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On the asymptotic stability of minimum-variance unbiased input and state estimation ☆

Huazhen Fang [1], Raymond A. de Callafon

Department of Mechanical & Aerospace Engineering, University of California, San Diego, La Jolla, CA 92037, USA

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ABSTRACT

In this note, we investigate the asymptotic stability of the filter for minimum-variance unbiased input and state estimation developed by Gillijns and De Moor. Sufficient conditions for the stability are proposed and proven, with inspiration from the Kalman filter stability analysis.

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1. Introduction

Motivated by a wide range of applications, optimal input and state estimation has been a topic of intense interest recently with the aim to use output measurements of a linear dynamic system to reconstruct both the state and input of the system. Gillijns and De Moor (2007a) proposed a recursive filter and designed the optimal gains using the principle of minimum-variance unbiased estimation (MVUE). The linear time-invariant system under consideration is

$$x_{k+1} = Ax_k + Gd_k + w_k, \tag{1}$$

$$y_k = Cx_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $d_k \in \mathbb{R}^m$ is the unknown input vector, and $y_k \in \mathbb{R}^p$ is the output measurement. The vectors $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^p$ are mutually uncorrelated white noise sequences, with known covariance matrices $Q \geq 0$ and $R > 0$, respectively. For the system in (1)–(2), a recursive linear filter of the following form is designed:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1}, \tag{3}$$

$$\hat{d}_{k-1} = M_k(y_k - C\hat{x}_{k|k-1}), \tag{4}$$

$$\hat{x}^*_{k|k} = \hat{x}_{k|k-1} + G\hat{d}_{k-1}, \tag{5}$$

$$\hat{x}_{k|k} = \hat{x}^*_{k|k} + K_k(y_k - C\hat{x}^*_{k|k}). \tag{6}$$

In the above filter, $\hat{x}_{k|k-1}$, which is the estimate of Ax_k, is obtained first. The estimate $\hat{d}_{k-1}$ is computed from y_k, since y_k is the first measurement that contains information about d_{k-1}. Then $\hat{d}_{k-1}$ can be used together with $\hat{x}_{k|k-1}$ to yield $\hat{x}^*_{k|k-1}$, which is an *a priori*, unbiased estimate of x_k. This step is based on the state equation in (1). Finally, the *a posteriori* estimate $\hat{x}_{k|k}$ is obtained by updating $\hat{x}^*_{k|k-1}$ with a correction term. Here, $M_k \in \mathbb{R}^{m\times p}$ and $K_k \in \mathbb{R}^{n\times p}$ are gain matrices adjustable to achieve optimal estimation performance. In Gillijns and De Moor (2007a), the determination of optimal M_k and H_k is based on the MVUE approach.

For input estimation, a necessary and sufficient condition for unbiasedness is required, that is characterized by

$$M_kCG = I \tag{7}$$

for all k. Then M_k is derived by minimum-variance estimation using least squares under the constraint (7). The state estimation is transformed into a standard Kalman filtering problem, with K_k determined by minimizing the estimation variance of $\hat{x}_{k|k}$. It is further proven in Gillijns and De Moor (2007a) that the optimal input estimate over the class of all linear unbiased estimates can be written in the proposed recursive form. Optimality of the state estimator has been proven in Kerwin and Prince (2000).

An important but unexplored problem in Gillijns and De Moor (2007a) is: *under what conditions the designed filter achieves time*

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E-mail addresses: hzfang@ucsd.edu (H. Fang), callafon@ucsd.edu (R.A. de Callafon).

[1] Tel.: +1 858 534 3166; fax: +1 858 822 3107.

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invariance and is asymptotically stable? It is well known that stability of the classical Kalman filter depends on detectability of (A, C) and stabilizability of $(A, Q^{\frac{1}{2}})$ — see more details in Anderson and Moore (1979) and Maybeck (1979). In this note, we will develop a parallel to establish the asymptotic stability properties of the filter in Gillijns and De Moor (2007a). The results can be extremely useful not only in behavior analysis of the input and state filter, but also in its practical implementation. We follow the notation and terminology from Gillijns and De Moor (2007a), to which the reader is encouraged to refer for more information.

2. Asymptotic stability analysis

Assume $X \in \mathbb{S} = \{S \in \mathbb{R}^{n\times n} | S = S^{\mathrm{T}}, S \geq 0\}$, $M \in \mathbb{R}^{m\times p}$ and $K \in \mathbb{R}^{n\times p}$. We define the operator

$$\phi(M, K, X) = \tilde{A}X\tilde{A}^{\mathrm{T}} + \tilde{F}Q\tilde{F}^{\mathrm{T}} + \tilde{G}R\tilde{G}^{\mathrm{T}},$$

where

$$\tilde{A}(M, K) = (I - KC)(I - GMC)A, \tag{8}$$

$$\tilde{F}(M, K) = -(I - KC)(I - GMC), \tag{9}$$

$$\tilde{G}(M, K) = (I - KC)GM + K. \tag{10}$$

Note that the filter in (3)–(6) can be rewritten as

$$\hat{d}_{k-1} = M_k(y_k - CA\hat{x}_{k-1|k-1}), \tag{11}$$

$$\hat{x}_{k|k} = A\hat{x}_{k-1|k-1} + G\hat{d}_{k-1} + K_k(y_k - CA\hat{x}_{k-1|k-1} - CG\hat{d}_{k-1}). \tag{12}$$

Define $\tilde{x}_k = x_k - \hat{x}_{k|k}$. It can be verified that

$$\tilde{x}_k = \tilde{A}(M_k, K_k)\tilde{x}_{k-1} - \begin{bmatrix}\tilde{F}(M_k, K_k) & \tilde{G}(M_k, K_k)\end{bmatrix}\begin{bmatrix} w_{k-1} \\ v_k \end{bmatrix}. \tag{13}$$

The associated covariance matrix is given by

$$P_{k|k} = \phi(M_k, K_k, P_{k-1|k-1}), \tag{14}$$

where ϕ can be determined from (13).

The filter designed in Gillijns and De Moor (2007a) has global optimality. First, it yields state estimation equivalent to the filters in Darouach and Zasadzinski (1997) and Kitanidis (1987) — both are globally optimal (Kerwin & Prince, 2000). Second, the global optimality of input estimation depends on the filter form and the state estimation. As the proposed recursive form is proven optimal and state estimation is optimal as aforementioned, the input estimation is also globally optimal. Thus we have the following lemma.

Lemma 1 (*Gillijns & De Moor, 2007a*)**.** *The gain matrices M_k and K_k designed in Theorems 2 and 7 in Gillijns and De Moor (2007a), denoted as M_k^* and K_k^*, respectively, minimize the estimation covariance $P_{k|k}$ subject to the unbiasedness constraint (7).*

If letting

$$\psi(X) = \phi(M^*, K^*, X) = \min_{\substack{M,K \\ \text{s.t. } MCG=I}} \phi(M, K, X), \tag{15}$$

where M^* and K^* are the optimal M and K, then Lemma 1 implies for the designed optimal filter that

$$P_{k|k} = \psi(P_{k-1|k-1}). \tag{16}$$

As pointed out in Anderson and Moore (1979), we are usually concerned with two properties for the filter. The first one, *asymptotic time invariance*, is whether the recursive computation in (16) leads to a fixed point $\bar{P}$, where

$$\lim_{k\to\infty} P_{k|k} = \bar{P}. \tag{17}$$

If this happens, $\bar{P}$ is also the asymptotically constant (i.e., limiting) solution of (16). That is, $\bar{P}$ satisfies

$$\bar{P} = \psi(\bar{P}). \tag{18}$$

The other one is *asymptotic stability*. It is about whether the filter is asymptotically stable given the limiting gain matrices associated with $\bar{P}$, denoted as $\bar{M}^*$ and $\bar{K}^*$. We are interested in deriving conditions that guarantee both the properties simultaneously.

Let us introduce the notation $\lambda_i(X)$ to denote the ith eigenvalue of a square matrix X. We give the lemma below as the initial step for the property analysis.

Lemma 2. *If there exist $M \in \mathbb{R}^{m\times p}$ and $K \in \mathbb{R}^{n\times p}$ such that M satisfies (7) and*

$$\left|\lambda_i\left[\tilde{A}(M, K)\right]\right| < 1 \tag{19}$$

for $i = 1, 2, \ldots, n$, then the sequence $\{P_{k|k}\}$ is bounded, i.e., $P_{k|k} < \infty$ for any k given any initial condition $0 \leq P_{0|0} \leq \infty$.

Proof. To assist in showing this, let us construct a suboptimal filter. Choose fixed $M^s \in \mathbb{R}^{m\times p}$ and $K^s \in \mathbb{R}^{n\times p}$ such that both (7) and (19) are satisfied. Following (11), consider the unbiased suboptimal filter

$$\hat{d}^s_{k-1} = M^s(y_k - CA\hat{x}^s_{k-1}),$$

$$\hat{x}^s_k = A\hat{x}^s_{k-1} + G\hat{d}^s_{k-1} + K^s(y_k - CA\hat{x}^s_{k-1} - CG\hat{d}^s_{k-1}).$$

Then following (13), the state estimation error $\tilde{x}^s_k = x_k - \hat{x}^s_k$ is given by

$$\tilde{x}^s_k = \tilde{A}(M^s, K^s)\tilde{x}^s_{k-1} - \begin{bmatrix}\tilde{F}(M^s, K^s) & \tilde{G}(M^s, K^s)\end{bmatrix}\begin{bmatrix} w_{k-1} \\ v_k \end{bmatrix},$$

with the associated covariance matrix P^s_{k+1} is given by

$$P^s_k = \phi(M^s, K^s, P^s_{k-1}).$$

Note that the suboptimal filter is asymptotically stable, since $\left|\lambda_i\left[\tilde{A}(M^s, K^s)\right]\right| < 1$ for all i. Thus P^s_k is bounded for any nonnegative initial condition. Comparing the above suboptimal filter to the designed optimal filter, we note that, if both of them are initialized by $P_{0|0}$, then the optimality suggests $P_{k|k} \leq P^s_k$. This proves the boundedness of $P_{k|k}$. □

The following theorem establishes conditions for asymptotic time invariance and stability of the consequent time-invariant filter, extending the ideas used in Kalman filter analysis in Anderson and Moore (1979) and Lancaster and Rodman (1995).

Theorem 1. *If there exist $M \in \mathbb{R}^{m\times p}$ and $K \in \mathbb{R}^{n\times p}$ satisfying (7) and (19), and if $(A, Q^{\frac{1}{2}})$ is stabilizable, then $P_{k|k}$ converges to a unique fixed point $\bar{P}$ for any initial condition $P_{0|0}$, where $\bar{P}$ is the unique positive semi-definite solution of $\bar{P} = \psi(\bar{P})$. Moreover, with the associated limiting gain matrices $\bar{M}^*$ and $\bar{K}^*$, the time-invariant filter is stable, i.e., $\left|\lambda_i\left[\tilde{A}(\bar{M}^*, \bar{K}^*)\right]\right| < 1$ for $i = 1, 2, \ldots, n$.*

The proof is organized as follows. First, it is illustrated that $P_{k|k}$ is monotonically increasing and converges to a fixed point with zero initial condition. Second, the asymptotic stability is shown by proving that the time-invariant steady-state filter is stable. Finally, the convergence of $P_{k|k}$ and the stability of the filter are proven to hold for arbitrary nonnegative initial conditions.

Proof of Theorem 1. If $P_{0|0} = 0$, $P_{k|k}$ is found monotonically increasing and thus convergent due to its boundedness proven in Lemma 2. To show this, we note that

$$P_{0|0} \leq P_{1|1},$$

when $P_{0|0} = 0$. Repeatedly using the fact that, if $X \leq Y$, then

$$\psi(X) = \phi(M_X^*, K_X^*, X) \leq \phi(M_Y^*, K_Y^*, X) \leq \phi(M_Y^*, K_Y^*, Y) = \psi(Y),$$

we obtain

$$P_{k|k} \leq P_{k+1|k+1}. \tag{20}$$

Thus $P_{k|k}$ is monotonically increasing for zero initial conditions. With the boundedness of $P_{k|k}$ proven in Lemma 2, we can conclude that $P_{k|k}$ converges to a fixed point $\bar{P}$, which is the solution of (14).

For notational convenience, define $\bar{\tilde{A}} = \tilde{A}(\bar{M}^*, \bar{K}^*)$, $\bar{\tilde{F}} = \tilde{F}(\bar{M}^*, \bar{K}^*)$ and $\bar{\tilde{G}} = \tilde{G}(\bar{M}^*, \bar{K}^*)$. Let us now show asymptotic stability, resorting to proof by contradiction. Assume that the time-invariant filter is unstable. Then there exist some eigenvalue $|\lambda| \geq 1$ and corresponding eigenvector ω such that $\bar{\tilde{A}}\omega = \lambda\omega$. From (16) we have

$$\begin{aligned}
\omega^*\bar{P}\omega &= \omega^*\bar{\tilde{A}}\bar{P}\bar{\tilde{A}}^{\mathrm{T}}\omega + \omega^*\bar{\tilde{F}}Q\bar{\tilde{F}}^{\mathrm{T}}\omega + \omega^*\bar{\tilde{G}}R\bar{\tilde{G}}^{\mathrm{T}}\omega \\
&= |\lambda|^2\omega^*\bar{P}\omega + \omega^*\bar{\tilde{F}}Q\bar{\tilde{F}}^{\mathrm{T}}\omega + \omega^*\bar{\tilde{G}}R\bar{\tilde{G}}^{\mathrm{T}}\omega,
\end{aligned}$$

where the superscript * denotes the complex conjugate as no confusion arises. Equivalently,

$$(1-|\lambda|^2)\omega^*\bar{P}\omega = \omega^*\bar{\tilde{F}}Q\bar{\tilde{F}}^{\mathrm{T}}\omega + \omega^*\bar{\tilde{G}}R\bar{\tilde{G}}^{\mathrm{T}}\omega.$$

Since $|\lambda| \geq 1$, then both sides must be 0 such that the equality holds. Then,

$$\left(\bar{\tilde{F}}Q^{\frac{1}{2}}\right)^{\mathrm{T}}\omega = -\left[\left(I - \bar{\tilde{G}}C\right)Q^{\frac{1}{2}}\right]^{\mathrm{T}}\omega = 0,$$

$$\bar{\tilde{G}}^{\mathrm{T}}\omega = 0,$$

from which we have

$$\bar{\tilde{A}}^{\mathrm{T}}\omega = \left(A - \bar{\tilde{G}}CA\right)^{\mathrm{T}}\omega = A^{\mathrm{T}}\omega = \lambda\omega,$$

$$Q^{\frac{1}{2}\mathrm{T}}\omega = 0,$$

since $Q \geq 0$ and $R > 0$. The above equations indeed imply $(A, Q^{\frac{1}{2}})$ is not stabilizable. The contradiction disproves the assumption that the time-invariant filter is unstable.

We now demonstrate that $P_{k|k}$ approaches $\bar{P}$ for any $P_{0|0} \geq 0$. Comparing $P_{k|k}$ and $\bar{P}$, we have

$$\begin{aligned}
P_{k|k} - \bar{P} &= \phi(M_k^*, K_k^*, P_{k-1|k-1}) - \phi(\bar{M}^*, \bar{K}^*, \bar{P}) \\
&\leq \phi(\bar{M}^*, \bar{K}^*, P_{k-1|k-1}) - \phi(\bar{M}^*, \bar{K}^*, \bar{P}) \\
&= \bar{\tilde{A}}(P_{k-1|k-1} - \bar{P})\bar{\tilde{A}}^{\mathrm{T}}.
\end{aligned}$$

As shown before, $\bar{\tilde{A}}$ is stable. The right hand side obviously approaches 0 as $k \to \infty$, thus proving

$$\lim_{k\to\infty} P_{k|k} \leq \bar{P}. \tag{21}$$

On the other hand,

$$P_{k|k} \geq \psi^k(0),$$

which implies

$$\lim_{k\to\infty} P_{k|k} \geq \lim_{k\to\infty} \psi^k(0) = \bar{P}. \tag{22}$$

Consequently, we can conclude from (21)–(22) that (17) holds for an arbitrary positive semi-definite $P_{0|0}$.

Finally, the uniqueness of $\bar{P}$ can be established. Assume that there exists another fixed point $\bar{P}'$, and that $P_{0|0} = \bar{P}'$. It is observed from the previous proof that $P_{k|k} \to \bar{P}$. Thus $\bar{P}' = \bar{P}$. □

For linear time-invariant systems *with direct feedthrough*, unbiased minimum-variance input and state estimation is studied analogously in Gillijns and De Moor (2007b). Consider a linear system

$$x_{k+1} = Ax_k + Gd_k + w_k, \tag{23}$$

$$y_k = Cx_k + Hd_k + v_k, \tag{24}$$

the filter for which is designed as

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + G\hat{d}_{k-1}, \tag{25}$$

$$\hat{d}_k = M_k(y_k - C\hat{x}_{k|k-1}), \tag{26}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C\hat{x}_{k|k-1}). \tag{27}$$

Here, M_k and L_k are gain matrices, which are determined via MVUE in Gillijns and De Moor (2007b). The global optimality of the above filter is proven in Hsieh (2010).

Accordingly, define the following operator

$$\begin{aligned}
\varphi(X) = {} & (A - AL^*C - GM^*C)X(A - AL^*C - GM^*C)^{\mathrm{T}} + Q \\
& + (AL^* + GM^*)R(AL^* + GM^*)^{\mathrm{T}},
\end{aligned}$$

where

$$M^* = \left[H^{\mathrm{T}}\tilde{R}^{-1}H\right]^{-1}H^{\mathrm{T}}\tilde{R}^{-1},$$

$$L^* = XC^{\mathrm{T}}\tilde{R}^{-1},$$

$$\tilde{R} = CXC^{\mathrm{T}} + R.$$

The property analysis results for the filter in (25)–(27) are presented in the next theorem.

Theorem 2. *For the linear time-invariant system in Gillijns and De Moor (2007b), if there exist $M \in \mathbb{R}^{m\times p}$ and $L \in \mathbb{R}^{n\times p}$ satisfying*

$$MH = I,$$

$$LH = 0,$$

$$\left|\lambda_i\left[A - (AL + GM)C\right]\right| < 1$$

for all i, and if $(A, Q^{\frac{1}{2}})$ is stabilizable, then $P_{k|k-1}$ converges to a unique fixed point $\bar{P}$ for any initial condition $P_{0|-1}$. $\bar{P}$ is the unique positive semi-definite solution of the equation $\bar{P} = \varphi(\bar{P})$. Moreover, with the associated limiting gain matrices, the time-invariant filter is stable.

The proof can be done along the same line as that of Theorem 1, so we only provide a sketch. The case of zero initial condition is first studied. It can be proven that $P_{k|k-1}$ will converge to a fixed point $\bar{P} = \varphi(\bar{P})$ if $P_{0|-1} = 0$. This is from the boundedness and increasing monotonicity of $P_{k|k-1}$ with $P_{0|-1} = 0$. The obtained time-invariant filter can also be shown to be stable. Next, convergence of $P_{k|k-1}$ and stability of the time-invariant filter can then be readily proven for an arbitrary initial condition, yielding the statement of Theorem 2.

3. Conclusion

In this note, we have analyzed the asymptotic stability properties for unbiased minimum-variance input and state estimation in Gillijns and De Moor (2007a), with results extended to Gillijns and De Moor (2007b). Sufficient conditions are proposed to ensure that the state estimation error covariance approaches to a unique fixed point, which is the solution of a matrix equation. Furthermore, it is shown that the time-invariant filter is stable. The results are an extension of the Kalman filter stability theory to the filter for input and state estimation in the minimum-variance unbiased sense.

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