ can be used to build convex sufficient conditions for robust stability, as we will see in the next sections.

Note that (9) and (11) are completely equivalent. For instance, values of Z_a and Z_b associated with the definition (9) can be obtained from (11) by computing

$$\begin{aligned} Z_a &= Z_\Omega(0) = Z_\xi\left(\frac{\omega_2}{\omega_2 - \omega_1}\right), \\ Z_b &= \frac{1}{\omega_2}\left[Z_\Omega(\omega_2) - Z_a\right] = \frac{1}{\omega_2}\left[Z_\xi(0) - Z_a\right] = \frac{1}{\omega_2}\left[Z_2 - Z_a\right]. \end{aligned}$$

Accordingly for Y_a and Y_b

$$\begin{aligned} Y_a &= Y_\Omega(0), \\ Y_b &= \frac{1}{\omega_2}\left[Y_\xi(0) - Y_a\right] = \frac{1}{\omega_2}\left[Y_2 - Y_a\right]. \end{aligned}$$

B. The Basic Idea

In order to understand the idea behind the results to be presented in the next sections consider, just for the moment, that $M(j\omega) = M \,\forall\, \omega \in \Omega = [\omega_1, \omega_2]$, that is, that M is constant and real. Now verify for some $\beta > 0$ whether the pair of inequalities

$$\begin{aligned} 0 &> M^T Z(\omega_1) M - j\left[M^T Y(\omega_1) - Y(\omega_1) M\right] - \beta^2 Z(\omega_1) \\ &= M^T Z_1 M - j\left[M^T Y_1 - Y_1 M\right] - \beta^2 Z_1, \\ 0 &> M^T Z(\omega_2) M - j\left[M^T Y(\omega_2) - Y(\omega_2) M\right] - \beta^2 Z(\omega_2) \\ &= M^T Z_2 M - j\left[M^T Y_2 - Y_2 M\right] - \beta^2 Z_2, \end{aligned}$$

have some feasible solution Z_1, $Z_2 \in \mathbf{Z}$, Y_1, $Y_2 \in \mathbf{Y}$. In the affirmative case, the sum of the first inequality multiplied by the positive scalar $\xi = (\omega_2 - |\omega|)/(\omega_2 - \omega_1) \in [0,1]$ with the second multiplied by $(1-\xi)$ implies that

$$M^T Z(\omega) M - j\left[M^T Y(\omega) - Y(\omega) M\right] - \beta^2 Z(\omega) < 0, \quad \forall\, \omega \in \Omega = [\omega_1, \omega_2]. \tag{12}$$

Now to prove that the result is also valid for negative values of ω, take the complex conjugate of the above inequality

$$M^T \overline{Z(\omega)} M + j\left[M^T \overline{Y(\omega)} - \overline{Y(\omega)} M\right] - \beta^2 \overline{Z(\omega)} < 0,$$

and use (10) to obtain

$$M^T Z(-\omega) M - j\left[M^T Y(-\omega) - Y(-\omega) M\right] - \beta^2 Z(-\omega) < 0,$$

which implies that (12) indeed holds for all $|\omega| \in \Omega$.

Feasibility of the pair of inequalities should be verified for a given value of β. A simple bisection algorithm can be used to compute the minimum value of $\beta^* = \beta$ for which these inequalities are feasible. This value is clearly an upper bound to $\rho_{\mathbf{\Delta}}$ for $\omega \in \Omega$.

C. Robust Stability Tests

Of course, the problem of the previous section where M is assumed to be constant, is not interesting at first[1]. The technical challenge is to introduce pairs of LMIs that produce upper bounds to $\rho_{\boldsymbol{\Delta}}$ for the case when M is a general proper rational function of ω, i.e., $M(j\omega) = C(j\omega I - A)^{-1}B + D$, where the real valued matrices (A, B, C, D) are assumed to have compatible finite dimensions.

To achieve this goal, frequency-dependent multipliers of the form (11) can be used and the following robust stability conditions sumamrize the main contributions of this paper.

Theorem 1: Let $M(j\omega) = C(j\omega I - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Let $\omega_1, \omega_2 \in \mathbb{R}$ such that $\omega_2 \geq \omega_1 \geq 0$. If there exists matrices $Z_1, Z_2 \in \mathbf{Z}$, $Y_1, Y_2 \in \mathbf{Y}$ and matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMIs

$$\begin{bmatrix} C^*Z_1C & C^*Z_1D - jC^*Y_1 \\ D^*Z_1C + jY_1C & D^*Z_1D - \eta Z_1 + \text{He}\{jY_1D\} \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (j\omega_1 I - A) & -B \end{bmatrix} \right\} < 0, \quad (13)$$

$$\begin{bmatrix} C^*Z_2C & C^*Z_2D - jC^*Y_2 \\ D^*Z_2C + jY_2C & D^*Z_2D - \eta Z_2 + \text{He}\{jY_2D\} \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (j\omega_2 I - A) & -B \end{bmatrix} \right\} < 0 \quad (14)$$

has feasible solutions then $\rho_{\boldsymbol{\Delta}}(M(j\omega), \Omega) \leq \sqrt{\eta}$ for all $|\omega| \in \Omega = [\omega_1, \omega_2]$.

It can be seen that indeed the frequency-dependent multipliers obtained with the above theorem are of the form (11). This theorem has difficulties to handle the extreme case $\omega_2 \to \infty$. This case is summarized in next main result, which is an alternative version of the previous theorem on the transformed frequency variable $\gamma = \omega^{-1}$.

Theorem 2: Let $M(j\omega) = C(j\omega I - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Let $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\gamma_2 \geq \gamma_1 \geq 0$. If there exists matrices $Z_1, Z_2 \in \mathbf{Z}$, $Y_1, Y_2 \in \mathbf{Y}$ and matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMIs

$$\begin{bmatrix} C^*Z_1C & C^*Z_1D - jC^*Y_1 \\ D^*Z_1C + jY_1C & D^*Z_1D - \eta Z_1 + \text{He}\{jY_1D\} \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (jI - \gamma_1 A) & -\gamma_1 B \end{bmatrix} \right\} < 0, \quad (15)$$

$$\begin{bmatrix} C^*Z_2C & C^*Z_2D - jC^*Y_2 \\ D^*Z_2C + jY_2C & D^*Z_2D - \eta Z_2 + \text{He}\{jY_2D\} \end{bmatrix} + \text{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (jI - \gamma_2 A) & -\gamma_2 B \end{bmatrix} \right\} < 0 \quad (16)$$

has feasible solutions then $\rho_{\boldsymbol{\Delta}}(M(j\omega), \Omega) \leq \sqrt{\eta}$ for all $\omega = \gamma^{-1} \in \Omega = [\gamma_2^{-1}, \gamma_1^{-1}]$.

Theorem 2 can handle the case $\omega \to \infty$ without further complications by making $\gamma_2 \to 0$. Note that Theorems 1 and 2 are not completely equivalent, and that they may produce different results for the very same frequency range. We will see in the next section that the multipliers produced by Theorem 2 are functions of $\gamma = \omega^{-1}$ of the form (11).

[1]Indeed, the pair of inequalities in the previous section, if feasible, can always be satisfied with $Z_1 = Z_2$ and $Y_1 = Y_2$.

V. Proof of the Main Result

The following Lemma is the central piece in the proof of Theorems 1 and 2. It establishes an equivalence between the frequency domain condition ρ_Δ and two conditions extended with matrix multipliers as in the lines of [12].

Lemma 1: Let $M(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Then the following statements are equivalent:

(i) $\rho_{\boldsymbol{\Delta}}(M) \leq \sqrt{\eta}$ on $\Omega = [\omega_1, \omega_2]$.

(ii) There exists multipliers $Z(\omega) \in \mathbf{Z}$, $Y(\omega) \in \mathbf{Y}$, $F(\omega) \in \mathbb{C}^{n\times n}$ and $G(\omega) \in \mathbb{C}^{p\times n}$ such that the inequality

$$Q_\eta(\omega) + \text{He}\left\{ \begin{bmatrix} F(\omega) \\ G(\omega) \end{bmatrix} \begin{bmatrix} (j\omega I - A) & -B \end{bmatrix} \right\} < 0 \quad (17)$$

where $Q_\eta(\omega)$ is given by

$$Q_\eta(\omega) := \begin{bmatrix} C^*Z(\omega)C & C^*Z(\omega)D - jC^*Y(\omega) \\ (\star)^* & D^*Z(\omega)D - \eta Z(\omega) + \text{He}\{jY(\omega)D\} \end{bmatrix} \quad (18)$$

and has a feasible solution evaluated for all $|\omega| \in \Omega = [\omega_1, \omega_2]$.

Proof: Equivalence of items (i) and (ii) can be proven using the multiplier methods of [12]. To show that (ii) implies (i), multiply inequality (17) by

$$\mathcal{B}^\perp(\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

on the right and by its transpose conjugate on the left to obtain

$$\Phi_{\sqrt{\eta}}(M(j\omega), Z(\omega), Y(\omega)) = \mathcal{B}^{\perp *}(\omega) Q_\eta(\omega) \mathcal{B}(\omega)^\perp < 0$$

To show that (i) implies (ii) one can invoke Finsler's Lemma (see [12] for more details) to establish the equivalence between feasibility of $\mathcal{B}^{\perp *}(\omega) Q_\eta(\omega) \mathcal{B}(\omega)^\perp < 0$ and the existence of matrix multipliers $\mathcal{X}(\omega) \in \mathbb{C}^{(n+p)\times n}$ such that $Q_\eta(\omega) + \mathcal{X}(\omega)\mathcal{B}(\omega) + \mathcal{B}(\omega)^*\mathcal{X}(\omega)^* < 0$, where in this case

$$\mathcal{B}(\omega) = \begin{bmatrix} (j\omega I - A) & -B \end{bmatrix}, \qquad \mathcal{X}(\omega) = \begin{bmatrix} F(\omega) \\ G(\omega) \end{bmatrix},$$

which is inequality (17). ■

Both items (i) and (ii) of the previous Lemma provide inequality conditions that must be checked for all $\omega \in \Omega$. It can be observed that Theorems 1 and 2 reduce the number of inequalities to be checked to two convex inequalities (Linear Matrix Inequalities) at the expense of some degree of conservativeness. We now show that these pairs of inequalities implies feasibility of the condition in item (ii) of the previous Lemma.

Assume that the pair of inequalities in Theorem 1 have feasible solutions and that $\omega_2 \geq \omega_1 > 0$. The sum of (13) multiplied by $\xi = (\omega_2 - |\omega|)/(\omega_2 - \omega_1)$ and of (14) multiplied by $(1 - \xi)$ produces

$$Q_\eta(\omega) + \mathrm{He}\left\{ \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} (j\omega I - A) & -B \end{bmatrix} \right\} < 0, \quad \forall \omega \in [\omega_1, \omega_2]. \quad (19)$$

Taking the conjugate and using the property in (10) we obtain the inequality

$$Q_\eta(-\omega) + \mathrm{He}\left\{ \begin{bmatrix} \overline{F} \\ \overline{G} \end{bmatrix} \begin{bmatrix} (-j\omega I - A) & -B \end{bmatrix} \right\} < 0, \quad \forall \omega \in [-\omega_2, -\omega_1]. \quad (20)$$

Therefore, by defining

$$F(\omega) = \begin{cases} \overline{F}, & \omega < 0, \\ F & \omega > 0, \end{cases} \qquad G(\omega) = \begin{cases} \overline{G}, & \omega < 0, \\ G & \omega > 0 \end{cases}$$

it can be concluded that (17) is feasible. Henceforth, $\sqrt{\eta}$ is an upper bound to $\rho_{\mathbf{\Delta}}$ on $\Omega = [\omega_1, \omega_2]$ when $\omega_1 > 0$.

The case $\omega_1 = 0$ can be proven by evaluating the previous expression in the limit as $\omega_1 \to 0$. It should be noted that $Z(\omega)$, $Y(\omega)$, $F(\omega)$ and $G(\omega)$ are discontinuous at $\omega = 0$, it is necessary to take the limits on the right and on the left separately. Also note that the arguments used above do not require continuity at $\omega = 0$, and that feasibility of (19) for $\omega_1 \to 0^+$ still implies feasibility of (20).

The proof of Theorem 2 can be found along the same pattern. The inequality in item (ii) is being satisfied for

$$F(\omega) = \begin{cases} \omega^{-1}\overline{F}, & \omega < 0, \\ \omega^{-1} F & \omega > 0, \end{cases} \qquad G(\omega) = \begin{cases} \omega^{-1}\overline{G}, & \omega < 0, \\ \omega^{-1} G & \omega > 0, \end{cases}$$

after imposing that $Z(\omega)$ and $Y(\omega)$ are affine functions of $\omega^{-1} = \gamma$ of the form (11) instead of ω.

VI. Illustrative Example

The proposed method for computing upper bounds for the structured singular value for performance robustness analysis is illustrated by a numerical example. Let an uncertain plant be known to belong to a set of input-multiplicative plant models described by

$$\{P(1 + \delta_1 W_u) : \ \delta_1 \in \mathcal{RH}_\infty, \quad \|\delta_1\|_\infty \leq 1\}$$

For the numerical example in this paper, the nominal plant model and uncertainty weight are respectively given by

$$P = \frac{10}{s(s+10)}, \quad W_u = \frac{10(s+5)}{(s+100)}$$

Furthermore, as illustrated in Figure 2, this uncertain plant operates under feedback with a PID controller

$$C = 5 + \frac{0.1}{s} + \frac{0.5s}{1 + 0.1s}$$

where the differentiator has been approximated by a proper transfer function.

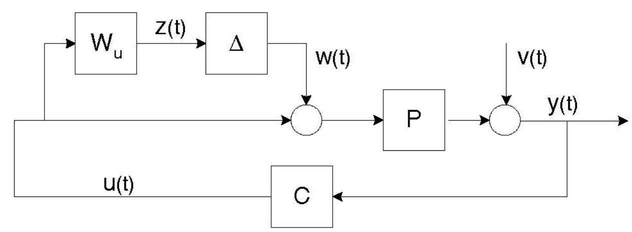

Fig. 2. System block diagram with input multiplicative uncertainty.

In order to analyze performance robustness of the feedback system in Figure 2 one can construct a generalized plant

$$\begin{pmatrix} z \\ y \end{pmatrix} = M \begin{pmatrix} w \\ v \end{pmatrix}$$

where

$$M = \begin{bmatrix} W_u CP/(1+CP) & W_u C/(1+PC) \\ P/(1+CP) & 1/(1+CP) \end{bmatrix},$$

and verify stability robustness with respect to the structured uncertainty

$$\Delta = \mathrm{diag}[\delta_1, \delta_2], \qquad \|\delta_1\|_\infty \leq 1, \quad \|\delta_2\|_\infty \leq \rho^{-1}.$$

according to the main-loop theorem [13].

We look for the minimum of ρ in $\Omega = [10^{-1}, 10^3]$ Hz for which M is robustly stable using four different methods. The results are presented in Figure 3. The four methods are: a) minimizing ρ using the Bounded Real Lemma (solid-thin line), b) solving the Generalized KYP Lemma [7] on Ω (dashed line), c) solving the pair of LMIs in Theorem 1 (dotted line), and d) solving problem (7) on a dense frequency grid with 200 logarithmically spaced frequencies (thick line). Note that for this example the uncertainty structure is such that $\mu_{\mathbf{\Delta}} = \rho_{\mathbf{\Delta}}$ at each frequency ω.

The extra freedom provided by the frequency-dependent multipliers in (13) and (14) allow for a much less conservative upper bound $\rho_{\mathbf{\Delta}}$ when compared to the BRL and the Generalized KYP Lemmas.

We now repeat the process by further subdividing $\Omega = \cup_{i=1}^{3} \Omega_i$ into three frequency ranges $\Omega_1 = [10^{-1}, 2]$, $\Omega_2 = [2, 50]$, $\Omega_3 = [50, 10^3]$. We solved the problem on each range using the Generalized KYP Lemma [7] and by solving

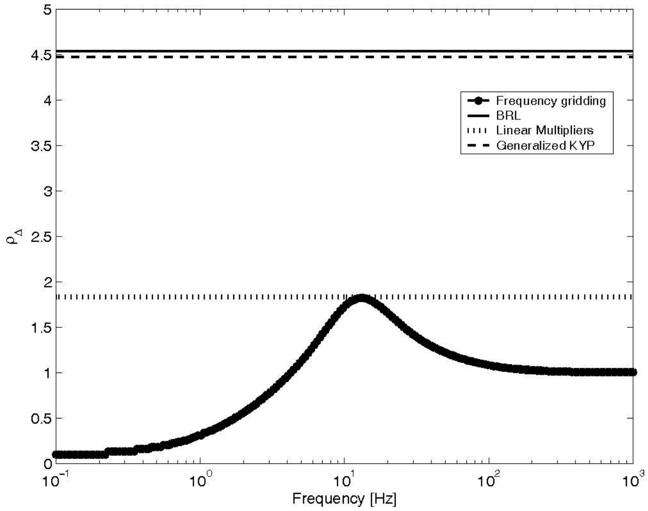

Fig. 3. Robust analysis upper bounds ρ_Δ computed via frequency grid $\omega \in [10^{-1}, 10^3]$ (thick solid line), BRL $\forall\ \omega \in \mathbb{R}$ (thin solid line), Generalized KYP lemma over $\omega \in [10^{-1}, 10^3]$ (dashed line), and Theorem 1 over $\omega \in [10^{-1}, 10^3]$ (dotted line).

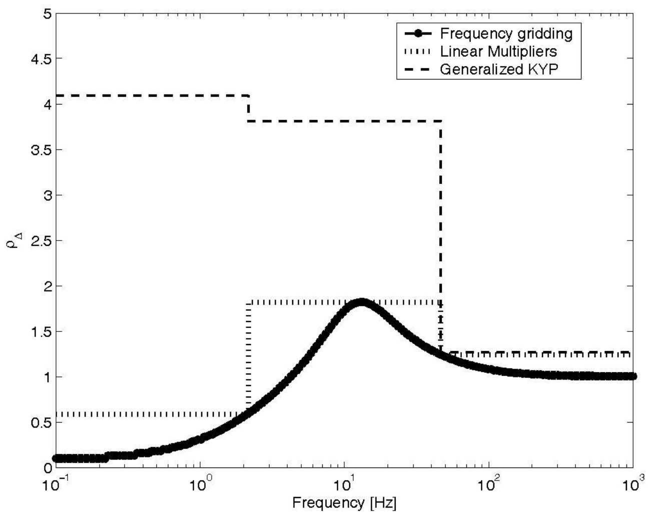

Fig. 4. Robust analysis upper bounds ρ_Δ computed via frequency grid $\omega \in [10^{-1}, 10^3]$ (thick solid line), Generalized KYP lemma over $\omega \in [10^{-1}, 2], [2, 50], [50, 10^3]$ (dashed line) and Theorem 1 over $\omega \in [10^{-1}, 2], [2, 50], [50, 10^3]$ (dotted line).

the pair of LMIs in Theorem 1. The results are shown in Figure 4.

Note that for frequency ranges beginning above $\omega_1 > 10Hz$ the generalized KYP lemma provides an upper bound similar to the method of Theorem 1, however for $\omega_1 < 10Hz$ the generalized KYP lemma provides a conservative upper bound. Smaller frequency intervals reduce the overall amount of conservatism for both methods. For this example, Theorem 1 requires two frequency intervals to reach the bound ρ_Δ determined by frequency gridding.

VII. Conclusions

A computationally efficient method for robustness analysis over a finite frequency interval has been proposed in this paper. The methodology utilizes frequency dependent multiplier relaxations and result into a pair of Linear Matrix Inequalities that can be solved numerically in polynomial time. The amount of conservatism in using frequency dependent multipliers parametrized linearly on ω for providing an upper bound for μ is reduced as the size of the frequency intervals are decreased. This effect is clearly demonstrated by a numerical example in this paper and the results are expected to be useful for analyzing robust stability and performance.

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