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Identification of Linear Time-Invariant Systems via Constrained Step-Based Realization

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Abstract: A constrained step-based realization algorithm is developed to produce linear, time-invariant, state-space system estimates. To match *a priori* knowledge of the system behavior, the eigenvalues of the estimate are required to be stable, real, and positive; the step response is required to have no undershoot or overshoot; and the steady-state gain is required to match a known value. The standard step-based realization method is augmented to become a convex optimization problem subject to a linear-matrix inequality that constrains eigenvalue location, and a subsequent convex optimization problem is developed to constrain time-domain behavior. Simulation results motivate the need for such constraints and are used for comparison with familiar alternative methods. Although the procedure is applied only to step-response data, it may be generalized to constrain eigenvalues to convex regions of the complex plain and is applicable to all subspace identification methods.

Keywords: System identification; Subspace methods; Step function responses; Convex programming

1. INTRODUCTION

When identifying models of systems from measured data, it is often desirable that the identified model behave in agreement with *a priori* knowledge of the system. This is often limited to basic knowledge of system stability or an assumed model order, but other times this knowledge is derived from first-principles laws that govern the underlying system dynamics.

A simple example is the placement of a warm metal object into a large bath of ice water. Both intuition and the laws of entropy tell us that the temperature of the object will be cooled to the freezing temperature of the water, but no lower. The temperature change in time of the metal object can naturally be modeled as the step response of a linear dynamical system. If this process is identified from noisy data, however, the resulting model may contain complex eigenvalues, potentially resulting a model with a step response that undershoots its initial value or overshoots its steady-state value. Such a model would clearly violate the laws of thermodynamics.

This simple example exposes another difficulty that arises when a system is constrained to certain inputs. For many systems, experiments are limited to impulse or step response measurements. Such experiments lack stationarity and do not easily fit into existing frameworks based on prediction error minimization or maximum likelihood estimation [Ljung, 1999].

Although the example of the metal object can likely be modeled with a single time constant, many higher-order processes have similar constraints but with far more complex dynamics. Examples include the heating of a room, the cooking of food, the charging of batteries, and numerous applications in the fields of chemical engineering and semiconductor design.

Systems with such behavioral constraints are often modeled as resistor-inductor-capacitor circuits or mass-spring-damper systems that constrain behavior by design. The parameters of each element of the representative system are then estimated. Such "white-box" approaches, in an effort to constrain system behavior, risk eliminating models that would produce more accurate system descriptions while maintaining the desired behavioral constraints. Our goal is instead to *construct a model from a measured step response with only the constraints necessary to conserve the desired behavior.*

The problem of identifying linear dynamical models from step responses has been previously studied throughout the history of dynamical systems, and a thorough overview of the progress from early graphical methods to more modern techniques may be found in the references of Ahmed et al. [2007]. One issue with existing step-based identification methods is that many are derived in continuous time and require numerical approximation of derivatives, which may be extremely inaccurate for noisy data.

The step-based realization (SBR) method, first developed by van Helmont et al. [1990] and restated in a subspace framework by Miller and de Callafon [2009], generalizes classical realization theory to construct state-space models from discrete-time step-response measurements. Only linearity of the target system is assumed, and the model order is derived during the identification process. We will show

that the computational simplicity of the method allows for the incorporation of convex constraints, resulting in models with the same desired behaviors that many of the aforementioned "white-box" methods were developed to provide.

In this paper, we develop a constrained step-based realization (CSBR) method which constrains the eigenvalues of the identified model to lie within general convex regions of the complex plane. The approach is a generalization of the method developed by Lacy and Bernstein [2003], which computes stable system estimates by re-parameterizing a Frobenius norm minimization present in all subspace methods to be affine in the desired parameters. This norm is then minimized subject to a linear matrix inequality (LMI), also affine in the parameters, which constraines the eigenvalues of the system estimates to be within the unit circle. The more generalized LMI framework developed in this paper relies on a similar Frobenius-norm minimization and is extendable to all standard subspace methods. We then construct a second constrained minimization problem to insure that the identified model exhibits no overshoot or undershoot and has a pre-determined steady-state gain. Results of simulated experiments are included throughout the paper.

2. PRELIMINARIES

We consider the identification of a linear, time-invariant, single-input-multi-output system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) + v(t) \end{aligned} \tag{1}$$

in which $u(t) \in \{0,1\}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^{n_y}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{n_y \times n}$, and $D \in \mathbb{R}^{n_y}$. The input $u(t)$ is a unit step beginning at time $t = 0$, and $x(0) = 0$. The noise signal $v(t)$ is generated by a stationary, stochastic process and may be either white or colored. This model includes the case in which white noise is added to the state and output.

The state-space system (1) has an alternative representation

$$y(t) = \sum_{k=0}^{\infty} G(k)u(t-k) + v(t) \tag{2}$$

in which

$$G(k) = \begin{cases} D, & k = 0, \\ CA^{k-1}B, & k > 0 \end{cases}$$

are the system Markov parameters.

The standard notation of positive definiteness and semi-definiteness as > 0 and ≥ 0, respectively, is used.

We assume the following:

(i) The system is stable.
(ii) The steady-state value of the step response is known.
(iii) The system has strictly real eigenvalues.
(iv) The step response has no undershoot or overshoot.

We will begin by presenting an SBR method that incorporates none of this knowledge and gradually incorporates the information into the algorithm as the paper progresses.

3. STEP-BASED REALIZATION

Kalman's original realization problem concerned the construction of state-space models from experimentally observed impulse responses [Kalman, 1963]. The Ho-Kalman algorithm provided a solution to the realization problem via the decomposition of block-Hankel matrices of Markov parameters [Ho and Kalman, 1966]. This was later refined by Kung [1978] to use the singular-value decomposition (SVD), which can be shown to be optimal in a rank-reducing sense. The SBR method is a generalization of Kung's method to step responses. In this section, we first review the standard Ho-Kalman-Kung procedure for constructing state-space realizations from block-Hankel matrices of Markov parameters and then extend the method to constructing realizations from step responses.

3.1 The Ho-Kalman-Kung Realization Algorithm

Suppose the first $r + l \geq 2n + 1$ Markov parameters of (2) are known exactly. The block-Hankel matrices

$$H = \begin{bmatrix} G(1) & G(2) & \cdots & G(l) \\ G(2) & G(3) & \cdots & G(l+1) \\ \vdots & \vdots & & \vdots \\ G(r) & G(r+1) & \cdots & G(r+l-1) \end{bmatrix} \tag{3}$$

and

$$\overline{H} = \begin{bmatrix} G(2) & G(3) & \cdots & G(l+1) \\ G(3) & G(4) & \cdots & G(l+2) \\ \vdots & \vdots & & \vdots \\ G(r+1) & G(r+2) & \cdots & G(r+l) \end{bmatrix} \tag{4}$$

will have rank n for all $rn_y \geq n$ and $l \geq n$. Moreover, if $\mathcal{O}_r$ and $\mathcal{C}_l$ are the extended observability and controllability matrices, respectively, given by

$$\mathcal{O}_r = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} \quad \text{and} \quad \mathcal{C}_l = \begin{bmatrix} B & AB & \cdots & AB^{l-1} \end{bmatrix},$$

then

$$H = \mathcal{O}_r\mathcal{C}_l \qquad \text{and} \qquad \overline{H} = \mathcal{O}_r A\mathcal{C}_l.$$

If the Markov parameters are not exact but noise-corrupted estimates, then (3) will be full rank but close to a rank-n matrix in a 2-norm sense, so long as the noise on $G(k)$ is small relative to the singular values of H. Let H have the singular-value decomposition (SVD)

$$H = [U_n \; U_s] \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_s \end{bmatrix} \begin{bmatrix} V_n^T \\ V_s^T \end{bmatrix}.$$

The rank-n estimate of H

$$\hat{H} = \underset{\text{rank}(\hat{H})=n}{\arg\min} \left|\left| \hat{H} - H \right|\right|_2$$

is then

$$\hat{H} = U_n \Sigma_n V_n^T.$$

If the model order n is unknown, it can be estimated from the range of the singular values at this point. Estimates of $\mathcal{O}_r$ and $\mathcal{C}_l$ are then taken as

$$\hat{\mathcal{O}}_r = U_n \Sigma_n^{1/2} \qquad \text{and} \qquad \hat{\mathcal{C}}_l = \Sigma_n^{1/2} V_n^T.$$

The state-space system parameters are estimated as

$$\hat{A} = (\hat{\mathcal{O}}_r)^{\dagger} \overline{H} (\hat{\mathcal{C}}_l)^{\dagger} = \Sigma_n^{-1/2} U_n^T \overline{H} V_n \Sigma_n^{-1/2}$$

$$\hat{B} = \hat{\mathcal{C}}_{l(:,1)} \qquad \hat{C} = \hat{\mathcal{O}}_{r(1:n_y,:)} \qquad \hat{D} = G(0)$$

in which $(\cdot)^\dagger$ represents the Moore-Penrose pseudoinverse and the parenthesized subscripts of $\hat{\mathcal{C}}_l$ and $\hat{\mathcal{O}}_r$ represent MATLAB-style indexing. Further discussion can be found in Kung [1978] and in the literature of the Eigensystem Realization Algorithm [Juang and Pappa, 1985].

3.2 The Step-Based Realization Algorithm

The preceding algorithm may be generalized to step responses following the procedure of van Helmont et al. [1990]. Let $\{y(0),\ y(1),\ \ldots,\ y(N)\}$ be a measured response of an LTI system to a unit-step input applied at $t = 0$ that is corrupted by some possibly-colored measurement noise $v(t)$. Construct the block-Hankel data matrices

$$Y = \begin{bmatrix} y(1) & y(2) & \cdots & y(l) \\ y(2) & y(3) & \cdots & y(l+1) \\ \vdots & \vdots & & \vdots \\ y(r) & y(r+1) & \cdots & y(N-1) \end{bmatrix}$$

and

$$\overline{Y} = \begin{bmatrix} y(2) & y(3) & \cdots & y(l+1) \\ y(3) & y(4) & \cdots & y(l+2) \\ \vdots & \vdots & & \vdots \\ y(r+1) & y(r+2) & \cdots & y(N) \end{bmatrix}.$$

These satisfy the equations

$$Y = HU_p + TU + V \tag{5}$$
$$\overline{Y} = \overline{H}U_p + \overline{T}U + \overline{V} \tag{6}$$

in which H and $\overline{H}$ are given by (3) and (4), respectively,

$$T = \begin{bmatrix} G(0) & & & \\ G(1) & G(0) & & \\ \vdots & \vdots & \ddots & \\ G(r-1) & G(r-2) & \cdots & G(0) \end{bmatrix}$$

and

$$\overline{T} = \begin{bmatrix} G(1) & G(0) & & \\ G(2) & G(1) & \ddots & \\ \vdots & \vdots & \ddots & G(0) \\ G(r) & G(r+1) & \cdots & G(1) \end{bmatrix}$$

are block-Toeplitz matrices of Markov parameters, U is an $(r-1)\times(l-1)$ matrix of '1's, and U_p is an $(l-1)\times(l-1)$ upper-triangular matrix of '1's.

The matrix U has rank 1, and cannot therefore be used to create a projector to separate the row spaces of T and H, as is done in traditional projector-based subspace identification. Because U is a matrix of '1's, however, each row of TU contains a sum of the columns of T. The result is a column-identical matrix

$$TU = \begin{bmatrix} G(0) & G(0) & \cdots \\ G(0)+G(1) & G(0)+G(1) & \cdots \\ \vdots & \vdots & \\ \sum_{k=0}^{r-1} G(k) & \sum_{k=0}^{r-1} G(k) & \cdots \end{bmatrix}.$$

An equivalent result holds for $\overline{T}U$. The products TU and $\overline{T}U$ cannot be calculated exactly for noisy measurements, but in the noise-free case, they are equal to $M = TU$ and $\overline{M} = \overline{T}U$, respectively, in which

$$M = \begin{bmatrix} y(0) & y(0) & \cdots \\ y(1) & y(1) & \cdots \\ \vdots & \vdots & \\ y(r-1) & y(r-1) & \cdots \end{bmatrix}, \quad \overline{M} = \begin{bmatrix} y(1) & y(1) & \cdots \\ y(2) & y(2) & \cdots \\ \vdots & \vdots & \\ y(r) & y(r) & \end{bmatrix}.$$

Hence we use the approximations $TU \approx M$ and $\overline{T}U \approx \overline{M}$ to approximate HU_p and $\overline{H}U_p$ as

$$HU_p \approx Y - M = R$$
$$\overline{H}U_p \approx \overline{Y} - \overline{M} = \overline{R}$$

and perform the realization algorithm on R and $\overline{R}$ instead of H and $\overline{H}$. Taking the SVD

$$R = [U_n\ U_s] \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_s \end{bmatrix} \begin{bmatrix} V_n^T \\ V_s^T \end{bmatrix},$$

an appropriate system order n may be found from the range of the singular values. Once a suitable n has been chosen, we compute the estimates

$$\hat{\mathcal{O}}_r = U_n \Sigma_n^{1/2} \qquad \hat{\mathcal{C}}_l U_p = \Sigma_n^{1/2} V_n^T$$

and estimate A as

$$\hat{A} = (\hat{\mathcal{O}}_r)^\dagger \overline{R} (\hat{\mathcal{C}}_r U_p)^\dagger = \Sigma_n^{-1/2} U_n^T \overline{R} V_n \Sigma_n^{-1/2}.$$

which is the solution of the Frobenius-norm minimization problem

$$\begin{aligned} \hat{A} &= \arg\min_{\hat{A}} \left|\left| \hat{\mathcal{O}}_r \hat{A} \hat{\mathcal{C}}_r U_p - \overline{R} \right|\right|_F \\ &= \arg\min_{\hat{A}} \left|\left| U_n \Sigma_n^{1/2} \hat{A} \Sigma_n^{1/2} V_n^T - \overline{R} \right|\right|_F. \end{aligned} \tag{7}$$

As with the standard realization algorithm, C is estimated as

$$\hat{C} = \hat{\mathcal{O}}_{r(1:n_y,:)}. \tag{8}$$

Because the first column of U_p is $[1\ 0\ 0\ \cdots]^T$, the first column of $\mathcal{C}_l U_p$ is B, and a possible estimate for B is

$$\hat{B} = \hat{\mathcal{C}}_{r(:,1)} U_p = (\Sigma_n^{1/2} V_n^T)_{(:,1)}$$

with D then estimated as $\hat{D} = y(0) \approx \hat{G}(0)$. Improved estimates of B and D, however, may instead be found via linear least-squares for a more optimal fit. Given estimates $\hat{A}$ and $\hat{C}$, let $\hat{B}$ and $\hat{D}$ be the solution of

$$\hat{B},\ \hat{D} = \arg\min_{\hat{B},\ \hat{D}} ||y - \hat{y}||_2 \tag{9}$$

$$y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N+i) \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} \hat{y}(0) \\ \hat{y}(1) \\ \vdots \\ \hat{y}(N+i) \end{bmatrix} \tag{10}$$

$$\hat{y}(t) = \begin{bmatrix} \sum_{k=0}^{t-1} \hat{C}\hat{A}^{t-k-1} & 1 \end{bmatrix} \hat{\theta} \qquad \hat{\theta} = \begin{bmatrix} \hat{B} \\ \hat{D} \end{bmatrix} \tag{11}$$

See, for instance, Verhaegen and Verdult [2007, Sec. 9.2] for additional discussion.

3.3 Simulation Experiments with SBR

To demonstrate the SBR algorithm, we apply it to a step response of the system

$$G(q) = \frac{0.004(q - 0.5)}{(q - 0.95)(q - 0.9)(q - 0.6)} \tag{12}$$

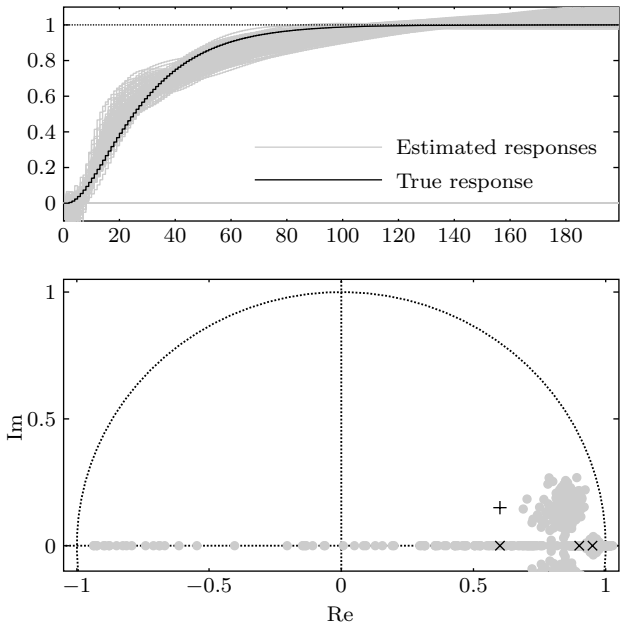

Fig. 1. Step responses of 300 SBR estimates (top) and pole locations (bottom). Estimate poles are gray dots. Poles of $G(q)$ are 'x' and those of $V(q)$ are '+'.

with signals

$$y(t) = G(q)u(t) + v(t),$$

where $v(t)$ is defined as

$$v(t) = V(q)e(t),$$

in which

$$V(q) = \frac{0.25(q^2 - 1.2q + 0.93)}{(q^2 - 1.2q + 0.3825)}$$

and $e(t)$ is a white noise signal with variance $\sigma^2 = 0.005$. The noise-generating system $V(q)$ has poles at $0.6 \pm i0.15$. This system has strictly real eigenvalues with both "slow" and "fast" response characteristics often seen in thermodynamic processes. We assume beforehand that the system has a single time delay, a steady-state value of 1, and meets the assumptions listed in Section 2; we wish for our model to have these characteristics as well.

Simulations of 300 step responses of length $N = 200$ were measured. The number of rows chosen for the SBR data matrices was $r = 15$. D was assumed to be 0. A sample identification result and pole locations for the 300 estimates are shown in Figure 1. Note that not every response is entirely shown due to the limits of the y-axis.

We conclude that the SBR method fails to consistently provide estimates with the desired behavior. We could repeat the experiment and average the responses until the model is accurate enough that the complex part of the poles may be ignored, but repeating experiments an arbitrary number of times is typically infeasible. We propose to instead apply constraints to the estimation procedure so that the estimate lies within a model set for which the desired behavior is guaranteed.

4. CONSTRAINED STEP-BASED REALIZATION

We begin by constraining the eigenvalues of the system estimate. We require that the eigenvalues of the identified model be (i) stable, (ii) real, and (iii) positive so as to not exceed the Nyquist sampling frequency of the system. To define these constraints as a convex optimization problem, we use the concept of LMI regions, first introduced in Chilali and Gahinet [1996]. We then develop additional constraints which incorporate all knowledge of our system into the identification procedure.

4.1 LMI Regions

An LMI region is a convex region $\mathcal{D}$ of the complex plane, defined in terms of a symmetric matrix α and a square matrix β, as

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) \geq 0\} \tag{13}$$

where

$$f_{\mathcal{D}}(z) = \alpha + \beta z + \beta^T \bar{z}.$$

The original definition of an LMI region in Chilali and Gahinet [1996] has $<$ in place of $\geq$. We adopt the above definition instead so that our results are straightforward to implement as a semi-definite program and because the real-number line cannot be parameterized as an LMI region if (13) uses a strict inequality.

We will call $f_{\mathcal{D}}(z)$ for a given $\mathcal{D}$ the describing function of $\mathcal{D}$. In Chilali and Gahinet [1996], it is shown that the eigenvalues of a matrix A lie within the region $\mathcal{D}$ if and only if

$$\mathcal{M}_{\mathcal{D}}(A,\ P) \geq 0, \qquad P = P^T \geq 0,$$

in which

$$\mathcal{M}_{\mathcal{D}}(A,\ P) = \alpha \otimes P + \beta \otimes (AP) + \beta^T \otimes (AP)^T. \tag{14}$$

LMI regions generalize standard notions of stability for both continuous and discrete-time systems. For example, the region formed by the unit disc and its interior is described by

$$f_{\mathcal{D}}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{z}.$$

This results in the LMI constraint

$$\mathcal{M}_{\mathcal{D}}(A,\ P) = \begin{bmatrix} P & AP \\ PA^T & P \end{bmatrix} \geq 0, \tag{15}$$

the Schur complement of which is, of course, the discrete-time Lyapunov equation.

The intersection of two LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ is also an LMI region, described by the matrix function

$$f_{\mathcal{D}_1 \cap \mathcal{D}_2}(z) = \begin{bmatrix} f_{\mathcal{D}_1}(z) & 0 \\ 0 & f_{\mathcal{D}_2}(z) \end{bmatrix}. \tag{16}$$

In general, the $(\alpha,\ \beta)$ pair that describes an LMI region is not unique.

4.2 LMI Regions for Stable, Real, and Positive Eigenvalues

We have already introduced the LMI region that constrains eigenvalues to lie on and within the unit disc. To provide a greater margin of stability for the identified models, we instead constrain eigenvalues to the disc of radius $1 - \delta_s$. The LMI region consisting of complex numbers for which

$$|z| \leq 1 - \delta_s$$

has a describing function with parameters

$$\alpha_s = (1 - \delta_s)I_2, \qquad \beta_s = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{17}$$

The LMI region that describes the real number line is given by the describing function

$$\mathbb{R} = \{z : f_{\mathbb{R}}(z) \geq 0\},$$
$$f_{\mathbb{R}}(z) = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} z \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \bar{z},$$

but a corresponding eigenvalue constraint would be computationally infeasible for optimization. Instead, we include a relaxation parameter to describe an arbitrarily small band around the real axis in the complex plane. The LMI region consisting of complex numbers for which

$$|\mathrm{Im}(z)| \leq \delta_r$$

has a describing function with parameters

$$\alpha_r = \delta_r I_2, \qquad \beta_r = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}. \tag{18}$$

The parameter δ_r can be made small enough so that the complex parts of the resulting identified eigenvalues are near machine precision.

Finally, we wish to construct an LMI region that describes the positive right-half plane. This region should also include a relaxation parameter so that the region begins some distance away from the imaginary axis. The LMI region consisting of complex numbers for which

$$\mathrm{Re}(z) \geq \delta_p$$

has a describing function with parameters

$$\alpha_p = \delta_p \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \beta_p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{19}$$

The three LMI regions described by (17), (18), and (19) may be combined using the identity (16) to form a single LMI constraint, generated by the function (14).

4.3 Incorporating Eigenvalue Constraints into SBR

To create the CSBR method by incorporating the LMI constraints, we first restate the minimization problem (7) to include a weighting term W as

$$J_0(\hat{A},\ W) = \left|\left|(\hat{\mathcal{O}}_r \hat{A} \hat{\mathcal{C}}_r U_p - \overline{R})W\right|\right|_F. \tag{20}$$

Let

$$W = (\mathcal{C}_r U_p)^{\dagger} P = V_n \Sigma_n^{-1/2} P,$$

so that (20) becomes

$$J_1(\hat{A},\ P) = \left|\left|U_n \Sigma_n^{1/2} \hat{A} P - \overline{R} V_n \Sigma_n^{-1/2} P\right|\right|_F. \tag{21}$$

Now re-parameterize (21) with the auxiliary term $Q = \hat{A}P$, to form the complete problem statement of the CSBR,

$$\begin{aligned} &\text{minimize} && J(Q,\ P) \\ &\text{subject to} && \mathcal{M}(Q,\ P) \geq 0, \quad P = P^T \geq 0 \end{aligned}$$

in which

$$\begin{aligned} J(Q,\ P) &= \left|\left|U_n \Sigma_n^{1/2} Q - \overline{R} V_n \Sigma_n^{-1/2} P\right|\right|_F \\ \mathcal{M}(Q,\ P) &= \mathrm{diag}(\mathcal{M}_s,\ \mathcal{M}_r,\ \mathcal{M}_p) \\ \mathcal{M}_s &= \begin{bmatrix} (1-\delta_s)P & Q \\ Q^T & (1-\delta_s) \end{bmatrix} \\ \mathcal{M}_r &= \begin{bmatrix} \delta_r P & 0.5(Q^T - Q) \\ 0.5(Q - Q^T) & \delta_r P \end{bmatrix} \\ \mathcal{M}_p &= \begin{bmatrix} 2\delta_p P & 0 \\ 0 & Q + Q^T - 2\delta_p P \end{bmatrix}. \end{aligned}$$

This can be shown to be equivalent to a convex linear programming problem with mixed equality, quadratic, and positive semidefinite constraints [Lacy and Bernstein, 2003]. Once solutions for Q and P are found, let $\hat{A} = QP^{-1}$ and take $\hat{C}$ from (8) as before.

4.4 Constraining Step-Response Undershoot, Overshoot, and Steady-State Gain

Returning to the least-squares problem of (9), we wish to constrain the step-response of the estimate to exhibit no overshoot, no undershoot, and have steady-state value y_∞, that is

$$0 \leq \hat{y}(t) \leq y_\infty \ \forall\ t \quad \text{and} \quad \hat{C}(I_n - \hat{A})^{-1}\hat{B} + \hat{D} = y_\infty \tag{22}$$

with $\hat{y}(t)$ defined as in (11). We thereby replace (9) with a constrained least-squares problem that enforces (22). Given estimates $\hat{A}$ and $\hat{C}$, solve the following problem:

$$\begin{aligned} &\text{minimize} && J(\theta) = ||y - \psi\theta||_2 \\ &\text{subject to} && \psi\theta \geq 0,\ \psi\theta \leq y_\infty,\ \gamma\theta = y_\infty \end{aligned}$$

in which

$$\psi = \begin{bmatrix} \psi(0) \\ \psi(1) \\ \vdots \\ \psi(N+i) \end{bmatrix} \quad \psi(t) = \begin{bmatrix} \sum_{k=0}^{t-1} \hat{C}\hat{A}^{t-k-1} & 1 \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \hat{C}(I_n - \hat{A})^{-1} & 1 \end{bmatrix} \qquad \theta = \begin{bmatrix} \hat{B} \\ \hat{D} \end{bmatrix}$$

This is a convex quadratic-programming problem easily solved by modern numerical optimization techniques.

4.5 Simulation Experiments with CSBR

The constrained SBR method was applied to the same 300 data sets generated for the experiments in Section 3.3. The results are shown in Figure 2. The minimization problem was solved using `CVX` [Grant and Boyd, 2011] with `SDPT3` [Toh et al., 2009] as the selected solver.

Also shown for comparison are ARX estimates in Figure 3 and output-error (OE) estimates in Figure 4 [Ljung, 1999]. The ARX estimates, while having comparatively low variance in pole location, are consistently biased due to the noise effects. The OE estimates have a high variation in pole location and suffer from severe convexity issues due to the short length and little excitation of the measured data sequences. For both methods, a sequence of 10 zeros was prepended to the input and output data to provide sufficient backward steps for the prediction-error criteria.

5. CONCLUSIONS

The method developed in this paper allows for the identification of systems with the eigenvalues of the estimate constrained to be real, stable, and positive. Additionally, a subsequent optimization constrains the response to have no overshoot or undershoot. Each step is a well-behaved convex optimization problem. By incorporating *a priori* assumptions concerning the underlying system behavior into the identification algorithm, we are able to compensate for the lack of excitation of a step-response measurement.

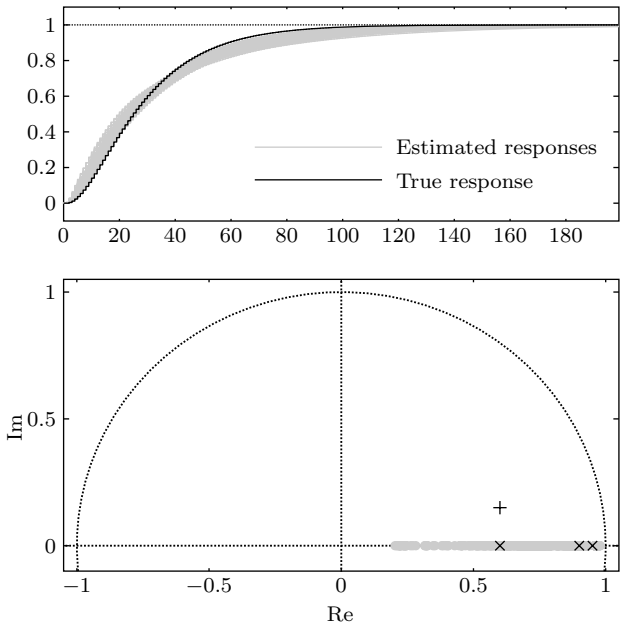

Fig. 2. Step responses of 300 CSBR estimates (top) and pole locations (bottom). Estimate poles are gray dots. Poles of $G(q)$ are 'x' and those of $V(q)$ are '+'.

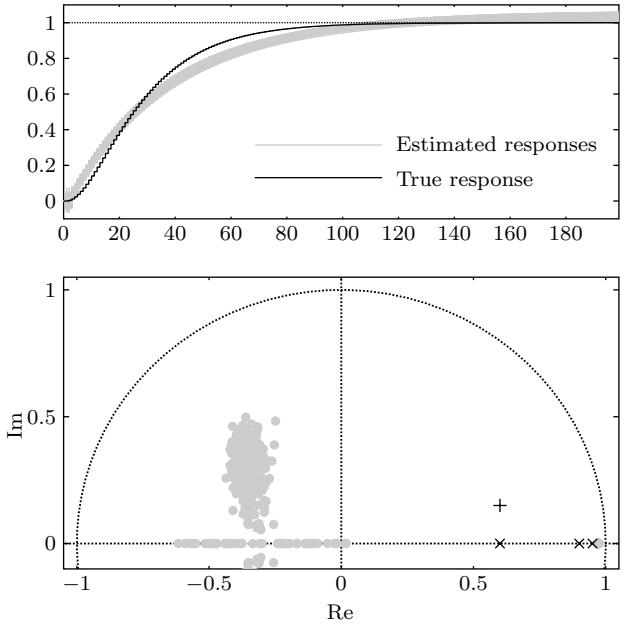

Fig. 3. Step responses of 300 ARX estimates (top) and pole locations (bottom). Estimate poles are gray dots. Poles of $G(q)$ are 'x' and those of $V(q)$ are '+'.

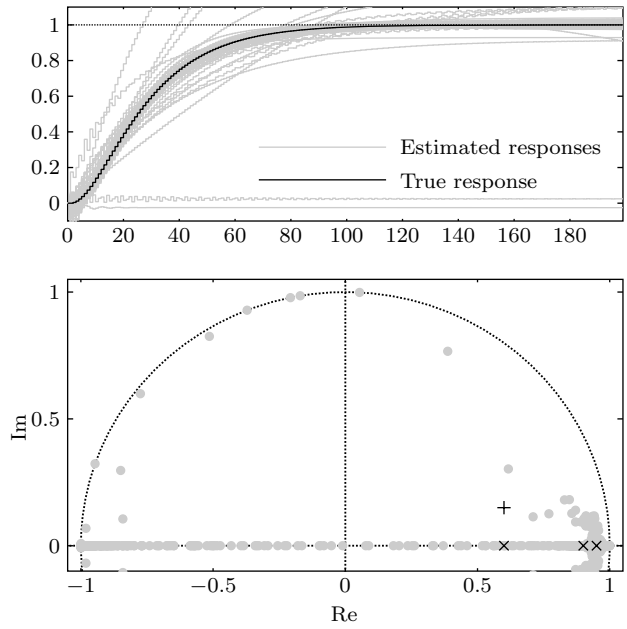

Fig. 4. Step responses of 300 OE estimates (top) and pole locations (bottom). Estimate poles are gray dots. Poles of $G(q)$ are 'x' and those of $V(q)$ are '+'.

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