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## A Unified Approach to Stability Robustness for Uncertainty Descriptions Based on Fractional Model Representations

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**Abstract**—The powerful standard representation for uncertainty descriptions in a basic perturbation model based on a standard plant representation can be used to attain necessary and sufficient conditions for stability robustness within various uncertainty descriptions. In this paper, these results are employed to formulate necessary and sufficient conditions for stability robustness of several uncertainty sets based on unstructured additive coprime factor uncertainty, gap-metric uncertainty, as well as the recently introduced  $\Lambda$ -gap uncertainty.

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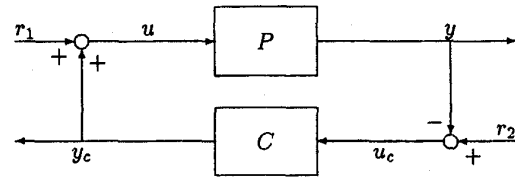


Fig. 1. Feedback connection structure  $T(P, C)$  of a plant  $P$  and a controller  $C$ .

## I. INTRODUCTION

In a model-based control design paradigm, the design is based on a (necessarily) approximative model  $\hat{P}$  of a plant to be controlled. An apparently successful control design leads to a controller  $C$  having some desired closed-loop properties for the feedback controlled model  $\hat{P}$ , but due to the mismatch between the actual plant  $P_o$  and the model  $\hat{P}$ , a verification of these desired closed-loop properties is preferred before implementing the controller  $C$  on the actual plant  $P_o$ . In this paper the discussion is directed toward the verification of one of the most important closed-loop properties: stability.

To evaluate stability when the controller  $C$  is being applied to the plant  $P_o$ , a characterization of the mismatch between the plant  $P_o$  and the model  $\hat{P}$  is indispensable. Since the real plant  $P_o$  is unknown, the discrepancy in general is characterized by a so-called uncertainty set, denoted with  $\mathcal{P}$ . Typically, an uncertainty set  $\mathcal{P}$  is defined by the (nominal) model  $\hat{P}$  which is found by physical modeling or identification techniques and some bounded "area" around it [4]. The uncertainty set  $\mathcal{P}$  itself reflects all possible perturbations of the (nominal) model  $\hat{P}$  that may occur.

By defining the uncertainty set in such a way that at least the plant  $P_o \in \mathcal{P}$ , stability robustness results for the set  $\mathcal{P}$  will reflect sufficient conditions under which the plant  $P_o$  will be stabilized by  $C$ ; see [4] or [5]. In this perspective, special attention will be given in this paper to an uncertainty set  $\mathcal{P}_{CF}$  which is characterized by additive perturbations on a coprime factor description of the nominal model  $\hat{P}$ . The specific application of such an uncertainty set description will be motivated by the favorable properties it has over a standard additive or multiplicative uncertainty set description.

Using the simple and powerful stability robustness results for a basic perturbation model in a standard plant configuration [4], [5], [15], several different uncertainty sets employing weighted and unstructured additive perturbations on a coprime factorization, gap-metric based uncertainty sets, and the recently introduced  $\Lambda$ -gap uncertainty sets will be shown to be closely related to each other. The contribution of this paper is in the unified treatment of these different uncertainty sets. While stability robustness results for uncertainty sets using additive perturbations on normalized (left) coprime factorizations [11] and gap-metric based uncertainty sets [10] have separately been derived before, this paper amplifies their relation, as well as the extension to a less conservative  $\Lambda$ -gap uncertainty set description [1], [2].

## II. PRELIMINARIES

Throughout this paper, the feedback configuration of a plant  $P$  and a controller  $C$  is denoted by  $T(P, C)$  and defined by the feedback connection structure depicted in Fig. 1.

A plant  $P$  and a controller  $C$  are assumed to be given by real rational transfer function matrices, and it will be assumed that the

feedback connection is well-posed, i.e., that  $\det[I + CP] \neq 0$ . Then the feedback system  $T(P, C)$  is defined to be internally stable if the mapping from  $\text{col}(r_2, r_1)$  to  $\text{col}(u_c, u)$  is bounded-input bounded-output BIBO stable, i.e., if the corresponding transfer function is in  $\mathbb{RH}_\infty$ , where  $\mathbb{RH}_\infty$  denotes the Hardy space of real rational transfer function matrices with bounded  $\mathcal{H}_\infty$ -norm [6]

$$\|G\|_\infty := \sup_{\omega \in [0, \pi]} \bar{\sigma}\{G(e^{i\omega})\} \quad (1)$$

with  $\bar{\sigma}$  the maximum singular value. Furthermore, the dynamics of the closed-loop system  $T(P, C)$  will also be described by the mapping from  $\text{col}(r_2, r_1)$  to  $\text{col}(y, u)$  which is given by the transfer function matrix  $T(P, C)$

$$T(P, C) := \begin{bmatrix} P \\ I \end{bmatrix} [I + CP]^{-1} \begin{bmatrix} C & I \end{bmatrix}. \quad (2)$$

Internal stability of  $T(P, C)$  is equivalent to the condition that  $T(P, C) \in \mathbb{RH}_\infty$ . Using the theory of fractional representations, e.g., as presented in [17], a plant  $P$  is expressed as a ratio of two stable transfer function matrices  $N$  and  $D$ . For two transfer functions  $N, D \in \mathbb{RH}_\infty$ , the pair  $(N, D)$  is called right coprime over  $\mathbb{RH}_\infty$  if there exist  $X, Y \in \mathbb{RH}_\infty$  such that  $XN + YD = I$ . The pair  $(N, D)$  is a right coprime factorization (rcf) of  $P$  if  $(N, D)$  is right coprime and  $P = ND^{-1}$ . An rcf  $(N, D)$  is called normalized (nrcf) if it satisfies  $N^*N + D^*D = I$ , where  $*$  denotes complex conjugate transpose. For (normalized) left coprime factorizations (lcf) dual definitions exist.

Fractional representations have a close relation with approximation in the graph topology. The graph topology is the weakest topology<sup>1</sup> in which a variation of the elements of a stable feedback configuration around their nominal values preserves stability of that closed-loop system [18]. The graph topology is known to be induced by several metrics, e.g., the graph metric introduced in [16] or the gap metric introduced in [19], expressed in the following way.

**Definition 2.1 [8]:** Consider two plants  $P_1, P_2$  with an nrcf  $(N_1, D_1), (N_2, D_2)$ , respectively. Then the gap between  $P_1$  and  $P_2$  is expressed by

$$\delta(P_1, P_2) := \max\{\bar{\delta}(P_1, P_2), \bar{\delta}(P_2, P_1)\}$$

with

$$\bar{\delta}(P_i, P_j) := \inf_{Q \in \mathbb{RH}_\infty} \left\| \begin{bmatrix} D_i \\ N_i \end{bmatrix} - \begin{bmatrix} D_j \\ N_j \end{bmatrix} Q \right\|_\infty.$$

Stability robustness results will be considered for two equivalent interconnection structures depicted in Fig. 2(a) and (b) (see [4] and [15]). Internal stability of the feedback system of Fig. 2(a) is equivalent to input-output stability of the upper linear fractional transformation  $\mathcal{F}(M, \Delta) := M_{22} + M_{21}\Delta[I - M_{11}\Delta]^{-1}M_{12}$ , where the decomposition of  $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$  is in accordance with Fig. 2(b).

If the transfer function  $M$  is BIBO stable, the small gain theorem can be applied to characterize stability results for the connection structure of Fig. 2(b), as formulated next.

**Lemma 2.2:** Let the stable transfer functions  $M, \Delta \in \mathbb{RH}_\infty$  construct a feedback connection  $\mathcal{F}(M, \Delta)$ . Then:

- a) A sufficient condition for BIBO stability of  $\mathcal{F}(M, \Delta)$  is given by

$$\|M_{11}\Delta\|_\infty < 1. \quad (3)$$

<sup>1</sup>Given two topologies  $\mathcal{O}_1$  and  $\mathcal{O}_2$ ,  $\mathcal{O}_1$  is said to be weaker than  $\mathcal{O}_2$  if  $\mathcal{O}_1$  is a subcollection of  $\mathcal{O}_2$ ; see also [18].

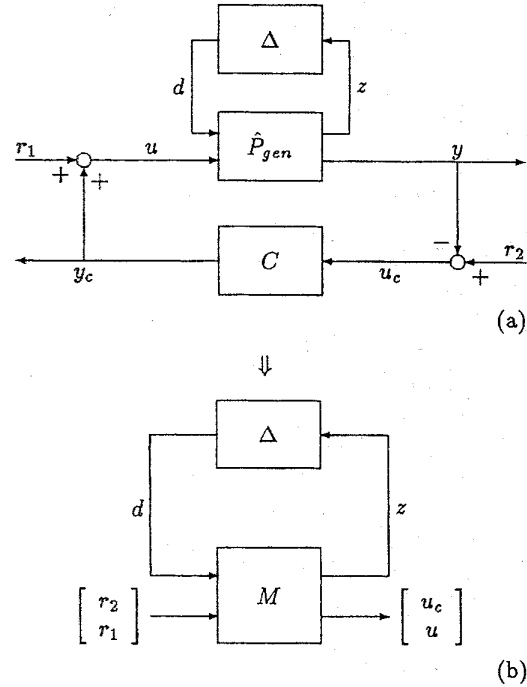


Fig. 2. (a) Feedback connection structure of a (perturbed) plant  $P_0$  and the controller  $C$ . (b) Basic perturbation model  $\mathcal{F}(M, \Delta)$ .

- b) Provided that for all  $\Delta$  with  $\|\Delta\|_\infty < \gamma$  the transfer function  $M_{21}\Delta[I - M_{11}\Delta]^{-1}M_{12}$  does not exhibit unstable pole/zero cancellations,<sup>2</sup>  $\mathcal{F}(M, \Delta)$  is BIBO stable for all  $\Delta$  with  $\|\Delta\|_\infty < \gamma$  if and only if

$$\|M_{11}\|_\infty \leq \gamma^{-1}. \quad (4)$$

Since  $M \in \mathbb{RH}_\infty$ , and thus  $M_{11}, M_{12}, M_{21}, M_{22} \in \mathbb{RH}_\infty$ , the small gain theorem directly leads to result a). Additionally, necessary conditions can be formulated on the stability of  $[I - M_{11}\Delta]^{-1}$  for all  $\Delta$  with  $\|\Delta\|_\infty < \gamma$ . Provided that unstable poles of  $[I - M_{11}\Delta]^{-1}$  are not cancelled in  $M$ , this leads to the necessary condition of (4). For a complete proof, see [11] or [17].

### III. STABILITY ROBUSTNESS FOR UNCERTAINTY DESCRIPTIONS BASED ON FRACTIONAL MODEL REPRESENTATIONS

The framework for stability robustness from the previous section can be directly applied to uncertainty sets based on coprime factor perturbations. As the uncertainty block  $\Delta$  is assumed to be stable, this implies that for simple additive or multiplicative uncertainty sets, the locations of all unstable poles of the plant  $P_0$  are assumed to be fixed. Additive perturbations on coprime factorizations are more flexible and allow changes in both the number and the locations of poles and zeros anywhere in  $\mathbb{C}$  [3]. Moreover, fractional representations have a close relation with approximation in the graph topology. First, an uncertainty set based on additive perturbations on a coprime factorization will be discussed.

<sup>2</sup>This additional condition which is often discarded in literature excludes trivial situations, e.g.,  $M_{21} = 0$  or  $M_{12} = 0$ . It can be shown to be satisfied for the common uncertainty classes based on additive, multiplicative, or coprime factor uncertainty.

*Corollary 3.1:* Consider a plant  $\hat{P}$  with *rcf*  $(\hat{N}, \hat{D})$ , stabilized by a given controller  $C$ , and consider the uncertainty set

$$\mathcal{P}_{CF}(\hat{N}, \hat{D}, V_D, V_N, W, \gamma) := \left\{ P | P = [\hat{N} + \Delta_N][\hat{D} + \Delta_D]^{-1} \right.$$

with

$$\left\| \begin{bmatrix} V_D & 0 \\ 0 & V_N \end{bmatrix} \begin{bmatrix} \Delta_D \\ \Delta_N \end{bmatrix} W \right\|_{\infty} < \gamma \right\}$$

for stable and stably invertible filters  $V_D, V_N, W$ . Then the feedback system  $T(P, C)$  is internally stable for all  $P \in \mathcal{P}_{CF}$  if and only if

$$\left\| W^{-1}[\hat{D} + C\hat{N}]^{-1} \begin{bmatrix} I & C \\ 0 & V_N^{-1} \end{bmatrix} \begin{bmatrix} V_D^{-1} & 0 \\ 0 & V_N^{-1} \end{bmatrix} \right\|_{\infty} \leq \gamma^{-1}.$$

Defining

$$\Delta := \begin{bmatrix} V_D & 0 \\ 0 & V_N \end{bmatrix} \begin{bmatrix} \Delta_D \\ \Delta_N \end{bmatrix} W \quad (5)$$

it can simply be shown that this leads to an uncertainty set according to Fig. 2(b) with

$$M_{11} = -W^{-1}[\hat{D} + C\hat{N}]^{-1} \begin{bmatrix} I & C \\ 0 & V_N^{-1} \end{bmatrix} \begin{bmatrix} V_D^{-1} & 0 \\ 0 & V_N^{-1} \end{bmatrix}. \quad (6)$$

For unity weightings  $V_D, V_N$ , and  $W$  it follows from [11] that the corresponding  $M$  is stable and pole/zero cancellations as mentioned in Lemma 2.2-b) do not occur for stable  $\Delta$ . These results can be extended directly to stable and stably invertible weighting functions  $V_D, V_N$ , and  $W$ , and thus Lemma 2.2 can be applied.

The corollary can alternatively be proven by employing stability results directly in terms of coprime factor representations of plant and controller. Here it has been stressed that the considered uncertainty set allows a description in terms of a standard perturbation model as depicted in Fig. 2.

The following equivalent formulations of the coprime factor uncertainty set discussed in Corollary 3.1 will appear to be useful in the sequel of the paper.

*Proposition 3.2:* The uncertainty set  $\mathcal{P}_{CF}(\hat{N}, \hat{D}, V_D, V_N, W, \gamma)$  as defined in Corollary 3.1 can alternatively be written in the following equivalent forms:

$$\begin{aligned} \text{a) } & \mathcal{P}_{CF}(\hat{N}, \hat{D}, V_N, V_D, W, \gamma) \\ & = \{ P | P = (\hat{N}W + V_N^{-1}\Delta_N)(\hat{D}W + V_D^{-1}\Delta_D)^{-1}, \\ & \quad \text{with } \left\| \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \right\|_{\infty} < \gamma \} \\ \text{b) } & \mathcal{P}_{CF}(\hat{N}, \hat{D}, V_N, V_D, W, \gamma) \\ & = \{ P | P = N_n D_n^{-1}, (N_n, D_n) \text{ an nrcf and} \\ & \quad \exists Q \in \mathbb{RH}_{\infty} \text{ such that} \\ & \quad \left\| \begin{bmatrix} V_N & 0 \\ 0 & V_D \end{bmatrix} \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} W - \begin{bmatrix} N_n \\ D_n \end{bmatrix} Q \right\|_{\infty} < \gamma \}. \end{aligned} \quad (7)$$

Part a) follows by simple calculation. The proof of part b) is more involved and is based on the fact that in any stable right, but not necessarily coprime, fractional representation  $(N, D)$  can be written as a right fractional representation  $(N_n Q, D_n Q)$  with  $Q \in \mathbb{RH}_{\infty}$  and  $(N_n, D_n)$  an nrcf. In this way the right, but not necessarily coprime, fractional representation of  $P$  in (7) can be written as  $(\hat{N}W + V_N^{-1}\Delta_N) = N_n Q$  and  $(\hat{D}W + V_D^{-1}\Delta_D) = D_n Q$  with  $(N_n, D_n)$  an nrcf and  $Q \in \mathbb{RH}_{\infty}$ . It follows then that  $\Delta_N = V_N[N_n Q - \hat{N}W]$  and  $\Delta_D = V_D[D_n Q - \hat{D}W]$  which proves the result. Note that the factor  $Q$  cancels in the representation of  $P$ .

#### IV. STABILITY ROBUSTNESS BASED ON DISTANCE MEASURES

Stability robustness results for gap-metric uncertainty sets can be considered in the same framework. It will be illustrated that the available stability robustness results for this situation, proven

separately in [10], can be considered as a special case of Corollary 3.1. To this end the following uncertainty sets are being considered:

$$\begin{aligned} \mathcal{P}_{\delta}(\hat{P}, \gamma) & := \{ P | \delta(\hat{P}, P) < \gamma \} \\ \mathcal{P}_{\delta}(\hat{P}, \gamma) & := \{ P | \delta(\hat{P}, P) < \gamma \} \end{aligned}$$

for which the following relation with the coprime factor uncertainty sets can be shown, as presented before.

*Lemma 4.1:* Let a plant  $\hat{P}$  and a controller  $C$  constitute an internally stable feedback system  $T(\hat{P}, C)$ . Consider the uncertainty set  $\mathcal{P}_{CF}(\hat{N}, \hat{D}, V_N, V_D, W, \gamma)$  under the additional conditions that  $(\hat{N}, \hat{D})$  is an nrcf of  $\hat{P}$ , and  $V_D = I, V_N = I$ , and  $W = I$ . Then:

- $\mathcal{P}_{CF}(\hat{N}, \hat{D}, V_N, V_D, W, \gamma) = \mathcal{P}_{\delta}(\hat{P}, \gamma)$ .
- For  $\gamma < 1$ ,  $\mathcal{P}_{\delta}(\hat{P}, \gamma) = \mathcal{P}_{\delta}(\hat{P}, \gamma)$ .

*Part a):* According to Proposition 3.2-b) and taking into account the specific choice of weighting functions in the lemma, it follows:

$$\begin{aligned} \mathcal{P}_{CF}(\hat{N}, \hat{D}, V_N, V_D, W, \gamma) & = \left\{ P | P = N_n D_n^{-1}, (N_n, D_n) \text{ an nrcf and} \right. \\ & \quad \left. \exists Q \in \mathbb{RH}_{\infty} \text{ such that } \left\| \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} - \begin{bmatrix} N_n \\ D_n \end{bmatrix} Q \right\|_{\infty} < \gamma \right\}. \end{aligned}$$

Since  $(\hat{N}, \hat{D})$  is chosen to be an nrcf of  $\hat{P}$ , it is straightforward to verify that  $\mathcal{P}_{CF} = \mathcal{P}_{\delta}$ .

*Part b):* This is proven in [10]. The restriction to  $\gamma < 1$  is caused by the fact that these sets with  $\gamma \geq 1$  cannot be stabilized by a single controller.  $\square$

Lemma 4.1 relates the set defined by a gap metric bound with the set of coprime factor perturbations by a special choice of the weighting functions  $V_D, V_N, W$  and the coprime factorization  $(\hat{N}, \hat{D})$  of the model  $\hat{P}$ . This gives rise to a unified approach to handle sets of plants that are bounded by a gap metric, and the stability robustness result for these sets follows directly from Corollary 3.1.

*Corollary 4.2:* Consider the situation of Lemma 4.1 with  $\gamma < 1$ . Then each of the three sets of plants  $\mathcal{P}_{CF}, \mathcal{P}_{\delta}$ , and  $\mathcal{P}_{\delta}, T(P, C)$  is internally stable for all  $P \in \mathcal{P}$  if and only if

$$\|T(\hat{P}, C)\|_{\infty} \leq \gamma^{-1}. \quad (9)$$

The proof follows simply by substituting the specific weightings in the result of Corollary 3.1, employing the fact that premultiplication of the expression within the norm by  $[\hat{N}^T \ \hat{D}^T]^T$  leaves the norm invariant, due to the normalization of the ref.  $\square$

Note that the result of this corollary is not new. It was shown already in [10], where a complete proof of the stability robustness result is given. It has been shown here that the stability robustness results in the standard form can simply be exploited, as formulated in Section II. Restricting attention to the situation that  $\gamma < 1$  is natural, as  $\|T(\hat{P}, C)\|_{\infty} \geq \|[I + C\hat{P}]^{-1}\|_{\infty} > 1$ , according to Bode's sensitivity integral, showing that stability robustness can only be achieved for sets with  $\gamma < 1$ .

Finally, it should be noted that the gap and graph metric are induced by the same topology and are uniformly equivalent [8]. Therefore, stability robustness in the graph metric yields a similar result as mentioned in Corollary 4.2, and their interrelation is discussed in [13].

#### V. STABILITY ROBUSTNESS IN THE $\Lambda$ -GAP

The results obtained in the previous section for gap-based stability robustness can be further extended for uncertainty sets based on the recently introduced  $\Lambda$ -gap [1], [2]. This  $\Lambda$ -gap is a distance measure that adds an additional frequency weighting in the expression

that is utilized in the gap-metric, while the frequency weighting is controller-dependent.

*Definition 5.1:* Let two plants  $P_1, P_2$  have nrcf's  $(N_1, D_1), (N_2, D_2)$ , respectively. Let  $C$  be a controller with nlcfc  $(\tilde{D}_c, \tilde{N}_c)$  such that  $T(P_1, C)$  is internally stable. Then the  $\Lambda$ -gap between the plants  $P_1, P_2$  is defined to be expressed by

$$\tilde{\delta}_\Lambda(P_1, P_2) = \inf_{Q \in \mathbb{RH}_\infty} \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} \Lambda^{-1} - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} Q \right\|_\infty$$

with  $\Lambda = [\tilde{D}_c D_1 + \tilde{N}_c N_1]$ .

The difference between  $\tilde{\delta}(P_1, P_2)$  and  $\tilde{\delta}_\Lambda(P_1, P_2)$  is the additional shaping of the nrcf of  $P_1$  with  $\Lambda^{-1}$  into a rcf  $(\tilde{N}, \tilde{D})$ . In this way  $\tilde{\Lambda} := \tilde{D}_c \tilde{D} + \tilde{N}_c \tilde{N} = I$ , with  $\tilde{N} = N_1 \Lambda^{-1}, \tilde{D} = D_1 \Lambda^{-1}$ , which is used to consider the closed-loop operation of  $P_1$  induced by the controller  $C$  being employed. This makes the distance between  $P_1$  and  $P_2$  dependent on the controller  $C$ . Note that the distance measure  $\tilde{\delta}_\Lambda(P_1, P_2)$  is not a metric, since  $\tilde{\delta}_\Lambda(P_1, P_2) \neq \tilde{\delta}_\Lambda(P_2, P_1)$  due to the influence of the controller  $C$  [2].

Accordingly, an uncertainty set based on  $\Lambda$ -gap uncertainty can be defined as

$$\mathcal{P}_{\tilde{\delta}_\Lambda}(\hat{P}, \gamma) := \{P | \tilde{\delta}_\Lambda(\hat{P}, P) < \gamma\}.$$

This uncertainty set can also be shown to be equivalent to an uncertainty set of coprime factor uncertainties, provided appropriate weighting functions are chosen.

*Lemma 5.2:* Let a plant  $\hat{P}$  and a controller  $C$  with nlcfc  $(\tilde{D}_c, \tilde{N}_c)$  constitute an internally stable feedback system  $T(\hat{P}, C)$ . Consider the uncertainty set  $\mathcal{P}_{CF}(\tilde{N}, \tilde{D}, V_N, V_D, W, \gamma)$  under the additional conditions that  $(\tilde{N}, \tilde{D})$  is an nrcf of  $\hat{P}$ , and  $V_D = I, V_N = I$ , and  $W = \Lambda^{-1}$  with  $\Lambda = [\tilde{D}_c \tilde{D} + \tilde{N}_c \tilde{N}]$ . Then:

- $\mathcal{P}_{CF}(\tilde{N}, \tilde{D}, V_N, V_D, W, \gamma) = \mathcal{P}_{\tilde{\delta}_\Lambda}(\hat{P}, \gamma)$ .
- $T(\hat{P}, C)$  is internally stable for all  $P \in \mathcal{P}_{CF}$  if and only if  $\gamma \leq 1$ .

The proof of a) is straightforward, along the same lines as the proof of Lemma 4.1-a). Result b) then follows directly from Corollary 3.1, employing the fact that  $\Lambda[\tilde{D} + C\tilde{N}]^{-1}[I \ C] = [\tilde{D}_c \ \tilde{N}_c]$  having an  $\infty$ -norm of one due to the fact that it is a normalized left coprime factorization.  $\square$

As said before, in case of the  $\Lambda$ -gap, the uncertainty set defined accordingly considers perturbations of the nominal plant  $\hat{P}$  that are controller dependent.

The introduction of weightings in the gap metric has also been studied in [7], [9], and [14]. In [7] a multiplicative uncertainty description on the nrcf  $(\tilde{N}, \tilde{D})$  of the model  $\hat{P}$  is being used, leading to an uncertainty structure  $\Delta$  having a diagonal form. Due to the diagonal form only necessary and sufficient conditions based on the structured singular value  $\mu\{\cdot\}$  can be obtained. The weightings in the weighted gap of [9] have to be defined *a posteriori* which makes the choice of the weighting functions, to access robustness issues on the basis of a weighted gap, not a trivial task. Information on the size of the coprime factor perturbations can be used in the weighted pointwise gap metric defined in [14], but an efficient computational method for pointwise gap metric is not available yet. The  $\Lambda$ -gap can simply be calculated. Controller synthesis in the  $\Lambda$ -gap, however, is more complicated and is a problem that is not solved yet.

## VI. CONSERVATISM ISSUES

All stability robustness results in this paper reflect necessary and sufficient conditions of an uncertainty set to be stabilized by a single controller. As such, no conservatism is introduced in the test for checking stability robustness itself. However, for a single given controller, different uncertainty sets contain a different portion of

the set of all systems that is stabilized by the controller. In this perspective, the concept of conservatism is an intrinsic property of the uncertainty set being used. As a result, an uncertainty set will be called more conservative than another if one controller stabilizes both sets, while the former set is contained in the latter.

*Theorem 6.1 [1]:* Consider a plant  $\hat{P}$  and a stabilizing controller  $C$  with nlcfc  $(\tilde{D}_c, \tilde{N}_c)$ . Consider the following two uncertainty sets resulting from the stability robustness results in the previous sections:

$$S_\delta(\hat{P}, C) := \{\cup \mathcal{P}_\delta(\hat{P}, b), \quad b < \|T(\hat{P}, C)\|_\infty^{-1}\}$$

$$S_{\tilde{\delta}_\Lambda}(\hat{P}, C) := \{\cup \mathcal{P}_{\tilde{\delta}_\Lambda}(\hat{P}, c), \quad c < 1\}$$

then

$$S_\delta(\hat{P}, C) \subset S_{\tilde{\delta}_\Lambda}(\hat{P}, C). \quad (10)$$

The following implication will be proven:

$$\bar{P} \in S_\delta(\hat{P}, C) \Rightarrow \bar{P} \in S_{\tilde{\delta}_\Lambda}(\hat{P}, C). \quad (11)$$

As  $\bar{P} \in S_\delta(\hat{P}, C)$ , there exists a  $\bar{U} \in \mathbb{RH}_\infty$  such that

$$\left\| \begin{bmatrix} \tilde{D}_n \\ \tilde{N}_n \end{bmatrix} - \begin{bmatrix} \tilde{D} \\ \tilde{N} \end{bmatrix} \bar{U} \right\|_\infty \leq \frac{1}{\|T(\hat{P}, C)\|_\infty}. \quad (12)$$

This implies that

$$\left\| \begin{bmatrix} \tilde{D}_n \\ \tilde{N}_n \end{bmatrix} - \begin{bmatrix} \tilde{D} \\ \tilde{N} \end{bmatrix} \bar{U} \right\|_\infty \cdot \|\Lambda^{-1}\|_\infty \leq \frac{\|\Lambda^{-1}\|_\infty}{\|T(\hat{P}, C)\|_\infty}. \quad (13)$$

As  $\|T(\hat{P}, C)\|_\infty = \|\Lambda^{-1}\|_\infty$ , this implies that

$$\left\| \begin{bmatrix} \tilde{D}_n \\ \tilde{N}_n \end{bmatrix} - \begin{bmatrix} \tilde{D} \\ \tilde{N} \end{bmatrix} \bar{U} \right\|_\infty \cdot \|\Lambda^{-1}\|_\infty \leq 1. \quad (14)$$

Lower bounding the left-hand term of this expression implies that

$$\left\| \begin{bmatrix} \tilde{D}_n \\ \tilde{N}_n \end{bmatrix} \Lambda^{-1} - \begin{bmatrix} \tilde{D} \\ \tilde{N} \end{bmatrix} \bar{U} \Lambda^{-1} \right\|_\infty \leq 1 \quad (15)$$

which proves the result.

The gap-metric uncertainty set can exhibit severe conservatism, as very well illustrated in, e.g., [12]. As the gap-metric does not take into account the closed-loop operation of the plant  $P$  in the set, induced by the controller  $C$  being used, this conservatism can intuitively be understood. In the situation that  $\Lambda = \alpha V$ , with  $\alpha \in \mathbb{R}$  and  $V$  a unitary matrix, it can be shown that the two sets in (10) are equal. For other situations, examples in [1] and [2] indicate a substantial decrease of conservatism when using  $\Lambda$ -gap uncertainty. The controller-relevant weighting within the  $\Lambda$ -gap is the basic reason for this.

## VII. CONCLUSIONS

The powerful standard representation for uncertainty descriptions in a basic perturbation model based on a standard plant configuration can be used to attain necessary and sufficient conditions for stability robustness within various uncertainty descriptions. In this paper these results are applied to uncertainty descriptions based on fractional model representations, leading to necessary and sufficient conditions for stability robustness in case of additive coprime factor uncertainties.

In this way a unified approach to handle additive coprime factor perturbations can be derived which yields a manageable and comprehensive way to relate gap and  $\Lambda$ -gap based uncertainty sets to (weighted) additive coprime factor perturbations. Based on this framework necessary and sufficient conditions for gap and  $\Lambda$ -gap based uncertainty sets are presented, and it is shown that in terms of stability robustness, the  $\Lambda$ -gap uncertainty set is less conservative than the gap uncertainty set.

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## New Square-Root Smoothing Algorithms

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**Abstract**—This paper presents new square-root smoothing algorithms for the three best-known smoothing formulas: 1) Rauch–Tung–Striebel (RTS) formulas, 2) Desai–Weinert–Yusypchuk (DWY) formulas, called backward RTS formulas, and 3) Mayne–Fraser (MF) formulas, called two-filter formulas. The main feature of the new algorithms is that they use unitary rotations to replace all matrix inversion and backsubstitution steps common in earlier algorithms with unitary operations; this feature enables more efficient systolic array and parallel implementations and leads to algorithms with better numerical stability and conditioning properties.

## I. INTRODUCTION

Square-root (or factorized, as they are sometimes called) algorithms for state-space estimation have been found to have several advantages over the conventional equation-based algorithms in terms of numerical stability, conditioning, and amenability to parallel and systolic implementation. While such algorithms for prediction and filtering have by now been studied quite extensively (see, e.g., [1]–[8]), the picture is not quite as complete for smoothing.

In the literature, there are two classes of square-root smoothing algorithms, both based on using quantities propagated by the square-root information filter algorithm (SRIF) presented by Dyer and McReynolds in 1969 [4]. In 1971, Kaminski [9] proposed the square-root information smoother (SRIS) of which Bierman in 1983 [10] gave a so-called UD (free of arithmetic square-root) version. The SRIF and SRIS propagate the square-root of the inverse of the filtering and smoothing error covariances, respectively, hence the name "information" form. In 1974, Bierman [11] proposed propagating the smoothing error covariance itself, using certain outputs from the SRIF to provide the coefficients of certain smoothing error covariance recursions. He called this the DMCS (Dyer–McReynolds Covariance Smoothing)—SRIF algorithm. A UD version of the DMCS–SRIF was given by Watanabe and Tzafestas [12]; see also McReynolds [13]. Watanabe [14] also gave a square-root form of certain smoothing formulas of Desai–Weinert–Yusypchuk (DWY) formulas [15], while Dobbins [16] derived a square-root version of the Mayne–Fraser (MF) (or two-filter) formulas.

These square-root algorithms have various advantages and disadvantages. However, all of them require certain matrix inversion and/or backsubstitution steps and, thus, none of them is particularly well-suited for parallel implementation. Recently, we have presented in [17] a new square-root smoothing algorithm for Bryson–Frazier (BF) formulas [18] (1963) that employs unitary rotations instead of matrix inversion and backsubstitution steps, thus simultaneously improving numerical stability and conditioning and also making parallel and systolic implementation easier—see, e.g., the discussion of these issues in [19] and [20].

There are essentially three more best-known smoothing formulas: those of Rauch–Tung–Striebel (RTS) [21] (1965), DWY [15] (1983), and Mayne [22] (1966) and Fraser [23] (1967). In this paper, we

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