Proc. 43rd IEEE Conference on Decision and Control,
pp. 2182-2187, Paradise Island, Bahamas, 2004

Time Domain Control Oriented Model Validation using Coprime Factor Perturbations

Marianne Crowder and Raymond de Callafon

***Abstract*— This paper addresses the time domain model validation problem for uncertainty models that are structured using coprime factorizations. A model validation technique is proposed in which measurement data is used to validate a derived uncertainty model. Newly proposed model validation techniques are based on a fractional representation approach and addresses the problem of correlation between the input and output signals inherent to closed-loop systems. The model validation problem for coprime factorizations is considered for closed-loop time domain data in the cases of noise-free and noisy measurements where the results rely on a linear relationship between input and output data.**

I. INTRODUCTION

Closed-loop model validation is a critical procedure to establish whether or not a model can reliably predict the output of a system measured under feedback controlled conditions. In previously developed model validation techniques [14], [4], models can be parametrized in a Linear Fractional Transformation (LFT) uncertainty set to account for modeling uncertainties. In [14] and [3] it was shown that model validation tests have a low level of computational complexity by formulating the model validation problem as a convex optimization. However, application of these time-domain based techniques to data obtained under closed-loop conditions is challenging for model validation purposes due to the presence of correlate noise on closed-loop signals and the implicit modeling uncertainty in the closed-loop transfer function.

In this paper a fractional representation approach is presented to address the model invalidation problem for measurements obtained under feedback controlled conditions. This approach allows the formulation of a unified method to estimate models for stable, marginally stable or unstable systems via the estimation of stable coprime factorizations on the basis of closed-loop data. The work on fractional model identification was initiated by [9] and further developed in the work by [11] [7] and [13]. The fractional approach forms an excellent framework to address the identification of systems on the basis of closed-loop data [1] and control oriented model validation [8].

A model validation problem using open-loop based frequency-response data in a coprime factor framework was presented in [2]. The closed-loop frequency-response data using both noisy and noise-free conditions was presented in [5] with the application to a flexible structure. The results of [5] show that coprime factorizations in the uncertainty model depend on the knowledge of a stabilizing feedback controller to facilitate the closed-loop (in)validation of the uncertainty model. Time domain model validation results for a variety of uncertainty models was presented in [14] where it was shown that the solutions are given in terms of a convex matrix optimization. By following the results given in [14], this paper develops alternative model invalidation results for uncertainty models characterized with coprime factorizations.

II. PROBLEM FORMULATION

A. Motivation for Closed-loop Model Validation

Validating models for control design purposes inherently requires closed-loop model validation techniques and closed-loop data. Closed-loop model validation is often preferred since open-loop model validation may invalidate a model that might be well-suited for control design purposes.

With a simple example it is easy to illustrate the benefits of a closed-loop model validation technique over that of an open-loop technique. Suppose a real plant is an integrator such that $P(s) = \frac{1}{s}$ and the model is described as $\hat{P}(s) = \frac{1}{s+\epsilon}$ where $\epsilon \geq 0$. Computing the additive and multiplicative uncertainty descriptions $\bar{\Delta}_a$ and $\bar{\Delta}_m$, respectively yields

$$\bar{\Delta}_a(s) := P(s) - \hat{P}(s) = \frac{\epsilon}{s(s+\epsilon)} \tag{1}$$

$$\bar{\Delta}_m(s) := \frac{P(s) - \hat{P}(s)}{\hat{P}(s)} = \frac{\epsilon}{s} \tag{2}$$

With $\epsilon > 0$ it is obvious that an ∞-norm for the additive and multiplicative uncertainties given in (1) and (2) will be unbounded and not suitable for open-loop model validation techniques. To overcome this problem, a closed-loop oriented uncertainty should be used to describe perturbations of a model for control design purposes.

B. Use of Fractional Models

For model validation purposes, the possible set of models is denoted by an uncertainty set $\mathcal{P}$ and is characterized by a fractional approach. A fractional based uncertainty set $\mathcal{P}$ in this paper is structured as follows:

$$\begin{array}{c}\mathcal{P} = \{P \mid P = ND^{-1} \text{ with} \\ N := \hat{N} + D_c\bar{\Delta},\ D = \hat{D} - N_c\bar{\Delta} \\ \text{and } \bar{\Delta} := V\Delta\}\end{array} \tag{3}$$

This research is supported by NASA GSRP, NGT4-52429

M. Crowder is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093, USA mcrowder@ucsd.edu

R. de Callafon is with Faculty of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093, USA callafon@ucsd.edu

where Δ is defined as the (unknown but bounded) perturbation

$$\mathbf{\Delta} = \{\Delta | \Delta \in \mathbb{R}H_\infty \text{ and } \|\Delta\|_\infty < 1\} \quad (4)$$

and where $(\hat{N}, \hat{D})$ and (N_c, D_c) respectively denote a right coprime factorization (*rcf*) of the nominal model $\hat{P}$ and the controller C that is stabilizes the nominal model $\hat{P}$. The weighting function V in (3) is used to normalize the unknown but bounded perturbation. The models $P \in \mathcal{P}$ are

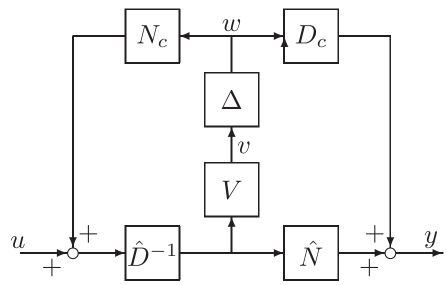

Fig. 1. Coprime factor based uncertainty model $\mathcal{P}$

visualized in the block diagram of Figure 1. Note that the uncertainty model given in (3) is different from standard additive coprime factor perturbations as used in [2].

In the uncertainty model given in (3), the perturbation $\bar{\Delta}$ is used to model a combined perturbation on the *rcf*$(\hat{N}, \hat{D})$ of the model $\hat{P}$ and the *rcf*(N_c, D_c) of the controller plays an important role in assigning the common perturbations in the *rcf*$(\hat{N}, \hat{D})$. Introducing a combined perturbation Δ on the *rcf*$(\hat{N}, \hat{D})$ of the model $\hat{P}$ in the uncertainty model $\mathcal{P}$, establishes a link with the Youla-Kucera parameterization [1] that facilitates closed-loop model validation of the uncertainty model. The Youla-Kucera parameterization is applicable to all models $P \in \mathcal{P}$ in (3), provided the nominal model $\hat{P}$ and the controller C form a stable feedback connection.

An alternative presentation of $\mathcal{P}$ in (3) can be given in terms of an LFT

$$\mathcal{P} = \{P \mid P = \mathcal{F}_u(Q, \Delta), \, \Delta \in \mathbf{\Delta}\} \quad (5)$$

where

$$\mathcal{F}_u(Q, \Delta) := Q_{22} + Q_{21}\Delta(I - Q_{11}\Delta)^{-1}Q_{12} \quad (6)$$

and where $\mathbf{\Delta}$ is defined in (4). The entries of the coefficient matrix Q in (5) dictate the way in which the set of models $\mathcal{P}$ is being structured and are given by

$$\begin{array}{rclrcl} Q_{11} & = & V\hat{D}^{-1}N_c & Q_{12} & = & V\hat{D}^{-1} \\ Q_{21} & = & D_c + \hat{N}\hat{D}^{-1}N_c & Q_{22} & = & \hat{N}\hat{D}^{-1} \end{array} \quad (7)$$

Although complicated at first glance, the uncertainty model $\mathcal{P}$ can be simplified if either the model $\hat{P}$ or the controller C is stable. In that case, the uncertainty set $\mathcal{P}$ illustrates the closed-loop oriented character of the allowable perturbation $\bar{\Delta}$.

As an example, consider $\hat{P}$ and C to be stable transfer functions with a *rcf* $\hat{N} = \hat{P}$, $\hat{D} = I$, $N_c = C$, $D_c = I$. This yields an uncertainty set $\mathcal{P}$ where for each model $P \in \mathcal{P}$ it can be verified that

$$\bar{\Delta} = (I + \hat{P}C)^{-1}(P - \hat{P}) \quad (8)$$

and illustrates that $\bar{\Delta}$ is a perturbation on a closed-loop transfer function. For an arbitrary *rcf*$(\hat{N},\ \hat{D})$ of $\hat{P}$ and *rcf*$(N_c,\ D_c)$ of C it can be shown that

$$\bar{\Delta} = D_c^{-1}(I + \hat{P}C)^{-1}(P - \hat{P})\hat{D} \quad (9)$$

which illustrates that the coprime factor uncertainty is a weighted closed-loop uncertainty, where the sensitivity function $(I + \hat{P}C)^{-1}$ plays an important role.

Reconsider the example described earlier where a PI controller $C(s) = K_p + K_i\frac{1}{s}$ is used for control a plant $P(s) = \frac{1}{s}$. With a nominal model $\hat{P} = \frac{1}{s+\varepsilon}$, the coprime factor uncertainty description is given by

$$\bar{\Delta}_{cf} = \frac{\epsilon}{s^2 + (K_p + \epsilon)s + K_i}. \quad (10)$$

As shown in (1) and (2), the additive and multiplicative uncertainty become unbounded when $\epsilon > 0$. However, $\bar{\Delta}_{cf}$ in (10) is stable and bounded for the case $K_p > -\epsilon$, $K_i > 0$, which is the condition for stability of the feedback connection of $\hat{P}$ and C. This simple example shows the clear merits of a coprime factor based uncertainty description over that of an additive or multiplicative uncertainty description.

C. Closed-Loop Model Validation Problem

For dealing with closed-loop data, consider a feedback connection of a system, denoted by P_o, and a feedback controller C, with $y = P_o u + d$, $u = r - Cy$, and where d is an additive noise on the output y. The signal r denotes an external reference signal that provides sufficient excitation of the closed-loop system and the signal d denotes an additive colored noise that will be present on bot the input u and output y signal. With this information, the input-output data $\{u, y\}$ of the closed-loop controlled system P_o can be described by

$$\left[\begin{array}{c} y \\ u \end{array}\right] = \left[\begin{array}{c} P_o \\ I \end{array}\right](I + CP_o)^{-1}r + \left[\begin{array}{c} I \\ -C \end{array}\right](I + P_oC)^{-1}d \quad (11)$$

where the additive noise d is assumed to be uncorrelated with the external reference signal r.

For closed-loop model validation purposes, the reference signal r is considered as an input signal. The signal u and/or y can be considered as a measurable closed-loop output signal. When only the output y is considered for closed-loop model purposes, the map S from the reference signal r to the output y for all models $P \in \mathcal{P}$ in (3) can be written as another LFT:

$$\mathcal{S} = \{S \mid S = \mathcal{F}_u(M, \Delta), \, \Delta \in \mathbf{\Delta}\ \} \quad (12)$$

with M given by

$$\begin{array}{rclrcl} M_{11} & = & 0 & M_{12} & = & V(\hat{D}+C\hat{N})^{-1} \\ M_{21} & = & D_c & M_{22} & = & \hat{P}(I+C\hat{P})^{-1} \end{array} \quad (13)$$

and where $\boldsymbol{\Delta}$ is defined in (4). The entries of M in (12) are all known quantities and determined by the coprime factor uncertainty set $\mathcal{P}$ in (3). It can be verified that all entries of M are stable *if and only if* the controller C internally stabilize the nominal model $\hat{P}$, as required by the construction of the uncertainty model $\mathcal{P}$ in (3).

With the LFT $\mathcal{F}_u(M,\Delta)$ given in (12) and (13) it is easy to see the benefits of the coprime factor based uncertainty model $\mathcal{P}$ in (3) for closed-loop model validation purposes. The closed-loop map from r to y is simplified as $\mathcal{F}_u(M,\Delta) = M_{22} + M_{21}\Delta M_{12}$ where the uncertainty Δ now appears linearly. The affine representation of the coprime factor perturbation Δ in the closed-loop map $\mathcal{F}_u(M,\Delta)$ can be exploited to formulate a time domain model validation techniques that rely on linearity of the uncertainty in the input-output map.

III. NOTATION AND MAIN RESULTS

Following the results given in [15], notation regarding discrete time domain model validation techniques are now described. These results are used to establish the framework for the closed-loop model validation discussed in this paper.

For a sequence of vectors $k = (k_1, k_2, \cdots, k_n \in \mathbf{R}^m)$, let $T_k \in \mathbf{R}^{mn\times n}$ denote the associated Toeplitz matrix defined as

$$T_k = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ k_2 & k_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ k_n & k_{n-1} & \cdots & k_1 \end{bmatrix}. \quad (14)$$

Further, let $\mathcal{S}^m$ denote the set of one-sided sequences with elements in $\mathbf{R}^m$ and define the n-step filter operator $\pi_n : \mathcal{S}^m \to \mathcal{S}^m$ such that

$$(\cdots, k_0, k_1, \cdots, k_n) \to (0, \cdots, 0, k_1, \cdots, k_n). \quad (15)$$

Let Δ be a stable, casual, time-invariant system with transfer matrix

$$\Delta(z) = h_1 + h_2 z^{-1} + h_3 z^{-2} + \cdots = \sum_{\ell=1}^{\infty} h_\ell z^{1-\ell} \quad (16)$$

where h_i, $i = 1, 2, \cdots$ are the matrix Markov parameters of the the transfer function $\Delta(z)$. Suppose the input sequence $a = (a_1, a_2, \ldots, a_n)$ is applied to a system and the output $b = (b_1, b_2, \ldots, b_n)$ is collected for the period $t = 1, 2, \cdots, n$. The input and output sequences are then related by a Toeplitz matrix such that the following holds:

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ h_2 & h_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ h_n & h_{n-1} & \cdots & h_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \quad (17)$$

This equation shows that the inputs and outputs uniquely determine the first n Markov parameters of the transfer function $\Delta(z)$. The existence of such a Δ, that is stable and satisfies $\|\Delta\|_\infty < \infty$, is the classical Carathédory-Fejér interpolation problem [15].

It can be observed from Figure 1 that v and w are the input-output signals of the coprime factor uncertainty Δ. For model validation, the problem is considered where a portion of this input-output data is used to determine the minimum norm causal operator Δ that could have produced the portion of data. In case the input-output signal v and w are available, the model validation problem could be summarized as follows.

Problem 1: Model Validation Problem
Given the signals $v = (v_1, v_2, \cdots, v_n \in \mathbf{R}^m)$ and $w = (w_1, w_2, \cdots, w_n \in \mathbf{R}^p)$ shown in Figure 1, the uncertainty model given in (3) is not invalidated by the data (v, w) if there exist a stable causal operator Δ with $\|\Delta\|_\infty \leq 1$ such that

$$(w_1, w_2, \cdots, w_n) = \Delta(v_1, v_2, \cdots, v_n). \quad (18)$$

Note that the inputs v_i and outputs w_i are allowed to be vectors. Problem 1 determines whether there exists an operator Δ such that the output of Δ for the period of $t = (1, 2, \cdots, n)$ is exactly $w = (w_1, w_2, \cdots, w_n)$ when the input of Δ is $v = (v_1, v_2, \cdots, v_n)$. If there does exist such a Δ, then the uncertainty model is not invalidated. In light of the problem formulation given in Problem 1, two key items need to be addressed in order to solve the model invalidation problem. First, it remains to determine how to access the signals v and w as measurements of u and y are the only signals available from the closed-loop experiments. Second, once the signals v and w have been established, a method must be developed to check the existence of a Δ with $\|\Delta\|_\infty < 1$ and is found from the Extension theorem in [14].

Theorem 1: Extension Theorem [14]
Given input sequences $\bar{v} = (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n \in \mathbf{R}^m)$ with $\bar{v} = \pi_n v$ and output sequences $\bar{w} = (\bar{w}_1, \bar{w}_2, \cdots, \bar{w}_n \in \mathbf{R}^p)$ with $\bar{w} = \Delta\bar{v}$. Then Δ is a stable, causal, linear, time-invariant operator with $\|\Delta\|_\infty < 1$ if and only if

$$T_{\bar{w}}^T T_{\bar{w}} \leq T_{\bar{v}}^T T_{\bar{v}} \quad (19)$$

where $T_{\bar{v}}$ and $T_{\bar{w}}$ are the associated Toeplitz matrices formed from $\bar{v}$ and $\bar{w}$, respectively.

Proof: See [14] ■

Corollary 1: Given the sequences $\bar{v}$ and $\bar{w}$, (19) is equivalent to

$$\bar{\sigma}(T_{\bar{w}}(T_{\bar{v}}^T T_{\bar{v}})^{-1/2}) \leq 1. \quad (20)$$

Proof: With $\bar{v}_1 \neq 0$, the matrix $T_{\bar{v}}$ has full column rank. It follows that $T_{\bar{v}}^T T_{\bar{v}} > 0$ which also implies $(T_{\bar{v}}^T T_{\bar{v}})^{-1/2} > 0$. Pre-multiplying and post-multiplying $T_{\bar{v}}^T T_{\bar{v}} - T_{\bar{w}}^T T_{\bar{w}} \geq 0$ by $(T_{\bar{v}}^T T_{\bar{v}})^{-1/2}$ results in $(T_{\bar{v}}^T T_{\bar{v}})^{-1/2}(T_{\bar{v}}^T T_{\bar{v}} - T_{\bar{w}}^T T_{\bar{w}})(T_{\bar{v}}^T T_{\bar{v}})^{-1/2} \geq 0$. Substituting $G = T_{\bar{w}}(T_{\bar{v}}^T T_{\bar{v}})^{-1/2}$ reduces the result to $1 - G^T G \geq 0$.

Since $\lambda(G^TG) = \sigma(G^2)$, it follows that $1 \geq \bar{\sigma}(G)$ which is equivalent to $1 \geq \bar{\sigma}(T_{\bar{w}}(T_{\bar{v}}^T T_{\bar{v}})^{-1/2})$. ■

The results in Theorem 1 and Corollary 1 provide tools to check the existence of Δ with $\|\Delta\|_\infty < 1$. However, the results only hold when the initial input signal is set to zero. When model validation must be done in real-time or the basis of short batches of data sequences, the effect of initial conditions must be taken into account. The effect of initial conditions on the output signal w can be expressed as

$$\rho = \sum_{k=t}^{\infty} \rho_k v(t-k) \tag{21}$$

where ρ is based on previous input measurements v. For the general case of determining the minimum norm casual operator Δ that could have produced a set of data at an arbitrary time after the beginning of the data measurements for the model validation, the following result can be used.

Theorem 2: Given the input measurements $v = (v_1, v_2, \cdots, v_n \in \mathbf{R}^m)$ and the output measurements $w = (w_1, w_2, \cdots, w_n \in \mathbf{R}^p)$. There exists a stable, causal, linear, time-invariant operator Δ with $\|\Delta\|_\infty < 1$ and a ρ defined in (21) such that

$$w - \rho = \Delta \pi_n v \tag{22}$$

if and only if there exists $\rho = (\rho_1, \rho_2, \cdots, \rho_n \in \mathbf{R}^p)$ with

$$\bar{\sigma}[(T_w - T_\rho)(T_v^T T_v)^{-1/2}] \leq 1 \tag{23}$$

where T_w, T_v, and T_ρ are formed from the Toeplitz matrices of w, v and ρ.

Proof: Substituting $\bar{w} = w - \rho$ and $\bar{v} = \pi_n v$ into the proof of Theorem 1 establishes the result. ■

The convex problem shown in Theorem 2 is possible because the sequence w is a linear combination of two signals such that $w = \bar{w} + \rho$.

As described earlier, the signal $\bar{w}$ is the response due to an input signal $\bar{v}$ with $v(t) = 0 \; \forall \; t < 1$ and ρ captures the mismatch due to initial conditions. To determine the effect of initial conditions, the result given in Theorem 2 requires the additional monitoring of past input samples of the signal v. However, the effect of the initial conditions reflected in ρ can be solved by a standard convex optimization.

Corollary 2: Given the Toeplitz matrices T_v, T_w, and unknown initial condition Toeplitz matrix T_ρ, the model validation test given in (23) can be written as the following linear matrix inequality (LMI):

$$\min_{\rho} \alpha, \text{ such that } \alpha \leq 1 \text{ and}$$
$$\begin{bmatrix} \alpha I & (T_w - T_\rho(T_v^T T_v)^{-1/2})^T \\ T_w - T_\rho(T_v^T T_v)^{-1/2} & I \end{bmatrix} \geq 0$$

Proof: The equation given in (23) can be rewritten as $\bar{\sigma}(Y) \leq \alpha I$ where $\alpha \leq 1$ and $Y = (T_w - T_\rho)(T_v^T T_v)^{-1/2}$. The inequality $\bar{\sigma}(Y) \leq \alpha I \Leftrightarrow 0 \leq \alpha I - YY^T$. Using the schur complement of $\alpha I - YY^T$ reduces the result to the following LMI:

$$\min \alpha \leq 1 \text{ such that}$$
$$\begin{bmatrix} \alpha I & Y^T \\ Y & I \end{bmatrix} \geq 0$$

Substituting $Y = T_w - T_\rho(T_v^T T_v)^{-1/2}$ and minimizing over the matrix T_ρ establishes the result. ■

The result given in Theorem 2 extends the results shown in [14] to the more general model validation case when the initial conditions are not zero. What remains to be done is the computation of v and w from closed-loop data $\{u, y\}$, which will be addressed in the next section.

IV. CLOSED-LOOP MODEL VALIDATION RESULTS

Consider the input signal $u = (u_1, u_2, \cdots, u_n; u_i \in \mathbf{R}^m)$ applied to the physical system P_0 where the output measurement signal $y = (y_1, y_2, \cdots, y_n; y_i \in \mathbf{R}^p)$ is observed. Using the signals (u, y) available from the closed-loop experiments it is possible to describe the auxiliary signals (v, w) as filtered versions of the signals (u, y). Without loss of generality, it is assumed that the input-output experiment is conducted immediately such that $u_1 \neq 0$. Following the results shown in [6], consider the following.

Lemma 1: Consider the uncertainty model given in (3) where the auxiliary signals v and w are described by

$$v = V(\hat{D} + C\hat{N})^{-1}[C \quad I] \begin{bmatrix} y \\ u \end{bmatrix} \tag{24}$$

and

$$w = (D_c + \hat{P}N_c)^{-1}[I \quad -\hat{P}] \begin{bmatrix} y \\ u \end{bmatrix} \tag{25}$$

then the closed-loop map in (11) can be rewritten as

$$w = \Delta v + \delta \tag{26}$$

where the signal δ is given by

$$\delta = D_c^{-1}(I + P_0 C)^{-1} d \tag{27}$$

and where v is uncorrelated with d.

Proof: See [6]. ■

The signals (v, w) can be considered as an input and a (possibly) disturbed output signal of the uncertainty Δ where the input v is uncorrelated with the disturbance acting on the signal w [6]. It can be noted that v is not perturbed by the additive noise d present in the closed-loop data. This is due to the fact that

$$[C \quad I] \begin{bmatrix} y \\ u \end{bmatrix} = r \tag{28}$$

and thus v is a function of r only. On the other hand, the signal w is perturbed by the additive noise d as indicated in (26) where the noise on w is characterized by the filtered noise δ in (27). For model validation purpose it is assumed that knowledge of a bound γ with $|\delta(t)| < \gamma \; \forall \; t$ is known.

By conducting an experiment on the closed-loop system, it is possible to characterize γ as a bound on the noise disturbance. When no reference signal is applied to the closed-loop system and the output signal is measured, the quantity $(1 + P_0C)^{-1}d$ is observed. Further, by filtering this observed signal with D_c^{-1}, the signal δ in (27) can be determined. For the general case of noisy measurements, consider the following result.

Lemma 2: Consider the uncertainty model given in (3), the signals v and w defined in (24) and (25) with $w = (w_1, w_2, \cdots, w_n;\ w_i \in \mathbf{R}^p)$, $v = (v_1, v_2, \cdots, v_n;\ v_i \in \mathbf{R}^m)$, the effect of initial conditions described in (21), and an upper bound for the filtered noise $\delta \in \mathbf{D}_\gamma$ with

$$\mathbf{D}_\gamma = \{\delta(t) \mid |\delta(t)| < \gamma\ \forall\ t\}. \tag{29}$$

Then the coprime factor uncertainty model given in (3) is not invalidated by the closed-loop data (u, y) if and only if the following convex problem is solvable:

Does there exist $q = (q_1, q_2, \cdots, q_n \in \pi_n\mathbf{D}_\gamma,\ q_i \in \mathbf{R}^p)$ and a $\rho = (\rho_1, \rho_2, \cdots, \rho_n \in \mathbf{R}^p)$ defined in (21) such that

$$\bar{\sigma}[(T_w - T_q - T_\rho)(T_v^T T_v)^{-1/2}] \leq 1 \tag{30}$$

where T_w, T_v, T_q, and T_ρ are formed from the Toeplitz matrices of w, v, q and ρ.

Proof: It can be observed that the uncertainty model in (3) is not invalidated if and only if there exists a $\Delta \in \mathbf{\Delta}$ with $\|\Delta\|_\infty \leq 1$ such that

$$w - q - \rho = \Delta\pi_n v$$

for some $q \in \pi_n\mathbf{D}_\gamma$ and some $\rho \in \mathbf{R}^p$. Invoking Theorem 1 establishes the result. ■

Similar to the noise-free case shown in Corollary 2, the convex optimization in (30) can be written as an LMI problem. The result has been summarized in the following.

Corollary 3: Given the Toeplitz matrices T_v, T_w, unknown initial condition Toeplitz matrix T_ρ, and unknown disturbance Toeplitz matrix T_q, the model validation test given in (30) can be written as the following linear matrix inequality (LMI):

$$min_{T_\rho, T_q}\alpha, \text{ such that } \alpha \leq 1 \text{ and}$$
$$\begin{bmatrix} \alpha I & Y^T \\ Y & I \end{bmatrix} \geq 0$$

where $Y = (T_w - T_q - T_\rho)(T_v^T T_v)^{-1/2}$.

Proof: The equation given in (30) can be rewritten as $\bar{\sigma}(Y) \leq \alpha I$ where $\alpha \leq 1$ and $Y = (T_w - T_q - T_\rho)(T_v^T T_v)^{-1/2}$. The inequality $\bar{\sigma}(Y) \leq \alpha I \Leftrightarrow 0 \leq \alpha I - YY^T$. Using the schur complement of $\alpha I - YY^T$ reduces the result to the following LMI:

minimize $\alpha \leq 1$ such that

$$\begin{bmatrix} \alpha I & Y^T \\ Y & I \end{bmatrix} \geq 0$$

Substituting $Y = (T_w - T_q - T_\rho)(T_v^T T_v)^{-1/2}$ and minimizing over the matrices T_ρ,T_q establishes the result. ■

The convex optimization results for the closed-loop model validation problem given in Lemma 2 are obtained due the fact that the coprime factor uncertainty Δ is affine in the closed-loop input-output map (r, y). Note that the Δ is not affine in the open-loop input-output map (u, y), but becomes affine when the controller C is applied to the coprime factor uncertainty model $\mathcal{P}$ in (3). Once the controller is applied to the system, the same input output data (u, y) can be used to perform model validation in closed-loop.

V. ILLUSTRATION OF MODEL VALIDATION TECHNIQUES

As described earlier, closed-loop model validation techniques are required to validate models used for control design purposes. The fractional approach discussed in this paper provides a method to deal with closed-loop experiments and takes into account the closed-loop data and the closed-loop model. To further illustrate the coprime factor model validation techniques presented in this paper, consider again the example described in Section II-A where $P = \frac{1}{s}$, $\hat{P} = \frac{1}{s+\epsilon}$, and $C = K_p + K_i\frac{1}{s}$.

Using knowledge of the controller C and the model $\hat{P}$, it is possible to describe the coprime factor uncertainty Δ_{cf} as developed in (10). For comparison to the open-loop uncertainty descriptions computed in (2) and (1), Figure 2 shows the uncertainty description for the coprime factor, multiplicative, and additive uncertainty descriptions. As

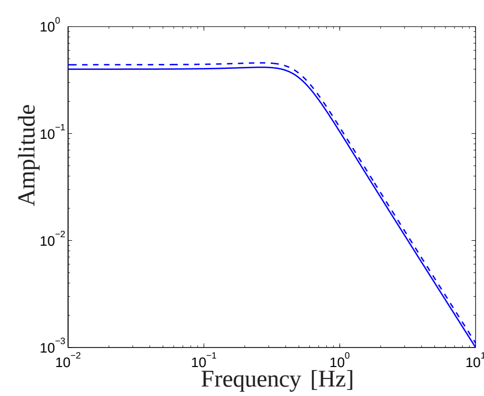

Fig. 2. Bounded coprime factor description Δ_{cf} (solid) and normalizing over bound V (dotted)

seen in Figure 2, the coprime factor uncertainty description Δ_{cf} is the only bounded uncertainty description for the closed-loop system. Since the open-loop uncertainty descriptions Δ_m and Δ_a are unbounded on the domain, only the closed-loop uncertainty description is suitable for closed-loop model validation.

For illustration purposes, we consider a white noise unit variance reference signal r. When the reference signal r is applied to the closed-loop system (P, C), the measurements of the closed-loop input/output signals (u, y) depicted in are available for model validation purposes. With the signals (u, y) and the uncertainty description Δ_{cf} overbounded by the weighting function V it remains to establish the filtered signals w and v.

Since both the closed-loop model $\hat{P}$ and the controller C are stable transfer functions, $(\hat{P}, C)$ can be described by a trivial choice of the *rcf*: $\hat{N} = \hat{P}$, $\hat{D} = I$, $N_c = C$, $D_c = I$. Using this *rcf*, the auxiliary signals v and w described by (24) and (25) can be obtained via filtering of the measured closed-loop input/output signals (u, y).

Following the procedure outlined in this paper, the Toeplitz matrices $T_{\bar{w}}$ and $T_{\bar{v}}$ are constructed for

$$\bar{w}(t),\ \bar{v}(t),\ t = n, n+1, \ldots, n+N$$

where $N = 200$ and $n = 1, 2, \ldots, 300$ indicates the starting index of the data for moel validation purposes. Applying the model validation result of Corollary 1 to the matrices $T_{\bar{w}}$ and $T_{\bar{v}}$ will show that the coprime factor based uncertainty model is not invalidated by the closed-loop input/output data (u, y), provided $\bar{\sigma}(T_{\bar{w}}(T_{\bar{v}}^T T_{\bar{v}})^{-1/2}) \leq 1$. As mentioned in Section III, the effect of initial conditions must be considered when performing model validation on a set of data obtained at a time later then the start of the experiment. The results shown in Figure 3 show the model validation results using data after the start of the experiment. As seen

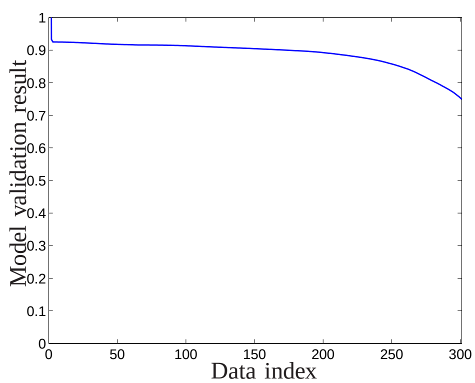

Fig. 3. Model validation result for different starting indices n of validation data

in Figure 3, the model validation result holds using data obtained from various starting points of the data collection.

This example demonstrates the simplicity of the model validation technique on the basis of closed-loop time domain data. Essential is the use of the coprime factor based uncertainty model, where perturbations on coprime factors are specified as (weighted) perturbations that involve knowledge of the feedback controller used during the closed-loop experiments. As a result, the coprime factor uncertainty appears linearly in the closed-loop input/output map, allowing for a straightforward application of a convex optimization routine to address the model validation problem in the time domain.

VI. CONCLUSIONS

In this paper, the model validation problem of a fractional representation has been studied with application to time domain data. It has been found that the LFT approach greatly facilitates manipulation and computation of linear systems and that the use of closed-loop data in using LFT's allow the formulation of affine closed-loop expressions for closed-loop model validation.

The model validation problem presented in this paper determines whether the uncertainty model is capable of reproducing data. Although prior knowledge of the system behavior and knowing how the model relates to observed data are important modeling considerations, as pertinent a factor in model validation is the appropriateness of the uncertainty model. As presented in this paper, uncertainty modeling using coprime factorizations allows one to perform time domain model validation techniques that reduce to a convex feasibility problem. The convex feasibility problems rely on the model information and the observed closed-loop data

REFERENCES

[1] B. Anderson, "From youla–kucera to identification, adaptive and non–linear control," *Automatica*, vol. 34, pp. 1485–1506, 1998.

[2] B. Boulet and B. Francis, "Consistency of open-loop experimental frequency-response data with coprime factor for plant models," *IEEE Transactions on Automatic Control*, pp. 1680–1691, 1998.

[3] J. Chen, "Frequency-domain tests for validation of linear fractional uncertain models," *IEEE Transation on Automatic Control*, vol. 42, pp. 748–760, 1997.

[4] J. Chen and S. Wang, "Validation of linear fractional uncertain models: Solutions via matrix inequalities," *IEEE Transaction on Automatic Control*, vol. 41, pp. 844–849, 1996.

[5] M. Crowder and R. de Callafon, "Coprime factor based closed-loop model validation applied to a flexible strucutre," in *Conference on Decision and Control*, Mauui, HI, USA, 2003.

[6] R. de Callafon, "Feedback oriented identificatio for enhanced and robust control: a fractional approach applied to a wafer stage," Ph.D. dissertation, Delft University, The Netherlands, 1998.

[7] R. de Callafon and P. Van den Hof, "Suboptimal feedback control by a scheme of iterative identification and control design," *Mathematical Modelling of Systems*, vol. 3, no. 1, pp. 77–101, 1997.

[8] R. de Callafon and P. Van Den Hof, "Closed-loop model validation using coprime factor uncertainty models," in *Prepr. 12th IFAC Symposium on System Identification*, Santa Barbara, CA, USA, 2000.

[9] F. Hansen, G. Franklin, and R. Kosut, "Closed-loop identification via the fractional representation: Experiment design," in *Proc. American Control Conference*, Pittsburgh, USA, 1989, pp. 1422–1427.

[10] A. Kumar and G. Balas, "An approach to model validation in the μ framework," in *Proc. American Control Conference*, Baltimore, MY, USA, 1994, pp. 3021–3026.

[11] W. Lee, B. Anderson, R. Kosut, and I. Mareels, "A new approach to adaptive robust control," *International Journal of Adaptive Control and Signal Processing*, vol. 7, no. 3, pp. 183–211, 1993.

[12] R. Lind and M. Brenner, *Robust Aeroservoelastic Stability Analysis*. London, UK: Springer Verlag, 1999.

[13] W. Lu, K. Zhou, and J. Doyle, "Stabilization of uncertain linear systems: An LFT approach," *IEEE Transactions on Automatic Control*, pp. 50–65, 1996.

[14] K. Poolla, P. P. Khargonekar, A. Tikku, J. Krause, and K. Nagpal, "A time domain approach to model validation," *IEEE Transaction on Automatic Control*, vol. 39, pp. 951–959, 1994.

[15] K. Zhou, *Essentials of Robust Control*. Upper Saddle River, NJ, USA: Prentice Hall, 1998.