COPRIME FACTOR PERTURBATION MODELS FOR CLOSED-LOOP MODEL VALIDATION TECHNIQUES

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Abstract: This paper addresses the problem of checking the consistency of experimental closed-loop frequency-domain data with uncertainty models that are structured using coprime factorizations. The uncertainty models presented in this paper use the knowledge of a stabilizing feedback controller to structure and formulate the uncertainty on a model. Subsequently, the controller dependent coprime factor uncertainty model can be used to formulate model (in)validation tests on the basis of closed-loop data. Closed-loop model validation results are developed for the cases of noise-free and noise perturbed closed-loop data. The model validation tests involve the computation of a structured singular value over a finite frequency grid. It is also shown that the computation of the structured singular value simplifies considerably when the feedback controller used for the closed-loop experiments is the same as the controller used for formulating the controller dependent coprime factor uncertainty model.

Keywords: identification; closed-loop; model validation; coprime factors

1. INTRODUCTION

Model validation is a critical procedure to establish whether or not a model can reliably predict the output of a system (Smith and Doyle 1992). In the last few years there has been much attention directed towards various techniques of performing uncertainty model validation. Specifically, the model validation of a general Linear Fractional Transformation (LFT) of discrete and continuous uncertain systems are studied in Poolla *et al.* (1994), Smith and Dullerud (1996) and Chen and Wang (1996). Model validation techniques using LFT's are applied to the frequency domain in Chen (1997) where the validation tests were illustrated to have a low level of computational complexity by formulating the model validation problem as a convex optimization.

In model (in)validation a distinction must be made between validating models on their open-loop and closed-loop behavior. A model suitable to predict and validate open-loop data may be different from a model that approximates the closed-loop behavior of a system. Closed-loop model validation techniques typically validates models on the basis of closed-loop data to verify the model for robust control applications. Control oriented or closed-loop model validation has been applied in Chen and Smith (1998) where it was observed that the convexity of the model validation problem can be preserved by using the knowledge of the controller.

In the line of development of closed-loop model validation techniques, a fractional representation approach is presented in this paper to address the control oriented identification and model validation problem. The fractional approach eliminates the effect of correlated noise on observed input and output signals that is unavoidable in feedback controlled systems. This approach allows enables a unified method to validate models for stable, marginally stable or unstable systems via the validation of stable coprime factorizations on the basis of closed-loop data. The work on fractional model identification was initiated by Hansen *et al.* (1989) and further developed in the work by Lee *et al.* (1993) de Callafon and Van den Hof (1997) and Lu *et al.* (1996). The fractional approach forms an excellent framework to address the identification of systems on the basis of closed-loop data (Anderson 1998) and control oriented model validation (de Callafon and Van den Hof 2000).

In this paper the problem of checking the consistency of experimental frequency-domain data is addressed with uncertainty models that are structured using coprime factorizations. Model validation techniques based on models formulated in a coprime factor framework have also been presented in Boulet and Francis (1998). The results of Boulet and Francis (1998) cover the noisy and noise-free conditions but are limited to open-loop frequency-response data that do not take into account the controller information. In this paper the coprime factorizations used in the uncertainty model depend on the knowledge of a stabilizing feedback controller to facilitate the closed-loop (in)validation of the uncertainty model.

The model validation tests presented in this paper involve the computation of a structured singular value $\mu(\cdot)$ over a finite frequency grid. Model validation techniques using (inverse) μ have also been studied in Lind and Brenner (1999) with applications towards aero-servoelastic systems. Model validation results using μ for SISO and MISO systems were also studied in Kumar and Balas (1994). Unfortunately, most of these results were applied to open-loop model validation and this paper extends these results to address the closed-loop model validation problem. It is also shown in this paper that the computation of the structured singular value for the model validation simplifies considerably when the knowledge of the feedback controller used for the closed-loop experiments is included in the coprime factor based uncertainty model.

2. PROBLEM FORMULATION

Given a nominal model $\hat{P}$ of a system P_0, an uncertainty structure Δ, and a set of input and output measurements (u, y) acting on the actual system P_0, the model (in)validation problem is to determine whether the measurements (u, y) could have been reproduced by the model with the uncertainty. The nominal model $\hat{P}$, along with the perturbation Δ will constitute an uncertainty model $\mathcal{P}$. Provided that $\mathcal{P}$ is constructed in such a way that $P_o \in \mathcal{P}$, such an uncertainty model $\mathcal{P}$ allows one to perform model (in)validation tests to verify the validity of $\mathcal{P}$. It is important to note that the model (in)validation test can be formulated as either an open-loop or closed-loop problem. The difference between open- and closed-loop data is not only determined by the data used for the model validation, but also depends on the way in which the uncertainty model $\mathcal{P}$ is structured. In this paper we consider a controller dependent coprime factor based uncertainty model that is presented in the following section.

2.1 *Use of Fractional Models*

An (upper) Linear Fractional Transformation (LFT)

$$\mathcal{F}_u(Q, \Delta) := Q_{22} + Q_{21}\Delta(I - Q_{11}\Delta)^{-1}Q_{12}$$

provides a general notation to represent all models $P \in \mathcal{P}$ as follows

$$\begin{array}{c}\mathcal{P} = \{P \mid P = \mathcal{F}_u(Q, \Delta) \\ \text{with } \Delta \in \mathbb{R}H_\infty \text{ and } \|\Delta\|_\infty < 1\}\end{array} \tag{1}$$

where Δ indicates an unknown (but bounded) uncertainty. The entries of the coefficient matrix Q in (1) dictate the way in which the uncertainty model $\mathcal{P}$ is being structured.

The uncertainty model $\mathcal{P}$ will be characterized by employing a fractional approach. A fractional based uncertainty model $\mathcal{P}$ is characterized by specifically using the knowledge of a controller C that stabilizes the nominal model $\hat{P}$. More specifically, the uncertainty model $\mathcal{P}$ proposed in this paper is structured as follows

$$\begin{array}{l}\mathcal{P} = \{P \mid P = ND^{-1} \text{ with} \\ N = \hat{N} + D_c\bar{\Delta}\,, D = \hat{D} - N_c\bar{\Delta} \text{ and} \\ \bar{\Delta} := V\Delta, \|\Delta\|_\infty < 1\}\end{array} \tag{2}$$

where (N_c, D_c) and $(\hat{N}, \hat{D})$ respectively denote a right coprime factorization (*rcf*) of the controller C and a nominal model $\hat{P}$. The weighting function V is used to normalize the unknown but bounded uncertainty.

The uncertainty model $\mathcal{P}$ in (2) is different from standard additive coprime factor perturbations as used in Boulet and Francis (1998). In the uncertainty model of (2), the perturbation $\bar{\Delta}$ is used to model a combined perturbation on the *rcf*(N, D) of the model P. It can be observed that $\hat{N}$ is perturbed by $\Delta_N = D_c\bar{\Delta}$ and $\hat{D}$ is perturbed by $\Delta_D = N_c\bar{\Delta}$ where the *rcf*(N_c, D_c) of the controller plays an important role in assigning the common perturbations in the *rcf*(N, D). From this representation, the coprime factors (N, D) can be expressed as

$$N = \hat{N} + \Delta_N \text{ and } D = \hat{D} - \Delta_D \tag{3}$$

where Δ_N and Δ_D are coupled and controller dependent additive perturbations on the coprime factorization $(\hat{N}, \hat{D})$ of the nominal model.

The reason to consider a combined perturbation Δ on *rcf*$(\hat{N}, \hat{D})$ of the nominal model $\hat{P}$ compared to independent perturbations (3) is two-fold. Firstly, independent additive perturbations of Δ_N and Δ_D in (3) would yield two components to the uncertainty, while it is the ratio of N and D that determines the model

P. One way to account for this effect is to choose a single weighting function V to bound both Δ_N and Δ_D in (3) as done in Boulet and Francis (1998). Such an approach might introduce conservatism in the uncertainty model, in case the perturbation Δ_N is significantly different from the perturbation Δ_D. The second reason to introduce a combined perturbation Δ in the uncertainty model (2) is to establish a link with the Youla-Kucera parameterization (Anderson 1998) that will facilitate closed-loop model validation of the uncertainty model.

A close relationship with the Youla-Kucera parameterization is obtained when the nominal model $\hat{P}$ is required to create a stable feedback connection with the controller C used in the uncertainty model (2). In this way, the coefficient matrix Q in (1) is formed by considering a model perturbation that is structured according to a Youla-Kucera parameterization as in Figure 1.

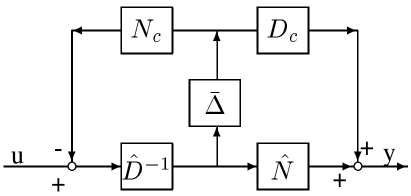

Fig. 1. Uncertainty model based on a controller related perturbations of coprime factorizations

With the uncertainty structure presented in Figure 1, the uncertainty model $\mathcal{P}$ can be written as follows

$$\begin{array}{c} \mathcal{P} = \{P \mid P = \mathcal{F}_u(Q,\Delta) \text{ with Q given by} \\ Q_{11} = V\hat{D}^{-1}N_c \qquad Q_{12} = V\hat{D}^{-1} \\ Q_{21} = D_c + \hat{N}\hat{D}^{-1}N_c \;\; Q_{22} = \hat{N}\hat{D}^{-1} \\ \text{and } \Delta \in \mathbb{R}H_\infty, \|\Delta\|_\infty < 1\} \end{array} \tag{4}$$

The coefficient matrix Q in (4) is formed by using the information of the *rcf* $(\hat{N},\hat{D})$ of the nominal model $\hat{P}$, the *rcf* (N_c, D_c) of the controller C and the perturbation given in(2). It can be observed that the uncertainty model is a generalization of an uncertainty description based on closed-loop transfer functions, in case either $\hat{P}$ or C is assumed to be stable and a trivial choice $(\hat{P}, I)$ or (C, I) is chosen for respectively the *rcf* of $\hat{P}$ or C.

2.2 *Closed-Loop Model Validation Problem*

Consider a feedback connection of the system P_o and a feedback controller $\bar{C}$, with $y = P_o u + v$ and $u = r - \bar{C}y$. It shoudl be noted that a distinction is made between the controller C and $\bar{C}$. The notation C is used to indicate the controller used in the construction of the uncertainty set (4), whereas the notation $\bar{C}$ is used to denote the controller used in the closed-loop experiments.

The input/output data $\{u, y\}$ of the system P_o controlled by the feedback $\bar{C}$ can be described by

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P_o \\ I \end{bmatrix} (I+\bar{C}P_o)^{-1} r + \begin{bmatrix} I \\ -\bar{C} \end{bmatrix} (I+P_o\bar{C})^{-1} v \tag{5}$$

where r denotes an external reference signal that provides sufficient excitation of the closed-loop system. The signal v denotes the additive noise on the output y that can be caused by sensor noise. For closed-loop model validation purposes, the reference signal r is considered as an input signal. The signal u and/or y can be considered as measurable closed-loop output signal. Without loss of generalization only the output signal y is considered here for closed-loop model validation to simplify the formulae.

Application of the feedback law $u = r - \bar{C}y$ to all models $P \in \mathcal{P}$ in (4) yields a set of closed-loop models $\mathcal{S}$ that is structured as follows.

$$\begin{array}{c} \mathcal{S} = \{S \mid S = \mathcal{F}_u(M,\Delta) \text{ with M given by} \\ \begin{aligned} M_{11} &= V(\hat{D}+\bar{C}\hat{N})^{-1}(\bar{C}-C)D_c \\ M_{12} &= V(\hat{D}+\bar{C}\hat{N})^{-1} \\ M_{21} &= (I+\hat{P}\bar{C})^{-1}(I+\hat{P}C)D_c \\ M_{22} &= (I+\hat{P}\bar{C})^{-1}\hat{P} \end{aligned} \\ \text{and } \Delta \in \mathbb{R}H_\infty, \|\Delta\|_\infty < 1\} \end{array} \tag{6}$$

The entries of M in (6) are all known quantities and found by a *rcf* of a nominal model $\hat{P}$ and a bound for the unstructured uncertainty Δ in (4). As C is required to internally stabilize $\hat{P}$, it can be verified that all entries of M are stable *if and only if* the controller $\bar{C}$ also internally stabilizes the nominal model $\hat{P}$. The entries of the coefficient matrix M form the set of closed-loop models $\mathcal{S}$ that is structured due to the uncertainty model posed in (4) and application of the feedback law $u = r - \bar{C}y$. Note that in (6) the controller C is the controller used to create the uncertainty structure of (4), whereas $\bar{C}$ is the controller used in the closed-loop experiments. The availability of closed-loop data $\{r, y\}$ and the characterization of the closed-loop uncertainty set $\mathcal{S}$ form the basis of the closed-loop model (in)validation problem in this paper.

In case the feedback controller $\bar{C}$ used in the closed-loop experiments $u = r - \bar{C}y$ equals the controller C used in constructing the uncertainty set (4), the entries of M in (6) can be greatly reduced. In case $\bar{C} = C$, the set of closed-loop model $\mathcal{S}$ reduces to

$$\begin{array}{l} \mathcal{S} = \{S \mid S = \mathcal{F}_u(M,\Delta) \\ \text{with } \Delta \in \mathbb{R}H_\infty \text{ and } \|\Delta\|_\infty < 1\} \end{array} \tag{7}$$

where the entries of M are given by

$$\begin{array}{ll} M_{11} = 0 & M_{12} = V(\hat{D}+\bar{C}\hat{N})^{-1} \\ M_{21} = D_c & M_{22} = \hat{P}(I+\bar{C}\hat{P})^{-1} \end{array}$$

Note that for $\bar{C} = C$, $M_{11} = 0$ and stability robustness is trivially satisfied for the uncertainty set $\mathcal{S}$ in (7). However, when the controller C used to parameterize the models is redesigned to $\bar{C}$, $M_{11} = \mathcal{F}_l(Q, -\bar{C}) \neq 0$ and stability robustness is not trivially satisfied. The fact that $M_{11} = 0$ is an immediate consequence of the Youla-Kucera parameterization used in constructing the fractional model based uncertainty set (4). In case $\bar{C} = C$, the controller $\bar{C}$ used in the closed-loop experiments, is the same as the controller C used to create the uncertainty set.

Choosing $\bar{C} = C$ also carries the interpretation of performing closed-loop model validation for a set of models $\mathcal{P}$ that is know to be stabilized by $\bar{C} = C$. This information is used beneficially, as only models that are stabilized by $\bar{C}$ are considered. As a final remark it can be observed that $M_{11} = 0$ yields an LFT $\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta M_{12}$ where the uncertainty Δ appears linearly in the closed-loop map from r to y. The affine representation of the uncertainty Δ can be exploited to formulate an affine model (in)validation problem on the basis of closed-loop data.

To account for noise during the observation of $y(t)$ we assume an unknown but bounded additive perturbation of the Discrete Fourier Transform $Y(\omega)$. In that case, the uncertainty model can be written as

$$Y(\omega) = (\mathcal{F}_u(M, \Delta) + W_{cl}(\omega)\delta(\omega))R(\omega) \qquad (8)$$

where $\mathcal{F}_u(M, \Delta)$ is given in (6) and $W_{cl}(\omega)$ is a stable and stably invertible weighting function. In (8) the effect of the additive noise is bounded by $\delta(\omega)$ with $|\delta(\omega)| < 1 \ \forall \ \omega$ or $\|\delta\|_\infty < 1$, while $W_{cl}(\omega)$ is a frequency dependent weighting function to model the spectral content of the noise. Using the above assumptions with the knowledge of the uncertainty model represented in M and the knowledge of the noise contribution in $W_{cl}(\omega)$, the closed-loop model validation problem can be summarized as follows.

Closed-loop model validation problem: consider the closed-loop measurements $Y(\omega)$ and $R(\omega)$, $\omega \in \Omega$. The closed-loop uncertainty model is not invalidated by the data if there exists a Δ with $\|\Delta\|_\infty < 1$ and δ with $\|\delta\|_\infty < 1$ such that (8) holds.

For the closed-loop model validation problem the objective is to determine whether there exists a stable perturbation Δ with $\|\Delta\|_\infty < 1$ and an additive noise perturbation δ with $\|\delta\|_\infty < 1$, such that (8) holds. The model validation results are presented in the next section.

3. CLOSED-LOOP FREQUENCY RESPONSE MODEL INVALIDATION

3.1 Noise-Free Case

For the closed-loop model validation problem we consider a closed-loop system where a reference signal r is applied and a noise free system response y is measured. The frequency domain data of the closed-loop system can be described by

$$Y(\omega) = \mathcal{F}_u(\hat{M}, \Delta)R(\omega), \ \omega \in \Omega \qquad (9)$$

where the entries of $\hat{M}$ are given by

$$\begin{aligned} \hat{M}_{11} &:= M_{11} \ \hat{M}_{12} := -M_{12}R(\omega) \\ \hat{M}_{21} &:= M_{21} \ \hat{M}_{22} := Y(\omega) - M_{22}(\omega)R(\omega) \end{aligned} \qquad (10)$$

In (10) the entries $\hat{M}_{ij}$ are frequency dependent functions where $\omega \in \Omega$ and $Y(\omega)$ and $R(\omega)$ are the respective Discrete Fourier Transforms of the (noise free) signals $y(t)$ and $r(t)$.

The model validation problem is performed frequency point-wise and decomposed into consistency problems evaluated over the frequency grid Ω. The consistency problems check the existence of $\Delta(w)$ with $\overline{\sigma}(\Delta(w)) < 1$ for $w \in \Omega$. In order to guarantee the existence of a $\Delta \in \mathbb{R}H_\infty$ with $\|\Delta\|_\infty < 1$, a boundary interpolation result (Chen 1997, Boulet and Francis 1998) should be used and the result is summarized in the following lemma.

Lemma 1. Let $\overline{\sigma}(\Delta(w)) < 1 \, \forall \, w \in \Omega$, then $\exists \ \Delta \in \mathbb{R}H_\infty$ with $\|\Delta\|_\infty < 1$.

One is referred to Boulet and Francis (1998) for a proof of this result. The boundary interpolation result is used to formalize the model (in)validation result for the closed-loop data in (9) in the following theorem.

Theorem 1. Let $Y(\omega)$, $R(\omega)$ with $\omega \in \Omega$ denote the noise free frequency response measurement of the closed-loop system $T(P_0, C)$ and let $\hat{M}$ be defined as in (10). The closed-loop uncertainty model $\mathcal{S}$ in (6) is not invalidated iff $\mu_\Delta(\hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}) > 1$ where $\mu_\Delta(\cdot)$ is computed with respect to the uncertainty structure Δ given in (6).

Proof: With Lemma 1 it suffices to show the existence of a $\Delta(w)$ with $\overline{\sigma}(\Delta(w)) < 1$ for $w \in \Omega$. This is equivalent to $\mathcal{F}_u(\hat{M}, \Delta_s) = 0$ for $w \in \Omega$. The inverse of $[\mathcal{F}_u(\hat{M}, \Delta_s)]^{-1} = \mathcal{F}_u(N, \Delta_s)$ where the entries of N are given by

$$\begin{aligned} N_{11} &= \hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21} \\ N_{12} &= -\hat{M}_{12}\hat{M}_{22}^{-1} \\ N_{21} &= \hat{M}_{22}^{-1}\hat{M}_{21} \\ N_{22} &= \hat{M}_{22}^{-1} \end{aligned}$$

With $\hat{M}_{22}$ invertible, it can be seen that $\mathcal{F}_u(N, \Delta_s)$ is ill-defined when $(I - \hat{M}_{11}\Delta + \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}\Delta)^{-1}$ is ill-defined. This condition can be replaced by the computation of the structured singular value $\mu_\Delta(\hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}) > 1$. □

The evaluation of $\mu_\Delta(\hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}) > 1$ is done frequency point-wise over $\omega \in \Omega$. In case Δ is unstructured, $\mu_\Delta(\cdot) > 1$ can be replaced with $\bar{\sigma}(\cdot) > 1$. The above model invalidation condition of $\mu_\Delta(\hat{M}_{11} - \hat{M}_{12}\hat{M}_{22}^{-1}\hat{M}_{21}) > 1$ is actually the inverse-μ result as derived in Lind and Brenner (1999) and is an adaptation of the result mentioned in Boulet and Francis (1998). Due to the noise free data, the value of $\Delta(\omega)$ can actually be computed over $\omega \in \Omega$. The result is summarized in the following corollary for an interpretation of the noise free closed-loop model validation problem.

Corollary 1. Let $\Phi_{cl}(\omega)$ denote the closed-loop noise free frequency response of the system and let Δ_V be defined as $\Delta_V = D_c{}^{-1}(I + \hat{P}C)^{-1}((I - \Phi_{cl}(\omega)\bar{C})^{-1}\Phi_{cl}(\omega) - \hat{P})\hat{D}$. Then the closed-loop uncertainty model $\mathcal{S}$ in (6) is not invalidated by $\Phi_{cl}(\omega)$ iff $|\Delta_V(\omega)| < |V(\omega)|$.

Proof: Let $P_0 = ND^{-1}$ with $N = \hat{N} + D_c\Delta_V$ and $D = \hat{D} - N_c\Delta_V$ where $(\hat{N}, \hat{D})$ and (N_c, D_c) are the respective right coprime factors of the model $\hat{P}$ and the controller C. Rearranging terms leads to the expression $(P_0\hat{D} - \hat{N}) = (\hat{P}N_c + D_c)\Delta$. Substituting $\hat{P} = \hat{N}\hat{D}^{-1}$ and $C = N_cD_c^{-1}$ and with further rearrangement leads to the uncertainty expression $\Delta_V = D_c{}^{-1}(I + PC)^{-1}(P_0 - \hat{P})\hat{D}$. In the case of noise free measurements, $(I - \Phi_{cl}(\omega)\bar{C})^{-1}\Phi_{cl}(\omega) = P_0$ and the result simplifies to $\Delta_V = D_c{}^{-1}(I + \hat{P}C)^{-1}((I - \Phi_{cl}(\omega)\bar{C})^{-1}\Phi_{cl}(\omega) - \hat{P})\hat{D}$. □

The result in Corollary 1 illustrates that an explicit and computable expression for the uncertainty Δ_V can be obtained. Note that the assumptions listed in Boulet and Francis (1998) for the noise-free open-loop model validation test are relaxed in our case. Since the validation test presented in this paper involves the measurements of $Y(\omega)$ and $U(\omega)$ explicitly, it is unnecessary to assume nonsingularity of $\Phi_{ol}(\omega) = Y(\omega)/U(\omega)$. The following corollary illustrates how the model validation results simplify when the controller $\bar{C}$ used to gather the closed-loop experiments is equal to the controller C used to characterize the uncertainty model $\mathcal{P}$.

Corollary 2. If the controller $\bar{C}$ used in the closed-loop experiments is equivalent to the controller C used in the uncertainty model $\mathcal{P}$ of (2) then the test on the structured singular value mentioned in Theorem 1 reduces to

$$\underline{\sigma}(V^{-1}D_c^{-1}\hat{D}\frac{P_0 - \hat{P}}{1 + P_0\bar{C}}) < 1 \tag{11}$$

Proof: Substituting $\hat{M}_{11}$, $\hat{M}_{12}$, $\hat{M}_{21}$, and $\hat{M}_{22}$ from (10) and letting $C = \bar{C}$ reduces the model invalidation test to $\bar{\sigma}(V(\hat{D} + \bar{C}\hat{N})^{-1}R(Y - (1 + \hat{P}\bar{C})^{-1}\hat{P}R)^{-1}D_c) > 1$. Note that $\bar{\sigma}(\cdot) = \mu_\Delta(\cdot)$ since we consider a full-block matrix Δ for the closed-loop noise-free case. Simplifying and noting that $\frac{Y}{R} = \frac{P_0}{1+P_0\bar{C}}$, the expression reduces to $\bar{\sigma}(VD_c\hat{D}^{-1}(1 + \bar{C}\hat{P})^{-1}(\frac{P_0}{1+P_0\bar{C}} - \frac{\hat{P}}{1+\hat{P}\bar{C}})^{-1} > 1$. Cancelling common terms and noting that $\bar{\sigma}(M) > 1 \Leftrightarrow \underline{\sigma}(M^{-1}) < 1$, the model invalidation test is written as $\underline{\sigma}(V^{-1}D_c^{-1}\hat{D}\frac{P_0-\hat{P}}{1+P_0\bar{C}}) < 1$. □

The above result is expected as we note a dependence on the closed-loop sensitivity function and a weighted dependence on the difference between the system and nominal model. As mentioned earlier in the case $\bar{C} = C$, the term $M_{11} = 0$ and robust stability is trivially satisfied. This follows directly from (7) and is the result of the dual-Youla parameterization.

3.2 Noisy Case

The noise free case was used to illustrate the main results and ideas of the closed-loop model validation problem. For practical application of the model validation techniques, the noisy measurements of the closed-loop data in (8) need to be considered. In (8) the quantity $W_{cl}(\omega)\delta(\omega)$ models an additive perturbation of the closed-loop measurement $Y(\omega)$ and incorporates both the effect of an additive noise during a measurement and also the effect of initial conditions. The noise $\delta(\omega)$ is unknown, but bounded by $\|\delta(\omega)\| < 1 \; \forall \, \omega \in \Omega$.

In structuring the closed-loop uncertainty model as in (6), the closed-loop model validation problem for the noisy case can be determined. In order to take into account the effect of the unknown but bounded additive noise, we can recast the uncertainty model as an LFT $\mathcal{F}_u(K, \Delta_s)$ where

$$\Delta_s = \begin{bmatrix} \Delta & 0 \\ 0 & \delta \end{bmatrix} \text{with } \|\Delta\|_\infty < 1 \text{ and } \|\delta\|_\infty < 1 \tag{12}$$

and the entries of K are given by

$$K_{11} = \begin{bmatrix} M_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad K_{12} = \begin{bmatrix} M_{12} \\ W_{cl} \end{bmatrix}$$
$$K_{21} = \begin{bmatrix} M_{21} & 1 \end{bmatrix} \quad K_{22} = \begin{bmatrix} M_{22} \end{bmatrix} \tag{13}$$

In structuring the closed-loop uncertainty model in this form, (8) can be written as

$$Y(\omega) = \mathcal{F}_u(\hat{K}, \Delta_s)R(\omega) \tag{14}$$

where the entries of $\hat{K}$ are given by

$$\begin{aligned} \hat{K}_{11} &:= K_{11}(\omega) \;\; \hat{K}_{12} := -K_{12}(\omega)R(\omega) \\ \hat{K}_{21} &:= K_{21}(\omega) \;\; \hat{K}_{22} := Y(\omega) - K_{22}(\omega)R(\omega). \end{aligned} \tag{15}$$

Similar as in the noise-free case, $\hat{K}_{ij}$ are frequency dependent functions of $\omega \in \Omega$. With this LFT formulation, a result similar to Theorem 1 can be formulated for the closed-loop model validation problem on the basis of noisy data.

Theorem 2. Let $Y(\omega)$ and $R(\omega)$ denote the closed-loop noisy frequency response measurement of the closed-loop system $T(P_0, C)$ and let $\hat{K}$ be defined as in (15). The closed-loop uncertainty model in (14) is not invalidated by the data iff $\mu_{\Delta_s}(\hat{K}_{11} - \hat{K}_{12}\hat{K}_{22}^{-1}\hat{K}_{21}) > 1$ where $\mu_{\Delta_s}(\cdot)$ is computed with respect to the uncertainty structure Δ_s given in (12).

The proof of Theorem 2 is similar to the proof of Theorem 1 and is omitted here for brevity. Consider again the case where the controller $\bar{C}$ used to gather the closed-loop experiments is equal to the controller C used to characterize the uncertainty model $\mathcal{P}$. With $C = \bar{C}$, a direct calculation of μ can be obtained by using the reduced rank μ problem (Fan and Tits 1985). Using the model invalidation result of Theorem 2, substitute the values of $\hat{K}_{12}$, $\hat{K}_{21}$ and $\hat{K}_{22}$ from (15) and note that $C = \bar{C} \Rightarrow M_{11} = 0 \Rightarrow \hat{K}_{11} = 0$. As a result, the model invalidation test is reduced to

$$\mu_{\Delta_s}\left(\begin{bmatrix} M_{12} \\ W_{cl} \end{bmatrix} R(Y - M_{22}R)^{-1}[M_{21}\ 1] \right) > 1 \quad (16)$$

where Δ_s is computed with respect to the uncertainty structure given in (12). Note that the dependency of ω is dropped for notational simplicity from the expression of $\mu_{\Delta_s}(\cdot)$ in (16). Since the argument of $\mu_{\Delta_s}(\cdot)$ in (16) is a diadic matrix (Fan and Tits 1985), it satisfies

$$\mu_{\Delta_s}\left(\begin{bmatrix} a \\ b \end{bmatrix} [c\ d] \right) = |ac| + |bd| \quad (17)$$

and the model invalidation test can be rewritten as

$$\begin{array}{c} |M_{12}R(Y - M_{22}R)^{-1}M_{21}| + \\ |W_{cl}R(Y - M_{22}R)^{-1}|) > 1 \end{array} \quad (18)$$

It can be seen from (18) that for $C = \bar{C}$ the closed-loop model invalidation test involves checking whether the sum of two transfer functions is less than 1, which greatly simplifies the computation of the structured singular value for the model (in)validation problem.

4. CONCLUSIONS

Uncertainty models that are structured using coprime factorizations are used to address closed-loop relevant model (in)validation on the basis of closed-loop frequency domain data. Important in the formulation of the uncertainty model is the knowledge on the controller used to create a closed-loop oriented model (in)validation of the uncertainty model.

The controller dependent uncertainty model is used to formulate model (in)validation tests on the basis of closed-loop data. The model validation tests involve the computation of a structured singular value over a finite frequency grid. It is also shown that the computation of the structured singular value reduces to the sum of two transfer functions when the feedback controller used for the closed-loop experiments is the same as the controller used for formulating the controller dependent coprime factor uncertainty model.

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