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Stability Analysis and Application of Kalman Filtering with Irregularly Sampled Measurements

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Abstract— **This paper analyzes the peak covariance stability properties of Kalman filtering for linear discrete-time systems with irregular time intervals for sampling of output measurements. Existing research on Kalman filtering with irregular sampling mostly builds on probabilistic description of sampling intermittence. In this work, we focus on the general case of irregular sampling without probabilistic assumptions. The obtained stability conditions show that the peak covariance stability is influenced by the eigenvalue distribution of the state matrix in the complex plane. The effectiveness of the analysis is illustrated via a simulation based study on ocean flow field estimation using submersible drogues that can measure position intermittently and acceleration incessantly.**

I. INTRODUCTION

The classical discrete-time KF is premised on the implicit assumption of regular sampling of the output measurements. However, it is sometimes unrealistic to obtain measurements regularly in a few situations. Especially in applications of process control, networked control systems, navigation and guidance, and ocean surveillance, irregular time sampling intervals are used to monitor a system. Thus the KF analysis under irregular time sampling has been given much emphasis recently.

When irregular sampling occurs, boundedness of the estimation error covariance in the KF has been studied extensively in the setting that the availability of measurements follows a stochastic process. It is common to employ an auxiliary binary random variable γ_k: $\gamma_k = 1$ or 0 denotes the measurement is available or not at time instant k, respectively. The sequence $\{\gamma_k\}$ are usually considered to belong to either of the following two categories:

- **Bernoulli process:** γ_k's are i.i.d. random variables, with the probability distribution of $p(\gamma_k = 1) = \lambda$ and $p(\gamma_k = 0) = 1 - \lambda$. A lead has been taken in [1] with the conclusion that there exists a critical probability λ_c. If $\lambda > \lambda_c$, then the *expectation* of the resulting estimation error covariance P_k is guaranteed bounded (given the usual stabilizability and detectability hypotheses), and divergent otherwise. Determination of the value of λ_c is investigated in [2] and [3]. A novel probability-based metric is proposed in [4] to evaluate the KF performance. Specifically, the upper and lower bounds of $\Pr(P_k \leq M)$ for a given M, are derived to assess the KF.
- **Two-state Markovian process:** Another interesting exploration is to consider the Gilbert-Elliott channel model, in which $\{\gamma_k\}$ is modeled as a two-state Markov chain. Sufficient conditions are derived in [5] for the boundedness of the expectation of the peak estimation error covariance. Some less conservative results are obtained in [6], [7]. It is shown in [8] that the estimation error covariance of the KF with Markovian intermittent measurements has a power-law decay, with the critical probability derived.

The following observations have motivated our research to develop results for KF with irregular time sampling schemes:

- **Observation 1:** Probability distributions for intermittent sampling are not easy to determine in practice. It is a big challenge to know the type of the probability distribution and the accurate probability values.
- **Observation 2:** Stochastic modeling may fail if the measurements are sampled irregularly but not randomly in many real-life applications.

In this paper, we will study the KF under general irregular sampling without stochastic modeling. Irregular sampling has been examined in the literature on system identification and nonlinear predictive control, see [9]–[11]. For a continuous-time system, state observability from irregularly sampled measurements is discussed in [12] and [13]. However, it is noted that analysis of the discrete-time KF with general irregular sampling schemes has seldom been discussed. We will hence contribute to this topic in this paper through studying the boundedess conditions for the KF's estimation error covariance.

II. PROBLEM FORMULATION

Consider a linear discrete-time system Σ_d with the following state equation:

$$\Sigma_d : \quad x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the input vector, and $w \in \mathbb{R}^n$ is the process noise vector, which is white Gaussian with zero mean and covariance matrix $Q \geq 0$. Here, the subscript k of a vector a_k denotes the k-th sampling time instant, i.e., $a_k := a(kT_s)$, where T_s is the standard sampling period. Sampling of the output measurement of Σ_d is not regular in temporal scale. The sampling durations are not fixed but the integer multiples of T_s. It is convenient to use the binary variable γ_k to denote the availability of the measurement at time instant k. If the measurement is sampled, $\gamma_k = 1$; and $\gamma_k = 0$

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otherwise. With the initial condition $\gamma_0 = 1$, let us introduce the following time sequence of measurement sampling:

$$t_0 = 1, \quad t_\ell = \inf\{k : k > t_{\ell-1}, \gamma_k = 1\} \text{ for } \ell = 1, 2, \cdots$$

The new time index t_ℓ denotes the time instant when the ℓ-th measurement is sampled. We define the set

$$\mathbb{T} = \{t_{\ell+1} - t_\ell : \ell = 0, 1, \cdots\}$$

which is the collection of all irregular sampling intervals. The output equation of Σ_d is given by

$$y_{t_\ell} = C x_{t_\ell} + v_{t_\ell} \tag{2}$$

where $C \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ is the output vector, and $v \in \mathbb{R}^m$ is the measurement noise vector, which is independent of w, also white and Gaussian with zero mean and covariance matrix $R > 0$.

For Σ_d, we adopt the following standard assumptions throughout the paper:

(A1) (A, C) is detectable;
(A2) $(A, Q^{\frac{1}{2}})$ is stabilizable;
(A3) $\sup \mathbb{T} < \infty$.

The assumptions (A1)-(A2) are well known to guarantee that, under regular single-rate sampling, the estimation error covariance of the Kalman filter converges to a unique fixed point from any initial condition [14]. We have (A3) established, since effective state estimation requires that the sampling intervals be finite. If the matrix A is stable, it is a trivial task to design a stable state estimator even under irregular sampling. Thus only the unstable case is particularly worth studying. In sequel, only an unstable A will be considered.

We use the lifting technique [15] to deal with irregular sampling. The lifted system Σ_l can be built from Σ_d:

$$\Sigma_l : \begin{cases} x_{t_{\ell+1}} = \Phi(t_{\ell+1}, t_\ell) x_{t_\ell} + \Gamma(t_{\ell+1}, t_\ell)\tilde{u}_{t_{\ell+1}, t_\ell} & \text{(3a)} \\ \qquad\qquad +\Omega(t_{\ell+1}, t_\ell)\tilde{w}_{t_{\ell+1}, t_\ell} & \text{(3b)} \\ y_{t_\ell} = C x_{t_\ell} + v_{t_\ell} & \text{(3c)} \end{cases}$$

where

$$\begin{aligned} \Phi(t_{\ell+1}, t_\ell) &= A^{t_{\ell+1}-t_\ell} \\ \Gamma(t_{\ell+1}, t_\ell) &= \begin{bmatrix} A^{t_{\ell+1}-t_\ell-1}B & \cdots & AB & B \end{bmatrix} \\ \Omega(t_{\ell+1}, t_\ell) &= \begin{bmatrix} A^{t_{\ell+1}-t_\ell-1} & \cdots & A & I \end{bmatrix} \\ \tilde{u}_{t_{\ell+1}, t_\ell} &= \begin{bmatrix} u_{t_\ell}^{\mathrm{T}} & \cdots & u_{t_{\ell+1}-2}^{\mathrm{T}} & u_{t_{\ell+1}-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ \tilde{w}_{t_{\ell+1}, t_\ell} &= \begin{bmatrix} w_{t_\ell}^{\mathrm{T}} & \cdots & w_{t_{\ell+1}-2}^{\mathrm{T}} & w_{t_{\ell+1}-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

We denote the covariance matrix of $\tilde{w}_{t_{\ell+1}, t_\ell}$ be

$$\Delta(t_{\ell+1}, t_\ell) = \mathrm{diag}(\underbrace{Q, \cdots, Q}_{t_{\ell+1}-t_\ell})$$

It is noted that the system Σ_l is linear time-varying. The KF can be applied to *optimal state estimation* for Σ_l. With the initializing condition $\hat{x}_{t_{-1}}^+ = \mathrm{E}(x_{t_{-1}})$, $P_{t_{-1}}^+ = E\left[(x_{t_{-1}} - \hat{x}_{t_{-1}}^+)(x_{t_{-1}} - \hat{x}_{t_{-1}}^+)^{\mathrm{T}}\right]$. the KF is given in a two-step procedure for $\ell \geq 1$:

- Step 1: Prediction

$$\begin{aligned} \hat{x}_{t_\ell}^- &= \Phi(t_\ell, t_{\ell-1})\hat{x}_{t_{\ell-1}}^+ + \Gamma(t_\ell, t_{\ell-1})\tilde{u}_{t_{\ell-1}} & (4) \\ P_{t_\ell}^- &= \Phi(t_\ell, t_{\ell-1})P_{t_{\ell-1}}^+\Phi^{\mathrm{T}}(t_\ell, t_{\ell-1}) \\ &\quad +\Omega(t_\ell, t_{\ell-1})\Delta(t_\ell, t_{\ell-1})\Omega^{\mathrm{T}}(t_\ell, t_{\ell-1}) & (5) \end{aligned}$$

- Step 2: Update

$$\begin{aligned} K_{t_\ell} &= P_{t_\ell}^- C^{\mathrm{T}}(C P_{t_\ell}^- C^{\mathrm{T}} + R)^{-1} & (6) \\ \hat{x}_{t_\ell}^+ &= \hat{x}_{t_\ell}^- + K_{t_\ell}(y_{t_\ell} - C\hat{x}_{t_\ell}^-) & (7) \\ P_{t_\ell}^+ &= (I - K_{t_\ell}C)P_{t_\ell}^- & (8) \end{aligned}$$

where the superscripts "-" and "+" denote *a priori* and *a posteriori*, respectively, and $\hat{x}$ and P are the state estimate and estimation error covariance matrix, respectively. At inter-sample time instants, due to the lack of output measurements, the propagation of state estimate is dependent only on the state equation (1). Hence,

$$\begin{aligned} \hat{x}_k^- &= A\hat{x}_{k-1}^+ + Bu_{k-1} & (9) \\ P_k^- &= AP_{k-1}^+A^{\mathrm{T}} + Q & (10) \\ \hat{x}_k^+ &= \hat{x}_k^- & (11) \\ P_k^+ &= P_k^- & (12) \end{aligned}$$

for $t_\ell < k < t_{\ell+1}$.

The lifted KF in (6)-(12) can be decomposed into single steps, resorting to using γ_k's. Its equivalent expression is given by the following equations:

$$\begin{aligned} \hat{x}_k^- &= A\hat{x}_{k-1}^+ + Bu_{k-1} & (13) \\ P_k^- &= AP_{k-1}^+A^{\mathrm{T}} + Q & (14) \\ K_k &= P_k^- C^{\mathrm{T}}\left(CP_k^-C^{\mathrm{T}} + R\right)^{-1} & (15) \\ \hat{x}_k^+ &= \hat{x}_k^- + \gamma_k K_k\left(y_k - C\hat{x}_k^-\right) & (16) \\ P_k^+ &= \left(I - \gamma_k K_k C\right)P_k^- & (17) \end{aligned}$$

If γ_k's are Bernoulli or two-state Markov random variables, there are a variety of works in the literature devoted to studying the behavior of the binary switching Kalman filter in (13)-(17), e.g., [1], [4], [5], to mention them. In this paper, we will explore a general situation, breaking away from the assumption that γ_k's are random variables.

III. Peak Covariance Stability Analysis

With the inspiration from the literature, e.g., [5], we shall focus on analysis of peak covariance stability defined in this section.

Definition 1: The peak covariance sequence $\{P_{t_\ell}\}$ is said to be stable if $P_{t_\ell} < \infty \ \forall \ell \geq 1$ for $P_{t_0} < \infty$. The filtering system satisfies peak covariance stability if $\{P_{t_\ell}\}$ is stable.

A. Pathological & Degenerate Sampling Intervals

It is easily verifiable that the KF performance may suffer seriously from the irregular sampling scheme. This indicates that the intervals between two consecutive samples can influence the KF behavior significantly.

Before proceeding further, let us investigate the eigenvalues of A. Suppose that A has l distinct eigenvalues, μ_i for $i = 1, 2 \cdots, l$, each with multiplicities h_i. Obviously, $\Sigma_{i=1}^{l} h_i = n$. These eigenvalues lie on different eigencircles centered around the original point, depending on their own magnitudes. If $|\mu_i| = |\mu_j|$, $\forall i, j \in \{1, 2, \cdots, l\}$, they are on the same eigencircle, the radius of which is given by $|\mu_i|$ or equally $|\mu_j|$. This leads to the definition about pathological sampling intervals:

Definition 2: The sampling interval $t_{\ell+1} - t_\ell$ for $\ell = 0, 1, 2, \cdots$ is pathological if $t_{\ell+1} - t_\ell \in \mathbb{P}$, where

$$\begin{aligned} \mathbb{P} \quad = \quad & \Big\{ \kappa \in \mathbb{N}\backslash\{0,1\} : \left(\frac{\mu_i}{\mu_j}\right)^{\kappa} = 1, |\mu_i| = |\mu_j| > 1, \\ & \forall i, j \in \{1, 2, \cdots, l\} \Big\} \end{aligned} \tag{18}$$

Otherwise, it is non-pathological.

Definition 2 can be briefly explained in the following way. Let μ_i and μ_j be two distinct unstable eigenvalues of A with $|\mu_i| = |\mu_j|$. Note that the eigenvalues of $A^{t_{\ell+1}-t_\ell}$ are equal to the $t_{\ell+1} - t_\ell$ power of the eigenvalues of A. Thus the sampling operation is equivalent to mapping the eigenvalues of A to new points on the complex plane. Lying on the same eigencircle, μ_i and μ_j will be mapped to one coincidence point and can not be distinguished any more, if the sampling interval $t_{\ell+1} - t_\ell$ is pathological. The next lemma further shows the property of the set $\mathbb{P}$.

Lemma 1: Assume that μ is any unstable eigenvalue of A ($|\mu| \geq 1$). For all $\kappa \in \mathbb{N}\backslash\{0,1\}$, if $\mu e^{j2\pi i/\kappa}$ for any $i = 1, 2, \cdots, \kappa - 1$ is an eigenvalue of A, then $\kappa \in \mathbb{P}$.

Proof: It is straightforward to see that

$$\left(\frac{\mu}{\mu e^{j2\pi i/\kappa}}\right)^{\kappa} = 1$$

implying $\kappa \in \mathbb{P}$. In addition, for all $\kappa \in \mathbb{P}$, there exists $i = 1, 2, \cdots, \kappa - 1$ such that $\mu e^{j2\pi i/\kappa}$ is an eigenvalue of A. ■

If any sampling interval is pathological, we continue to check if it is 'degenerate'. It is known that there always exists a similarity transformation M such that $A_J := M^{-1}AM$, where A_J is the Jordan canonical form of A. Accordingly, denote $C_J := CM$. For all $\kappa \in \mathbb{P}$, define $\mathbb{G}_{\kappa,\alpha}$ as the set of eigenvalues with the same modulus α and suffering from the pathological sampling interval κ:

$$\begin{aligned} \mathbb{G}_{\kappa,\alpha} \quad = \quad & \Big\{ \mu_i, \mu_j : \left(\frac{\mu_i}{\mu_j}\right)^{\kappa} = 1, |\mu_i| = |\mu_j| = \alpha, \kappa \in \mathbb{P}, \\ & \forall i, j \in \{1, 2, \cdots, h\} \Big\} \end{aligned} \tag{19}$$

Let $A_{J,\mathbb{G}_{\kappa,\alpha}}$ be the block in A_J, with the diagonal entries being the elements of $\mathbb{G}_{\kappa,\alpha}$, and $C_{J,\mathbb{G}_{\kappa,\alpha}}$ be the associated block in C_J.

Definition 3: The sampling interval $t_{\ell+1} - t_\ell$ for $\ell = 0, 1, 2, \cdots$ is degenerate if $t_{\ell+1} - t_\ell \in \mathbb{D}$, where

$$\mathbb{D} := \left\{ \kappa \in \mathbb{P} : C_{J,\mathbb{G}_{\kappa,\alpha}} \text{ is rank deficient} \right\} \tag{20}$$

Otherwise, it is non-degenerate.

It is obvious that $\mathbb{D} \subseteq \mathbb{P}$. Indeed, a non-degenerate sampling interval is non-pathological, but not vice versa.

B. Peak Covariance Stability

We have the following main theorem.

Theorem 1: If $\mathbb{T} \cap \mathbb{D} = \emptyset$, the Kalman filter with irregularly sampled measurements satisfies peak covariance stability.

To prove Theorem 1, the following lemmas will be needed.

Lemma 2: If $\mathbb{T} \cap \mathbb{P}$, the pair (A^κ, C) is detectable for any $k \in \mathbb{T}$.

Proof: The reader may refer to [15] [16] [17] for some existing discussion on the lemma. Here we give a proof for completeness.

Let μ be any unstable eigenvalue of A. Define the complex function

$$f(s) := \frac{s^\kappa - \mu^\kappa}{s - \mu}$$

Note that $f(s)$ is analytic everywhere. Its zeros are given by $\mu e^{j2\pi i/\kappa}$ for $i = 1, 2, \cdots, \kappa - 1$. By the *spectral mapping theorem*, the eigenvalues of $f(A)$ are the values of $f(s)$ at the eigenvalues of A. According to the definition of $\mathbb{P}$ in (18), none of the points $\mu e^{j2\pi i/\kappa}$, where $i = 1, 2, \cdots, \kappa - 1$, is an eigenvalues of A. Hence, 0 is also not an eigenvalue of $f(A)$, implying that $f(A)$ is invertible.

It follows from the invertibility of $f(A)$ that

$$\begin{bmatrix} A^\kappa - \mu^\kappa I \\ C \end{bmatrix} = \begin{bmatrix} f(A) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \mu I \\ C \end{bmatrix} \tag{21}$$

where μ^κ is the eigenvalue of A^κ. By (A1), we have $\operatorname{rank}\left[\, A^{\mathrm{T}} - \mu I \quad C^{\mathrm{T}} \,\right] = n$. Hence, we obtain from (21) that $\operatorname{rank}\left[\, (A^\kappa)^{\mathrm{T}} - \mu^\kappa I \quad C^{\mathrm{T}} \,\right] = n$ This shows that (A^κ, C) is detectable. ■

The lemma below can be developed on the basis of Lemma 2.

Lemma 3: If $\mathbb{T} \cap \mathbb{D} = \emptyset$, the pair (A^κ, C) is detectable for any $k \in \mathbb{T}$.

Proof: For simplicity and without loss of generality, assume that all l eigenvalues of A are unstable, and that $\mathbb{G}_\kappa = \{\mu_{l-1}, \mu_l\}$, that is, μ_{l-1} and μ_l are the only eigenvalues affected by the pathological κ. Then we drop the subscript 'α' here for convenience. It is obvious from Lemma 2 that $\operatorname{rank}\left[\, (A^\kappa)^{\mathrm{T}} - \mu_i^\kappa I \quad C^{\mathrm{T}} \,\right] = n$ for $i = 1, 2, \cdots, l-2$. After Jordan canonical transformation, A_J and C_J are given by

$$\begin{aligned} A_J &= \operatorname{diag}\left([\, A_{J,1} \quad \cdots \quad A_{J,l-1} \quad A_{J,l} \,]\right) \\ C_J &= [\, C_{J,1} \quad \cdots \quad C_{J,l-1} \quad C_{J,l} \,] \end{aligned}$$

where $A_{J,i}$ is the Jordan block associated with $\mu_i$and $C_{J,i}$ is the corresponding block in C. Define

$$\begin{aligned} A_{J,\bar{\mathbb{G}}_\kappa} &:= \operatorname{diag}\left([\, A_{J,1} \quad \cdots \quad A_{J,l-2} \,]\right) \\ C_{J,\bar{\mathbb{G}}_\kappa} &:= [\, C_{J,1} \quad \cdots \quad C_{J,l-2} \,] \\ A_{J,\mathbb{G}_\kappa} &:= \operatorname{diag}\left([\, A_{J,l-1} \quad A_{J,l} \,]\right) \\ C_{J,\mathbb{G}_\kappa} &:= [\, C_{J,l-1} \quad C_{J,l} \,]. \end{aligned}$$

For $i = l-1, l$, $\left[\, \left(A^\kappa_{J,\bar{\mathbb{G}}_\kappa}\right)^{\mathrm{T}} - \mu_i^\kappa I \quad C^{\mathrm{T}}_{J,\bar{\mathbb{G}}_\kappa} \,\right]$ is of full rank by Lemma 2, and so is $\left[\, \left(A^\kappa_{J,\mathbb{G}_\kappa}\right)^{\mathrm{T}} - \mu_i^\kappa I \quad C^{\mathrm{T}}_{J,\mathbb{G}_\kappa} \,\right]$ due to the full rank of $C_{J,\mathbb{G}_\kappa}$. We additionally have $A^\kappa_{J,\bar{\mathbb{G}}_\kappa} - \mu_i^\kappa I$ is of full rank as its diagonal elements are nonzero. Consequently, for $i = l-1, l$, $\mathrm{rank}\,[\, A_J^{\kappa} - \mu_i^{\kappa} I \quad C_J^{\mathrm{T}} \,] = n$. Thus we have $\mathrm{rank}\,[\, (A^{\kappa})^{\mathrm{T}} - \mu^{\kappa} I \quad C^{\mathrm{T}} \,] = n$, where μ is any eigenvalue of A. ■

The proof of Theorem 1 is as follows.

Proof of Theorem 1: If $\mathbb{T} \cap \mathbb{P} = \emptyset$, it follows from Lemmas 2 and 3 that, (A^{κ}, C) is detectable for $\kappa \in \mathbb{T}$. In other words, $(\Phi(t_{\ell+1}, t_{\ell}), C)$ is detectable for $\ell = 0, 1, \cdots$. Thus there always exists $K_{t_\ell}^{\mathrm{sub}}$ such that $\Phi(t_{\ell+1}, t_{\ell}) - K_{t_\ell}^{\mathrm{sub}} C$ is stable. We can then construct a suboptimal estimator $\hat{x}_{t_{\ell+1}}^{\mathrm{sub}} = \Phi(t_{\ell+1}, t_{\ell}) \hat{x}_{t_\ell}^{\mathrm{sub}} + K_{t_\ell}^{\mathrm{sub}} (y_{t_\ell} - C \hat{x}_{t_\ell}^{\mathrm{sub}})$. The estimation error is given by

$$\begin{aligned} \tilde{x}_{t_{\ell+1}}^{\mathrm{sub}} &:= x_{t_{\ell+1}} - \hat{x}_{t_{\ell+1}}^{\mathrm{sub}} \\ &= \left[\Phi(t_{\ell+1}, t_{\ell}) - K_{t_\ell}^{\mathrm{sub}} C\right] \tilde{x}_{t_\ell}^{\mathrm{sub}} + w_{t_\ell} - K_{t_\ell}^{\mathrm{sub}} v_{t_\ell} \end{aligned}$$

It is observed that the above error performance system is stable. Then the associated estimation error covariance $P_{t_\ell}^{\mathrm{sub}}$ is upper bounded. Furthermore, given the same initial condition, $P_{t_\ell} \le P_{t_\ell}^{\mathrm{sub}}$, due to the KF's optimality. Thus if $P_{t_0} < \infty$, then $P_{t_\ell} < \infty$. The proof is complete. ■

Theorem 1 shows that, the peak covariance stability of the filtering system with irregular sampling is guaranteed if the sampling intervals are non-degenerate (or futhermore, non-pathological). From Theorem 1, some corollaries can be derived to further reveal the stability properties.

Corollary 1: If the matrix A has no two or more distinct unstable eigenvalues with the same modulus, the peak covariance sequence is stable.

Proof: For the given condition, $\mathbb{P} = \emptyset$ and thus $\mathbb{T} \cap \mathbb{P} = \emptyset$ is satisfied, ensuring the peak covariance stability. ■

Corollary 2: For any eigenvalues of A that have the same modulus, if the corresponding block of the matrix C in Jordan canonical form is of full column rank, then the peak covariance sequence is stable.

Proof: Note that $\mathbb{D} = \emptyset$ is satisfied in this case, so the peak covariance stability is ensured. ■

A question of interest here is: *Can we determine the accurate upper and lower bounds of the peak covariance?* Currently, there is no definite answer for the general case. However, if narrowing the scope to scalar systems, we can obtain some desirable conclusions.

For an scalar system ($n = 1$), the peak covariance stability is always guaranteed, given the assumptions (A1)-(A3). Define $\tau := \inf \mathbb{T}$ and $\eta := \sup \mathbb{T}$, where $1 \le \tau \le \eta < \infty$. Suppose there are two associated linear multi-rate systems, which are generated by sampling the output of the system Σ_d every τT_s and ηT_s, respectively. Implementation of the KF to both systems yields two prediction error covariance matrices, denoted by $P_{\tau,\ell}$ and $P_{\eta,\ell}$, respectively.

To proceed further, let us define the following mappings

$$f(X) := AXA^{\mathrm{T}} + Q \tag{22}$$
$$g(X) := X - XC^{\mathrm{T}}(CXC^{\mathrm{T}} + R)^{-1}CX \tag{23}$$

where $X \ge 0$. It is noted that

$$P_{t_{\ell+1}} = \underbrace{f \circ f \circ \cdots \circ f}_{t_{\ell+1} - t_\ell} \circ g(P_{t_\ell}) =: f^{t_{\ell+1} - t_\ell} \circ g(P_{t_\ell})$$

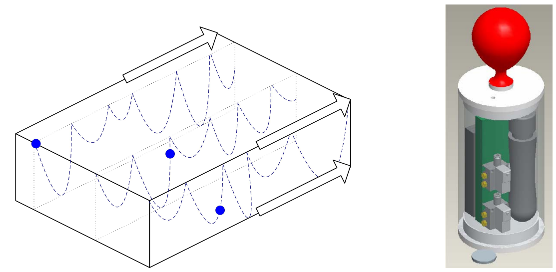

Fig. 1. (a) Diagrammatic sketch of ocean flow field estimation (arrows: flow direction, filled circles: drogues, dashed lines: depth profiles of drogues); (b) the prototype of the drogue to be used.

where '$\circ$' denotes function composition. Similarly, $P_{\tau,\ell+1} = f^{\tau} \circ g(P_{\tau,\ell})$, and $P_{\eta,\ell+1} = f^{\eta} \circ g(P_{\eta,\ell})$.

Theorem 2: Assume $n = 1$. If the initial condition satisfies $0 \le P_{\tau,0} \le P_{t_0} \le P_{\eta,0}$, then

$$P_{\tau,\ell} \le P_{t_\ell} \le P_{\eta,\ell} \tag{24}$$

Furthermore, let $\bar{P}_{\tau}$ and $\bar{P}_{\eta}$, respectively, be the unique positive solutions to the auxiliary equations $X = f^{\tau} \circ g(X)$ and $X = f^{\eta} \circ g(X)$. Then

$$\bar{P}_{\tau} \le \lim_{\ell \to \infty} P_{t_\ell} \le \bar{P}_{\eta} < \infty \tag{25}$$

Proof: It is proven by [1] and [4] that $f(X)$ and $g(X)$ are monotonically increasing with X. In addition, for $a \le b$, we have $f^{a}(X) \le f^{b}(X)$, since $n = 1$ and A is unstable. Given $0 \le P_{\tau,0} \le P_{t_0} \le P_{\eta,0}$, it follows that

$$0 \le f^{\tau} \circ g(P_{\tau,0}) \le f^{t_1 - t_0} \circ g(P_{t_0}) \le f^{\eta} \circ g(P_{\eta,0})$$

or equivalently

$$0 \le P_{\tau,1} \le P_{t_1} \le P_{\eta,1}$$

Repeating the above procedure inductively until ℓ, we obtain (24). For the associated multi-rate systems, there exist unique $\bar{P}_{\tau}$ and $\bar{P}_{\eta}$ such that $\bar{P}_{\tau} = f^{\tau} \circ g(\bar{P}_{\tau})$ and $\bar{P}_{\eta} = f^{\eta} \circ g(\bar{P}_{\eta})$, respectively, in light of the theories of solutions to Riccati equations. Hence, (25) is obtained from (24). ■

IV. Application Example

In this section, we show the application of the KF with irregularly sampled measurements to ocean flow estimation. The scenario is shown in Fig. 1(a). A three-dimensional ocean flow velocity field is considered. It is assumed to be only dependent on depth but independent of time. A few drogues are released at different locations, and then travels along the flow through the field. Capable of arbitrary vertical migration behaviors, each drogue has a motion of submersion and surfacing, featuring different depth profiles. During the process, the acceleration information of each drogue is recorded, but the position information is available only when the drogue is at water surface. Incessant acceleration and intermittent position measurements will be used for reconstruction of the velocity profile of the drogue to estimate the flow field.

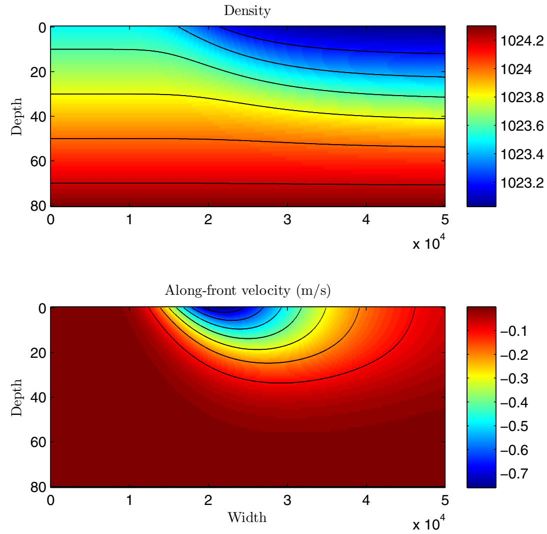

Fig. 2. The density and flow field profiles at a cross section.

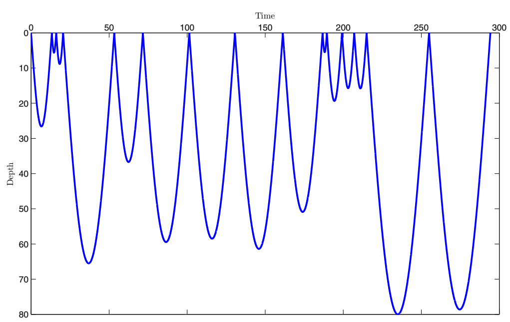

Fig. 3. The depth profile of the drogue released at 2.5×10^4m.

The prototype of the drogue is shown in Fig. 1(b). We have the drogue dynamics described by the differential equation [18]

$$m\ddot{d}(t) = c \cdot \left(v_d(z,t) - \dot{d}(t)\right) \cdot \left|v_d(z,t) - \dot{d}(t)\right| \tag{26}$$

where m is the constant rigid mass plus added mass, d is the displacement of the drogue, and c is the drag parameter that quantifies the drag or resistance applied on the drogue in the flow field. It is understood that the drogue has the irregular sampling feature. When both underwater and at surface, the drogue samples its acceleration $\ddot{d}$ and depth z regularly. However, the displacement d is irregularly measurable – it can be obtained only when the drogue is at the surface. Here, the objective is to estimate $v_d(z,t)$, using (26) and the measurements.

Define the states $x_1 := d$ and $x_2 := \dot{d}$; also define the acceleration term as the input to system because the acceleration is available every time instant, that is,

$$u := \frac{c}{m} \cdot \left(v_d(z,t) - \dot{d}(t)\right) \cdot \left|v_d(z,t) - \dot{d}(t)\right| \tag{27}$$

It then follows that

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

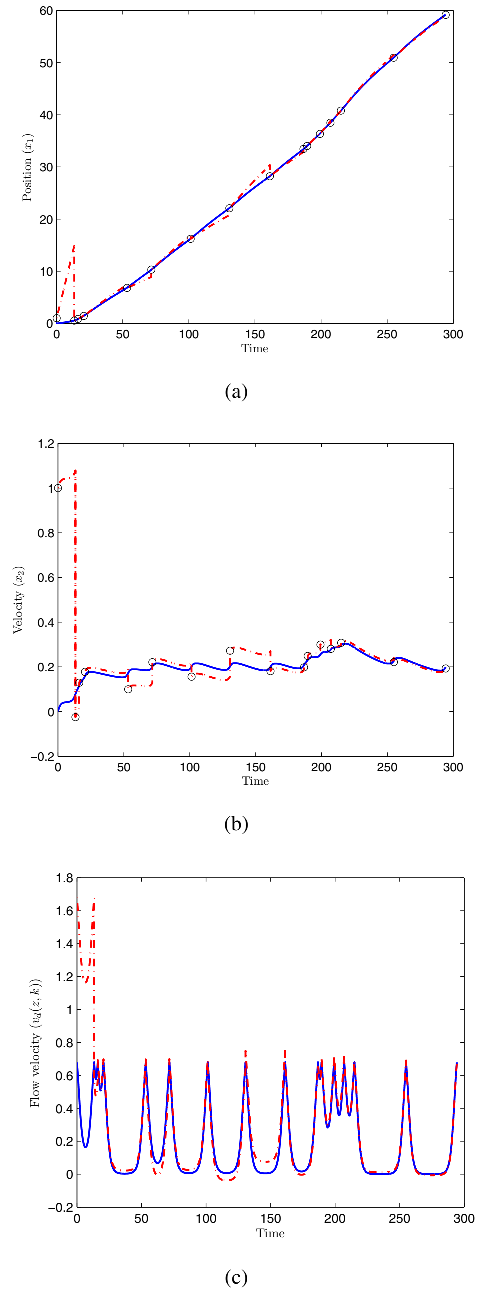

Fig. 4. (a) x_1 vs. $\hat{x}_1$ (solid line: true values, dashed line: estimated values, circle: the time instant when the sampling occurs); (b) x_2 vs. $\hat{x}_2$; (c) estimation of the along-front velocity.

Its discrete-time representation, by assuming zero-order hold for the input variable u and using step invariant transformation to approximate the differentiation over half open intervals $[kT, (k+1)T)$, can be written as

$$x_{k+1} = Ax_k + Bu_k + w_k$$

where

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}$$

Here, the term w_k is added to reflect the impact of the process noise. Now the output y is the displacement that is irregularly

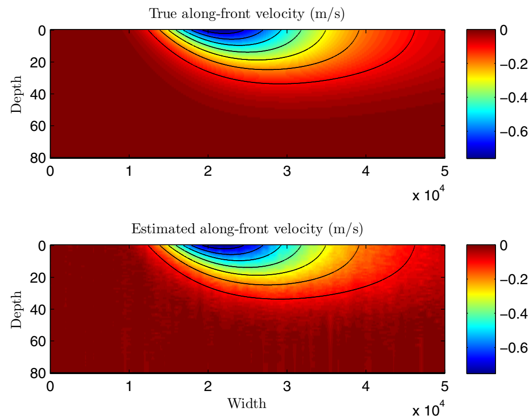

Fig. 5. Comparison between the true and reconstructed flow field.

measured at time instant t_ℓ:

$$y_{t_\ell} = Cx_{t_\ell} + v_{t_\ell}$$

where

$$C = [1 \quad 0]$$

It is verifiable that $\mathbb{P} = \emptyset$ for the discrete-time state-space model. Thus for any sequence of sampling intervals $\mathbb{T}$, the KF satisfies peak covariance stability when applied to estimate the state variables x_1 and x_2 from the irregularly sampled measurements y. Then the flow velocity $v_d(z, k)$ can be computed using the inverse of (27).

A cross section of the along-front velocity field under consideration is shown in Fig 2. The fronts are regions of strong horizontal density gradient in the ocean. Inherently unstable, they are usually sites of strong currents and eddy formation. Due to the density gradient and Coriolis force, the along-front velocity is yielded as illustrated. The field is assumed as wide as 5×10^4m and as deep as 80m.

Suppose no priori knowledge about the flow field is available. Thus for simplicity, the drogues are released every 500m, though uneven distribution introduces no more difficulties to this application. Each drogue can have a different depth profile, with irregular diving and surfacing patterns. The KF is implemented to every single drogue to estimate the along-front velocities. Let us take a close look at the drogue released at the position of 2.5×10^4m. Its depth profile is shown in Fig. 3. As is seen, the time intervals of underwater traveling are unequal, resulting in the irregular sampling of the position measurement. Fig. 4 illustrates the estimation results, including the displacement and velocity of the drogue and the along-front flow velocity. We observe that the estimated values approach the true ones. In particular, the flow estimation in Fig. 4(c) exhibits satisfactory performance, with even small variations captured.

Finally, the flow field can be reconstructed by combining the flow estimation results of all drogues. The reconstructed flow field is compared with the true one in Fig. 5. Despite the existence of minor differences, the reconstruction agrees well with the original flow field, verifying the effectiveness of the application of the KF with irregular sampling.

V. Conclusion

In this paper, we study the KF with irregularly sampled measurements. When irregular sampling occurs, the KF is still the optimal state estimator for linear discrete-time systems. Sufficient conditions that guarantee the defined peak covariance stability are derived. It is found that the stability is closely related with the distribution of eigenvalues of A in the complex plane. As a demonstration example, the KF is applied to address ocean flow field estimation, which is a compelling problem in oceanography and involves irregular sampling, with adequate accuracy achieved.

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