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Eigenvalue Constraints for Realization-Based Identification

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Successful modeling of aeroelastic and structural dynamics requires the use of system identification techniques for model development and validation. Experimental methods for such applications often generate large datasets of high input and output dimensions, and identification methods that minimize maximum likelihood criteria are ill-suited for such cases due to the non-convexity of the model parameters in the error. Realization-based identification methods such as the Eigensystem Realization Algorithm and subspace identification methods have consequently gained widespread use for these applications, because they require only a fixed number of linear algebra operations. Such methods, however, do not guarantee certain system properties, such as stability, without modification.

Although realization-based identification methods are typically formulated as a sequence of linear algebra equations with analytical solutions, they all nonetheless minimize some Frobenius norm, the argument of which is affine in the parameters being identified. The convexity of this minimization has so far been mostly unexploited. We present a method for incorporating constraints into realization-based identification methods that requires the eigenvalues of the identified model to lie within arbitrary convex regions of the complex plane based on linear matrix inequalities. This results in a convex optimization problem that remains easily solvable even for large datasets with high input and output dimensions. Some specific constraints for particularly important regions of the complex plane are presented and motivated with numerical examples.

I. Introduction

For a linear, time-invariant (LTI) system, output measured from some time $t = 0$ may be expressed as the linear combination of past input (for $t < 0$) and input measured from the same initial time (for $t \geq 0$). For a finite-dimensional system, the mapping from past input to future output is a finite-rank linear operator, and the effect of the past input may be stored in a finite-dimensional vector; this vector is the *state* of the system. The central idea of realization theory is to factor this mapping from past input to future output into two parts: a map from the input to the state, and another from the state to the output. This factorization provides a complete description of the system dynamics and guarantees the representation is both causal and finite-dimensional; thus it can be physically constructed, or *realized*.

Realization-based identification refers to system identification methods that construct system models by identifying the mapping from past input to future output and constructing a state-space representation via a rank-reducing factorization. The non-deterministic nature of the experimental process requires that both these steps be carefully considered. The many ways of treating stochastic signals has resulted in the development of many different realization-based identification methods, but all share a common ancestor in the Ho-Kalman realization algorithm, which constructs a state-space representation of a system from deterministic impulse-response coefficients, or *Markov parameters*.

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An often overlooked characteristic of realization-based identification methods is that the key step of estimating the system dynamics, though often stated as a simple matrix product, can also be stated as the minimization of the Frobenius-norm of a matrix expression that is linear in the system parameters. Though this minimization has an exact analytical solution in the unconstrained case, it is nonetheless a minimization of a convex cost function, and it can consequently be modified with convex constraints to form a well-posed convex optimization problem.

One property of an LTI system often of interest is the location of the system eigenvalues, or poles. For a particular matrix, constraints on the location of its eigenvalues in the complex plane may be stated as linear matrix inequalities (LMIs). These constraints are not only convex, but linear in the parameters. Additionally, the minimization of a Frobenius-norm expression that is linear in the parameters subject to LMI constraints can be stated as a linear program with semidefinite constraints. Such problems can be easily solved using interior-point methods that are guaranteed to converge to a unique minimizing solution in a relatively small number of iterations.[1]

Examples of subspace identification methods which incorporate LMI constraints include Lacey and Berstein,[2] in which an LMI framework is proposed to constrain the eigenvalues of system estimates to be stable, and Hoagg et. al.[3] and McKelvey and Moheimani,[4] which use similar frameworks to restrict estimates to positive real systems. Hoagg and collaborators[5] later extend this framework to provide a lower bound on the natural frequencies of the poles of the identified model, creating a convex optimization procedure which restricts the eigenvalues to a *non*-convex region of the convex plane; the parameterization used, however eliminates the possibility of also restricting the eigenvalues to lie within convex regions of the complex plane, such as the unit circle. A method for constrained step-based realization procedures has also been recently proposed by the authors.[6]

This paper presents a unifying model for performing realization-based identification subject to constraints on eigenvalue locations via LMIs. The method is developed in a general form that may be applied to specific realization-based identification methods by substitution of various parameters. A simulation example shows how restricting pole locations can be interpreted as reducing the variance of a system estimate by the incorporation of prior knowledge into the identification procedure.

The rest of the paper is organized as follows: Section II provides some preliminary definitions and notation used throughout the rest of the paper. Section III restates several popular system identification methods in a general framework of structured matrices and Frobenius-norm minimization. Section V applies the technique to the identification of aeroelastic modes, and Section VI concludes with some suggestions on future work.

II. Preliminaries

This section presents definitions and notation that are used throughout the rest of the paper and reviews some results from realization theory that are important to the identification problem. The identification methods presented in this paper are restricted to those which identify discrete-time, linear, time-invariant (LTI) systems. The dynamics of a discrete-time LTI system may be stated in several equivalent representations. One common representation of LTI systems is the state-space representation, in which a set of matrices (A, B, C, D) defines the relationship

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}^{n_u}$ is the input signal, $y(t) \in \mathbb{R}^{n_y}$ is the output signal, and $x(t) \in \mathbb{R}^n$ is the system state. The time variable t is an integer signal index.

The system (1) is stable if A has all eigenvalues inside the unit circle. It is *controllable* if the state $x(t)$ may achieve an arbitrary value with a proper selection of $u(t)$ in n time steps. This is true if and only if the *controllability matrix*

$$\mathcal{C}_n = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has rank n. The system is *observable* if the initial state $x(0)$ may be determined from n observations of the

output. This is true if and only if the *observability matrix*

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n. Systems that are both controllable and observable are *minimal* and have the property that the state dimension n cannot be reduced.[7] In this paper, all systems are assumed minimal. The state-space representation is guaranteed to be causal, and so it is called a *realizable* representation of a system since it can always be constructed in reality.[8]

A discrete-time LTI system also has a representation of a convolution with an infinite series

$$y(t) = \sum_{k=0}^{\infty} G(k)u(t-k) + v(t) \tag{2}$$

where $G(k) \in \mathbb{R}^{n_y \times n_u}$ are the system Markov parameters. Although this representation also guarantees causal systems since the summation starts at 0, there are no tests for stability using the Markov parameters that do not require knowledge of the entire sequence. It is easily found that the Markov parameters may be constructed from the state-space parameters as

$$G(k) = \begin{cases} D & k = 0, \\ CA^{k-1}B & k > 0. \end{cases} \tag{3}$$

The inverse problem of constructing a state-space representation from a sequence of Markov parameters is more complicated. This is the problem of *realization*.

A positive definite (semidefinite) matrix is a matrix with real and strictly positive (nonnegative) eigenvalues. The notation $P > 0$ will be used to represent that a matrix is positive definite, and $P \geq 0$ to represent that a matrix is positive semidefinite. The symbol $\otimes$ denotes the Kronecker product.

III. Realization-Based Identification

The realization problem begins with the construction of structured data matrices. We first present the Ho-Kalman algorithm, which constructs a state-space realization from a finite-length deterministic sequence of Markov parameters. The method is then extended to noise-corrupted estimates of Markov parameters. Two popular subspace identification methods are then presented that generalize the ideas of realization-based identification to matrices of input and output data. Finally, hybrid subspace method is presented that generalizes to the original Ho-Kalman algorithm when applied to a deterministic impulse response.

III.A. Realization from Exact System Markov Parameters

Suppose for now that the Markov parameters of a system are known perfectly. The matrix

$$H_k = \begin{bmatrix} G(1) & G(2) & \cdots & G(l) \\ G(2) & G(3) & \cdots & G(l+1) \\ \vdots & \vdots & & \vdots \\ G(k) & G(k+1) & \cdots & G(k+l-1) \end{bmatrix} \tag{4}$$

is the system Hankel matrix with k block rows. It maps past input to future output, and in some respects its 2-norm $||H||_2$ represents a "gain" of an LTI system. (Details can be found in the literature of robust control[9]). Assume that $k > n$. Substitution of the state-space parameters from (3) reveals that (4) is the

product of the *extended* observability matrix

$$\mathcal{O}_k = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}$$

with $k \geq n$ and the *extended* controllability matrix

$$\mathcal{C}_l = \begin{bmatrix} B & AB & A^2B & \cdots & A^{l-1}B \end{bmatrix}.$$

with $l \geq n$ so that

$$H_k = \mathcal{O}_k \mathcal{C}_l.$$

Because $\text{rank}(\mathcal{O}_k) = \text{rank}(\mathcal{C}_l) = n$ for a minimal system,

$$\text{rank}(H_k) = n.$$

Additionally, if the indices of H_k are shifted forward by 1 to form a shifted Hankel matrix

$$H'_k = \begin{bmatrix} G(2) & G(3) & \cdots & G(l+1) \\ G(3) & G(4) & \cdots & G(l+2) \\ \vdots & \vdots & & \vdots \\ G(k+1) & G(k+2) & \cdots & G(k+l) \end{bmatrix}, \tag{5}$$

then substitution of (3) results in

$$H'_k = \mathcal{O}_k A \mathcal{C}_l. \tag{6}$$

The index l is omitted from the subscripts of H_k and H'_k because the column dimension is less significant in the identification algorithms.

If H_k is known exactly, then *any* factorization

$$H_k = \mathcal{O}_k \mathcal{C}_l$$

with valid dimensions will result in an $\mathcal{O}_k$ and $\mathcal{C}_l$ for some arbitrary state basis. If H'_k is also known exactly, the parameter A in the same basis as $\mathcal{O}_k$ and $\mathcal{C}_l$ may be found from

$$A = \mathcal{O}_k^\dagger H'_k \mathcal{C}_l^\dagger, \tag{7}$$

where $(\cdot)^\dagger$ is the Moore-Penrose pseudoinverse. Then with C taken from the top n_y rows of $\mathcal{O}_k$, B taken from the first n_u columns of C_l, and $D = G(0)$, a complete and minimal state-space realization may be found from a deterministic sequence of Markov parameters. The state-basis of the resulting realization will of course depend on the factorization used to find $\mathcal{O}_k$ and $\mathcal{C}_l$.

At times, only the extended observability matrix $\mathcal{O}_k$ is available. In these cases, A can be estimated from the shift-invariance of $\mathcal{O}_k$ alone as follows: Let $\mathcal{O}_{2|k}$ denote block rows 2 through k of the extended observability matrix, so that

$$\mathcal{O}_{2|k} = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix}.$$

Because

$$\mathcal{O}_{2|k} A = \mathcal{O}_{k-1},$$

A may be found from

$$A = \mathcal{O}_{2|k}^\dagger \mathcal{O}_{k-1}. \tag{8}$$

This relationship is the basis of the Eigensystem Realization Algorithm (ERA)[10] and the Multivariable Output-Error State-sPace (MOESP) methods.[11] A similar method may be used to solve for A by shifting the columns of the controllability matrix.[12]

III.B. Realization from Estimated System Markov Parameters

We have not yet addressed the effects of noise on the estimate of H_k or on the factorization $\mathcal{O}_k\mathcal{C}_l$. Let $\hat{H}_k$ be the estimate of H_k constructed from estimated Markov parameters. If $\hat{H}_k$ has an error term

$$\hat{H}_k = H_k + E,$$

where E is the result of a stochastic process, then $\hat{H}_k$ will have full rank instead of rank n if $k > n$, and a factorization into valid-dimensioned $\mathcal{O}_k$ and $\mathcal{C}_k$ will only be possible if $n = k$. Generally either the order of the system is unknown, or we would like to use more than the first $2n + 1$ Markov parameters to estimate the system so that we may have an estimate $\hat{H}_k$ with 2-norm closer to H_k. This requires reducing the rank of $\hat{H}_k$. The obvious tool for reducing the rank of matrices is the *singular-value decomposition* (SVD). Estimating the system parameters this way sometimes referred to as the method of Kung,[12] who performed the first rigorous analysis of the method, though in a model reduction setting.

Assume for now that n is known. The SVD of $\hat{H}_k$ is

$$\hat{H}_k = U\Sigma V^T$$

where U and V^T are orthogonal matrices and Σ is a diagonal matrix containing the nonnegative *singular values* σ_i ordered from largest to smallest. The SVD for a matrix is unique and guaranteed to exist, and the number of nonzero singular values of a matrix is equal to its rank.[13] Because U and V^T are orthogonal, it also satisfies

$$\hat{H}_k = \left|\left|U\Sigma V^T\right|\right|_2 = ||\Sigma||_2 = \sigma_1 \tag{9}$$

where $||\cdot||_2$ is the induced matrix 2-norm, and

$$\hat{H}_k = \left|\left|U\Sigma V^T\right|\right|_F = ||\Sigma||_F = \left(\sum_i^l \sigma_i^2\right)^{1/2} \tag{10}$$

where $||\cdot||_F$ is the Frobenius norm. From (9) and (10), we can directly see that if the SVD of H_k is partitioned into

$$\hat{H}_k = \begin{bmatrix} U_n & U_s \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_s \end{bmatrix} \begin{bmatrix} V_n^T \\ V_s^T \end{bmatrix},$$

where U_n is the first n columns of U, Σ_n is the upper-left $n \times n$ block of Σ, and V_n^T is the first n rows of V^T, the solution to the rank-reduction problem is[13]

$$Q = \underset{\text{rank}(Q)=n}{\arg\min} \left|\left|Q - \hat{H}_k\right|\right|_2 = \underset{\text{rank}(Q)=n}{\arg\min} \left|\left|Q - \hat{H}_k\right|\right|_F = U_n\Sigma_n V_n^T.$$

Additionally,

$$\left|\left|Q - \hat{H}_k\right|\right|_2 = \sigma_{n+1},$$

which suggests that if the rank of H_k is not known beforehand, it can be determined from examining the nonzero singular values in the deterministic case and from searching for a significant drop-off in singular values if only a noise-corrupted estimate is available.

From the rank-n matrix Q, any factorization

$$Q = \hat{\mathcal{O}}_k\hat{\mathcal{C}}_l$$

can be used to estimate $\mathcal{O}_k$ and $\mathcal{C}_l$. The error in the state-space realization, however, will depend on the chosen state basis. Generally we would like to have a state variable with a norm $||x_k||_2$ in between $||u_k||_2$ and $||y_k||_2$. Choosing the factorization

$$\hat{\mathcal{O}}_k = U_n\Sigma_n^{1/2} \qquad \text{and} \qquad \hat{\mathcal{C}}_l = \Sigma_n^{1/2}V_n^T \tag{11}$$

results in

$$\left|\left|\hat{\mathcal{O}}_k\right|\right|_2 = \left|\left|\hat{\mathcal{C}}_l\right|\right|_2 = \sqrt{\left|\left|\hat{H}_k\right|\right|_2}, \tag{12}$$

and thus, from a functional perspective, the choice of (11) will result in mappings from input to state and state to output with equal norms, and the scalar entries of the state vector x_k will have similar magnitudes. State-space realizations that satisfy (12) are sometimes called *internally balanced* realizations.[7] (Alternative definitions of a "balanced" realization exist, however, and it is generally wise to verify the definition in each context.)

With $\hat{\mathcal{O}}_k$ and $\hat{\mathcal{C}}_l$ known, an estimate $\hat{A}$ may be calculated by finding an estimate of the shifted Hankel matrix H'_k. If $\hat{H}'_k$ is an estimate of H'_k, then choosing the factorization (11) simplifies (7) to

$$\begin{aligned}\hat{A} &= \left(\hat{\mathcal{O}}_k\right)^{\dagger} \hat{H}'_k \left(\hat{\mathcal{C}}_l\right)^{\dagger} \\ &= \Sigma_n^{-1/2} U_n^T \hat{H}'_k V_n \Sigma_n^{-1/2}.\end{aligned}$$

By estimating $\hat{B}$ as the first block column of $\hat{\mathcal{C}}_l$, $\hat{C}$ as the first block row of $\hat{\mathcal{O}}_k$, and $\hat{D}$ as $G(0)$, a complete state-space realization $(\hat{A},\ \hat{B},\ \hat{C},\ \hat{D})$ may be identified from estimates of the system Markov parameters. Alternatively, if only $\hat{\mathcal{O}}_k$ is known, (8) may be used to estimate the system dynamics.

A number of realization-based identification methods construct system estimates from estimated Markov parameters. The most notable of these is the Eigensystem Realization Algorithm (ERA) of Juang.[10] Most often the Markov parameter estimates are constructed from the inverse Fourier transform of an estimated frequency response. An extension of the ERA uses an Observer/Kalman-Filter Identification (OKID) algorithm to produce improved Markov parameter estimates for the identification procedure.[14]

III.C. Subspace Identification Data-Matrix Equations

Realization-based identification methods that generate a system estimate from a Hankel matrix constructed of estimated Markov parameters have numerous difficulties when applied to noisy measurements. Measuring an impulse response directly is often infeasible; high-frequency damping may result in a measurement that has a very brief response before the signal-to-noise ratio becomes prohibitively small, and a unit pulse will often excite high-frequency nonlinearities that degrade the quality of the resulting estimate. Taking the inverse Fourier transform of the frequency response guarantees that the estimates of the Markov parameters will converge as the dataset grows only so long as the input is broadband. Generally input signals decay at higher frequencies, and calculation of the frequency response by inversion of the input spectrum will amplify high-frequency noise.

To overcome these difficulties, realization-based methods that identify systems directly from measured data have been developed. These include the so-called subspace identification methods, which estimate the system dynamics from block-Hankel matrices of input-output data. The effects of the state on the output are then isolated using geometrically inspired projection operations. Before presenting some popular algorithms, we overview the data-matrix equations that they share in common.

Suppose the state-space system (1) has an additive noise signal $v(t) \in \mathbb{R}^{n_y}$ such that

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) + v(t).\end{aligned} \tag{13}$$

$v(t)$ may be white or colored, which also includes the case of additive state-noise. We will assume that $v(t)$ and $u(t)$ are uncorrelated, which is the case when the system is operating in open loop. Suppose input data $u(t)$ and output data $y(t)$ have been measured over the course of some experiment. Consider a block-Hankel matrix of r block rows and l columns of measured output data starting with the sample $y(0)$,

$$Y_{0|r-1} = \begin{bmatrix} y(0) & y(1) & \cdots & y(l-1) \\ y(1) & y(2) & \cdots & y(l) \\ \vdots & \vdots & & \vdots \\ y(r-1) & y(r) & \cdots & y(r+l-2) \end{bmatrix} \in \mathbb{R}^{n_y r \times l}, \tag{14}$$

and a block-Hankel matrix of r block rows and l columns of measured input data starting with the sample

$u(0)$,

$$U_{0|r-1} \begin{bmatrix} u(0) & u(1) & \cdots & u(l-1) \\ u(1) & u(2) & \cdots & u(l) \\ \vdots & \vdots & & \vdots \\ u(r-1) & u(r) & \cdots & u(r+l-2) \end{bmatrix} \in \mathbb{R}^{n_u r \times l}. \tag{15}$$

These data matrices satisfy

$$Y_{0|r-1} = H_r U_p + T_{0|r-1} U_{0|r-1} + V_{0|r-1}, \tag{16}$$

where

$$H_r = \begin{bmatrix} G(1) & G(2) & G(3) & \cdots \\ G(2) & G(3) & G(4) & \cdots \\ \vdots & \vdots & \vdots & \\ G(r) & G(r+1) & G(r+2) & \cdots \end{bmatrix} \in \mathbb{R}^{n_y r \times \infty},$$

$$U_p = \begin{bmatrix} u(-1) & u(0) & \cdots & u(l-2) \\ u(-2) & u(-1) & \cdots & u(l-3) \\ u(-3) & u(-2) & \cdots & u(l-4) \\ \vdots & \vdots & & \vdots \end{bmatrix} \in \mathbb{R}^{\infty \times l},$$

$$T_{0|r-1} = \begin{bmatrix} G(0) & & & \\ G(1) & G(0) & & \\ \vdots & \vdots & \ddots & \\ G(r-1) & G(r) & \cdots & G(0) \end{bmatrix} \in \mathbb{R}^{n_y r \times n_u r},$$

and $V_{0|r-1}$ is a matrix of noise $v(t)$ with the same dimensions and indices as $Y_{0|r-1}$.

The column dimensions of H_r and H'_r and the row dimension of U_p may be infinite in the above equation, but their product is finite. To express (16) with terms of finite dimensions, we note that a matrix of system states satisfies

$$X = \begin{bmatrix} x(0) & x(1) & \cdots & x(l-1) \end{bmatrix} = \mathcal{C}_l U_p \in \mathbb{R}^{n \times l},$$

where $\mathcal{C}_l$ has infinite column dimension. Hence (16) may be alternatively stated as

$$Y_{0|r-1} = \mathcal{O}_r X + T'_{0|r-1} U_{0|r-1} + V_{0|r-1}. \tag{17}$$

Assume that the input data matrix $U_{0|r-1}$ has full row rank. The matrix

$$\Pi_{U^\perp} = I_l - U^T_{0|r-1} (U_{0|r-1} U^T_{0|r-1})^{-1} U_{0|r-1} \tag{18}$$

is the projector for the null-space of $U_{0|r-1}$ and satisfies the property

$$U_{0|r-1} \Pi_{U^\perp} = 0.$$

Multiplication of $Y_{0|r-1}$ on the right by $\Pi_{U^\perp}$ results in

$$Y_{0|r-1} \Pi_{U^\perp} = \mathcal{O}_r X \Pi_{U^\perp} + V_{0|r-1} \Pi_{U^\perp} \tag{19}$$

Because $U_{0|r}$ has full row rank, the dimension of the null space of $U_{0|r}$ is the difference between its columns and its rows:[15]

$$\dim(\text{null}(U_{0|r})) = l - r,$$

and thus $\text{rank}(\Pi_{U^\perp}) = l - r$. A necessary condition to preserve the rank of $\mathcal{O}_r X$ and $\mathcal{O}_r A X$ when multiplying by $\Pi_{U^\perp}$ is that $\Pi_{U^\perp}$ have rank n. Hence

$$l \geq n + r$$

is a necessary condition to preserve the rank of $\mathcal{O}_r X$ and $\mathcal{O}_r A X$. This provides a necessary condition on the dimensions of the data matrices. We must also satisfy

$$\text{rank}(X \Pi_{U^\perp}) = n,$$

so that the state dimension is preserved under the projection. This may safely be assumed true for nearly all input signals. For an extensive analysis of both necessary and sufficient conditions to preserve the rank of $\mathcal{O}_r X$ under a null space projection and the relationship of the conditions to persistency of excitation see Willems.[16]

III.D. Realization from the Extended Observability Matrix

Let W be some weighting matrix such that $V_{0|r-1}W \to 0$. Because $u(t)$ and $v(t)$ are uncorrelated, this also implies $V_{0|r-1}\Pi_{U^\perp}W \to 0$. Then the right-hand multiplication

$$\frac{1}{N}Y_{0|r-1}\Pi_{U^\perp}W = \frac{1}{N}\mathcal{O}_r X\Pi_{U^\perp} + \frac{1}{N}V_{0|r-1}\Pi_{U^\perp}W$$

will converge to

$$\frac{1}{N}Y_{0|r-1}\Pi_{U^\perp}W = \frac{1}{N}\mathcal{O}_r X\Pi_{U^\perp}W. \tag{20}$$

In this case, the SVD

$$\frac{1}{N}Y_{k|r-1}\Pi_{U^\perp}W = \begin{bmatrix}U_n & U_s\end{bmatrix}\begin{bmatrix}\Sigma_n & 0\\ 0 & \Sigma_s\end{bmatrix}\begin{bmatrix}V_n^T\\ V_s^T\end{bmatrix}$$

has the property that row space of U_n converges to the row space of $\mathcal{O}_r$. Thus we may use the estimate

$$\hat{\mathcal{O}}_r = U_n$$

to estimate the system dynamics from

$$\hat{A} = \hat{\mathcal{O}}_{2|r}^{\dagger}\hat{\mathcal{O}}_{r-1}. \tag{21}$$

with $\hat{\mathcal{O}}_{r-1}$ and $\hat{\mathcal{O}}_{2|r}$ defined similarly to $\mathcal{O}_{k-1}$ and $\mathcal{O}_{2|k}$ in (8).

Identification methods that use (21) are referred to as MOESP-type methods. The weighting W is usually formed from past output (PO-MOESP) or past input (PI-MOESP) depending on the assumptions regarding the content of $v(t)$. Overviews of the MOESP family of subspace methods may be found in the book of Verhaegen and Verdult.[11]

III.E. Realization from an Estimated State Sequence

If the noise $v(t)$ is the result of additive white noise on the state and the output, then the SVD

$$Y_{0|r-1}\Pi_{U^\perp} = \begin{bmatrix}U_n & U_s\end{bmatrix}\begin{bmatrix}\Sigma_n & 0\\ 0 & \Sigma_s\end{bmatrix}\begin{bmatrix}V_n^T\\ V_s^T\end{bmatrix}$$

may be interpreted as the factorization

$$\hat{\mathcal{O}}_r = U_n S_n^{1/2} \qquad \tilde{X} = S_n^{1/2}V_n^T$$

in which $\tilde{X} = X\Pi_{U^\perp}$ is a bank of Kalman filter states for the free response of the system. The state dynamics may then be estimated using the shift-invariant structure of $\tilde{X}$. In fact, all parameters (A, B, C, D) as well as a noise parameter K may be estimated via the least squares problem

$$\min_{\left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]}\left\|\begin{bmatrix}\tilde{X}'\\ Y\end{bmatrix} - \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\tilde{X}\\ U\end{bmatrix} - \begin{bmatrix}K\\ I\end{bmatrix}E\right\|_F$$

where $\tilde{X}'$ is $\tilde{X}$ shifted forward by one index and E is a column-wise sequence of noise in innovations form. This method is known as N4SID (an acronym for "Numerical Methods for Subspace System Identification.") Additionally, the null-space projection may be replaced with an *oblique* projection onto the null space of $U_{0|r-1}$ along a subspace of past output and past input. Details for these methods may be found in the book of Van Overschee and De Moor.[17]

III.F. Realization from Shift-Invariance of Time-Domain and Covariance-Function Data

The dynamics of A may also be estimated from shifting the data itself. A forward-shifted block-Hankel output data matrix

$$Y_{1|r} = \begin{bmatrix} y(1) & y(2) & \cdots & y(l) \\ y(2) & y(3) & \cdots & y(l+1) \\ \vdots & \vdots & & \vdots \\ y(r) & y(r+1) & \cdots & y(r+l-1) \end{bmatrix} \in \mathbb{R}^{n_y r \times l} \tag{22}$$

may be expressed in terms of a shifted block-Toeplitz matrix

$$\begin{aligned} T_{1|r} &= \begin{bmatrix} G(1) & G(0) & & & \\ G(2) & G(1) & G(0) & & \\ \vdots & \vdots & \vdots & \ddots & \\ G(r) & G(r-1) & G(r-2) & \cdots & G(0) \end{bmatrix} \\ &= \left[\begin{array}{c|c} G(1) & \\ \vdots & T_{0|r-1} \\ G(r) & \end{array}\right] \in \mathbb{R}^{n_y r \times n_u (r+1)} \end{aligned}$$

as

$$Y_{1|r} = \mathcal{O}_r A X + T_{1|r} U_{0|r} + V_{1|r}, \tag{23}$$

in which

$$U_{0|r} = \left[\begin{array}{c} U_{0|r-1} \\ \hline u(r) \quad \cdots \quad u(r+l-1) \end{array}\right] \in \mathbb{R}^{n_u (r+1) \times l} \tag{24}$$

and $V_{1|r}$ is a block-Hankel matrix of noise terms $v(t)$ with the same indices as $Y_{1|r}$. Note that the dynamics matrix A appears in the output data itself in (23).

An important quality of (16) is that the index r in $U_{0|r-1}$ may be replaced with any index $\bar{r} \geq r$ without invalidating the equations; causality and dimensional consistency are retained by appending '0's to the right-hand side of $T_{0|r-1}$. Thus $U_{0|r-1}$ may be replaced with $U_{0|r}$ and $T_{0|r-1}$ with

$$T'_{0|r-1} = \left[\begin{array}{c|c} & 0 \\ T_{0|r-1} & \vdots \\ & 0 \end{array}\right].$$

in (16) so that both equations contain the same input matrix:

$$Y_{0|r-1} = \mathcal{O} X + \underbrace{T'_{0|r-1} U_{0|r}}_{= T_{0|r-1} U_{0|r-1}} + V_{0|r-1}$$

Forming the projector matrix from the extended input matrix $U_{0|r}$ results in

$$\Pi_{U^\perp} = I_l - U_{0|r}^T (U_{0|r} U_{0|r}^T)^{-1} U_{0|r}. \tag{25}$$

If we choose a weighting matrix W that satisfies the same conditions assumed in (20), then multiplication of $Y_{0|r-1}$ and $Y_{1|r}$ on the right by $\Pi_{U^\perp} W$ results in

$$Y_{0|r-1} \Pi_{U^\perp} W = \mathcal{O}_r X \Pi_{U^\perp} W + V_{0|r-1} \Pi_{U^\perp} W \to \mathcal{O}_r X \Pi_{U^\perp} W \tag{26}$$

$$Y_{1|r} \Pi_{U^\perp} W = \mathcal{O}_r A X \Pi_{U^\perp} W + V_{1|r} \Pi_{U^\perp} W \to \mathcal{O}_r A X \Pi_{U^\perp} W. \tag{27}$$

By taking the SVD

$$Y_{0|r-1} \Pi_{U^\perp} W = \begin{bmatrix} U_n & U_s \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_s \end{bmatrix} \begin{bmatrix} V_n^T \\ V_s^T \end{bmatrix},$$

we form the estimates

$$\hat{\mathcal{O}}_r = U_n \Sigma_n^{1/2} \qquad \text{and} \qquad \hat{X}\Pi_{U^\perp} W,$$

and estimate $\hat{A}$ as

$$\begin{aligned}\hat{A} &= \hat{\mathcal{O}}_r^\dagger Y_{1|r} \Pi_{U^\perp} W \left(\hat{X}\Pi_{U^\perp} W\right)^\dagger \\ &= \Sigma_n^{-1/2} U_n^T Y_{1|r} \Pi_{U^\perp} W V_n \Sigma_n^{-1/2}. \end{aligned} \tag{28}$$

This method has been referred to as the Generalized Realization Algorithm due to its similarity with the original realization method of Ho and Kalman.[18] This method may also be applied to covariance functions as well, which eliminates the need for a weighting matrix W. Such an approach was recently applied to the identification of the structural modes of an F/A-18 from in-flight data by the authors.[19]

IV. Eigenvalue Constraints for Realization-Based Identification

In this section, we augment the realization-based methods with convex constraints with the goal of producing models that meet *a priori* requirements. Although the constrained solutions can no longer be formulated as a finite sequence of linear algebra operations, they can be solved via convex optimization techniques with guaranteed convergence. We first introduce the concept of LMI regions, which transform statements about the location of the eigenvalues of a matrix into questions of feasibility. The constructed LMIs are then incorporated into the realization problem as convex constraints.

An LMI region is a convex region $\mathcal{D}$ of the complex plane, defined in terms of a symmetric matrix α and a square matrix β, as

$$\mathcal{D} = \{z \in \mathbb{C} : f_{\mathcal{D}}(z) \geq 0\} \tag{29}$$

where

$$f_{\mathcal{D}}(z) = \alpha + \beta z + \beta^T \bar{z}. \tag{30}$$

We will call $f_{\mathcal{D}}(z)$ for a given $\mathcal{D}$ the characteristic function of $\mathcal{D}$. LMI regions generalize Lyapunov notions of stability for continuous and discrete time systems, and the describing-function parameters α and β may be used to form Lyapunov-type inequalities.

LMI regions were first introduced by Chilali and Gahenet,[20] and we repeat the central theorem of LMI regions here for future reference.

Theorem 1. *The eigenvalues of a matrix A lie within an LMI region with characteristic function* (30) *if and only if there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$P = P^T > 0, \qquad \mathcal{M}_{\mathcal{D}}(A,\ P) \geq 0 \tag{31}$$

in which

$$\mathcal{M}_{\mathcal{D}}(A,\ P) = \alpha \otimes P + \beta \otimes (AP) + \beta^T \otimes (AP)^T. \tag{32}$$

The intersection of two LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ is also an LMI region, described by the matrix function

$$f_{\mathcal{D}_1 \cap \mathcal{D}_2}(z) = \begin{bmatrix} f_{\mathcal{D}_1}(z) & 0 \\ 0 & f_{\mathcal{D}_2}(z) \end{bmatrix}. \tag{33}$$

Note that in general the (α, β) pair that describes an LMI region is not unique.

IV.A. Some LMI Regions Useful for Identification

In the following, we derive some LMI regions useful for identification purposes. Of course the user need not be limited by these; LMI regions can be constructed for any convex intersection of half-spaces, ellipsoids, and parabolas symmetric about the real axis. The following regions are straightforward to verify by algebraically solving for the eigenvalues of (30).

IV.A.1. Discrete-Time Stable Eigenvalues

Stable system estimates are often desirable in the identification problem. Realization-based identification methods, however, do not guarantee stability of the identified model. To provide some known degree of stability for the identified models, we may constrain eigenvalues to the disc of radius $1 - \delta_s$.

Proposition 1. *The set*

$$\mathcal{S} = \{z \in \mathbb{C} : |z| \leq 1 - \delta_s,\ 0 \leq \delta_s \leq 1\}$$

is equivalent to the LMI region $f_{\mathcal{S}}(z) \geq 0$*,*

$$f_{\mathcal{S}}(z) = (1 - \delta_s) I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{z}. \tag{34}$$

Theorem 1 applied to this region with $\delta_s = 0$ results in

$$P > 0, \qquad \text{and} \qquad \begin{bmatrix} P & AP \\ PA^T & P \end{bmatrix} > 0,$$

which, by means of Schur complements, is equivalent to the familiar discrete-time Lyapunov condition

$$P > 0, \qquad \text{and} \qquad P - APA^T > 0.$$

This is also similar, though not identical, to the constraint in the method proposed by Lacey and Bernstein.[2] In (34), however, the relaxation parameter δ_s has a specific interpretation in the complex plane.

IV.A.2. Eigenvalues with Positive Real Parts

It is also generally desirable to avoid models with poles that have negative real parts. Such systems cannot be transformed to continuous time by inverting a zero-order hold without increasing the model order, since the matrix logarithm is undefined for matrices with negative real eigenvalues. Consequently, we wish to construct an LMI region that describes the positive right-half plane. This region should also be parameterized so that the region begins some distance away from the imaginary axis.

Proposition 2. *The set*

$$\mathcal{P} = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq \delta_p,\ \delta_p \geq 0\}$$

is equivalent to the LMI region $f_{\mathcal{P}}(z) \geq 0$*,*

$$f_{\mathcal{P}}(z) = \delta_p \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{z}. \tag{35}$$

IV.A.3. Eigenvalues with Zero Imaginary Parts

If it is known that a process has strictly real eigenvalues (such as with many thermodynamic processes), then it may be desirable to constrain the eigenvalues of the estimate to the real number line.

Proposition 3. *The real number line* $\mathbb{R}$ *is equivalent to the LMI region* $f_{\mathbb{R}}(z) \geq 0$*,*

$$f_{\mathbb{R}}(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{z}.$$

This constraint, however, is computationally unfriendly for many numerical optimization procedures, since it is effectively using two inequalities to define an equality, which can create problems for interior-point-based solvers. Instead, we include a parameter to describe an arbitrarily small band around the real axis in the complex plane.

Proposition 4. *The set*

$$\mathcal{R} = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \leq \delta_r,\ \delta_r \geq 0\}$$

is equivalent to the LMI region $f_{\mathcal{R}}(z) \geq 0$*,*

$$f_{\mathcal{R}}(z) = 2\delta_r I_2 + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bar{z}. \tag{36}$$

The parameter δ_r can be made small enough so that the complex parts of the resulting identified eigenvalues are near machine precision.

The geometric interpretation of each region in the complex plane is shown in Figure 1.

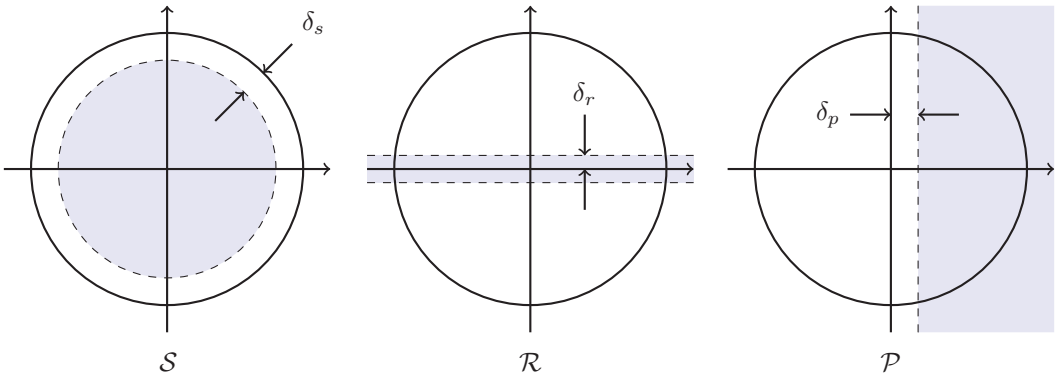

Figure 1. Sample LMI Regions in the Complex Plane

IV.B. Incorporation of LMI Constraints into Realization-Based Identification

We now present a method for incorporating constraints based on LMI regions into realization-based identification methods. An interesting quality of *all* realization-based methods is that they may be stated as the minimization of a linear expression inside a Frobenius norm:

$$\hat{A} = \arg\min_{A} ||R_1 A R_2 - M||_F \,, \tag{37}$$

where the matrices R_1, R_2, and M vary depending on the method. Parameters for the methods presented in Section III are shown in Table 1.

Method	R_1	R_2	M
ERA	$\hat{\mathcal{O}}_{2\|r}$	I	$\hat{\mathcal{O}}_{r-1}$
MOESP	$\hat{\mathcal{O}}_{2\|r}$	I	$\hat{\mathcal{O}}_{r-1}$
N4SID	I	$-\tilde{X}$	$KE - \tilde{X}'$
GRA	$\hat{\mathcal{O}}_r^\dagger$	$\hat{X}\Pi_{U^\perp}W$	$Y_{1\|r}\Pi_{U^\perp}W$

Table 1. Parameters of Equation (37) for Realization-Based Methods

From the general form (37), we will develop a method of modifying any realization-based identification method with convex constraints that constrain the eigenvalues of the resulting model to specific regions of the complex plane. We start by defining the cost function

$$J_r(\hat{A}) = \left|\left| R_1 \hat{A} R_2 - M \right|\right|_F . \tag{38}$$

The unconstrained minimum of (38) is the analytic solution of the various identification methods when R_1, R_2, and M are replaced with the values in Table 1. Note that (38) is affine in the parameter $\hat{A}$. Theorem 1

contains the product AP, however, so we augment (38) to contain $\hat{A}P$ via a right-hand weighting $W_r = R_2^\dagger P$,

$$\begin{aligned} J_r'(\hat{A},\ P) &= \left|\left|\left(R_1\hat{A}R_2 - M\right)W_r\right|\right|_F \\ &= \left|\left|\left(R_1\hat{A}R_2 - M\right)R_2^\dagger P\right|\right|_F \qquad (39) \\ &= \left|\left|R_1\hat{A}P - MR_2^\dagger P\right|\right|_F. \qquad (40) \end{aligned}$$

Though (40) has the same global minimum as (38), namely

$$\hat{A} = R_1^\dagger M R_2^\dagger, \tag{41}$$

it does not necessarily have the same minimum over an arbitrary convex set. (This may be seen by calculating the matrix differential in $\hat{A}$ for both costs.) We must also ensure that P does not become arbitrarily small during the minimization procedure, so we include the constraint

$$\mathrm{tr}(P) = 1$$

to ensure numerical stability.

Though (40) now contains $\hat{A}P$, it is no longer affine in the parameters $\hat{A}$ and P. We thereby re-parameterize the cost function with an auxiliary term $Q = \hat{A}P$, to create the final convex optimization problem:

Given R_1, R_2, and M; and an LMI region parameterized by α and β,

$$\begin{array}{rl} \text{minimize} & J_c(Q,\ P) \\ \text{subject to} & \mathcal{M}(Q,\ P) \geq 0, \\ & P = P^T > 0 \\ & \mathrm{tr}(P) = 1 \end{array} \tag{42}$$

in which

$$J_c(Q,\ P) = \left|\left|R_1 Q - MR_2^\dagger P\right|\right|_F,$$

and

$$\mathcal{M}(Q,\ P) = \alpha \otimes P + \beta \otimes Q + \beta^T \otimes Q^T.$$

The solution for A is then given by $\hat{A} = QP^{-1}$. Note that the strictly positive definite constraint for P guarantees that P is invertible.

At this point we should remark that although the global minimizer (41) might be in the set of feasible points, numerical optimization tools may not be able to find it exactly. Optimization routines based on primal-dual gap methods[1] may deviate from (41) even when it is feasible and supplied as an initial value. This is because, although the analytic solution to primal and dual problems is the same, the numerical solution might not be. Such numerical difficulties become more common as the row dimension of R_1 increases. In practice, it is best to confirm that the eigenvalues of (41) do not satisfy the LMI region's characteristic equation before solving the convex optimization problem.

V. Application to the Identification of Aeroelastic Dynamics

In this section, the eigenvalue-constrained procedure is applied to a high-order linear model of the structural modes of the NASA Active Aeroelastic Wing F/A-18. A 249^{th}-order linear, continuous-time model of the acceleration of the airframe due to differential aileron deflection was discretized to 400 Hz via zero-order hold. The output signal consists of measurements from 53 accelerometers, shown in Figure 2. The input signal was chosen to be white noise of variance 1. Prior to identification, each output signal was scaled to have a variance of 1, and independent white noise signals of variance 0.1 were added to each output.

The eigenvalue LMI constraints were incorporated with the realization-based identification method presented in Section III.F to form a constrained realization-based identification method, as outlined in Section IV.B. The number of block rows r was chosen to be 20, and an 18^{th}-order model was identified. 1000

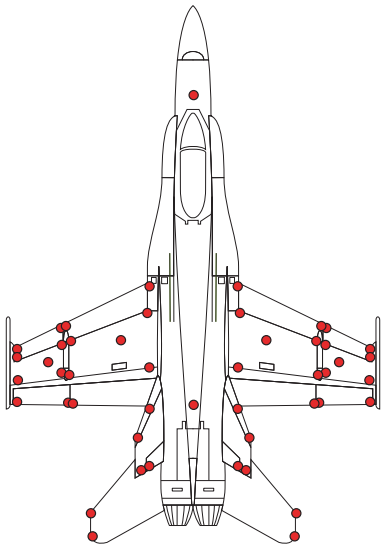

Figure 2. AAW Accelerometer Locations

realizations of data of 1000 points each were generated, of which 81 resulted in unstable estimates of A. Constrained estimates were generated using YALMIP[21] with SeDuMi[22] as the selected solver.

Pole locations for estimates generated by the unconstrained and constrained methods from the same realization are shown in Figure 3. It can be seen that the poles outside the unit circle for the unconstrained case are within the unit circle for the constrained case.

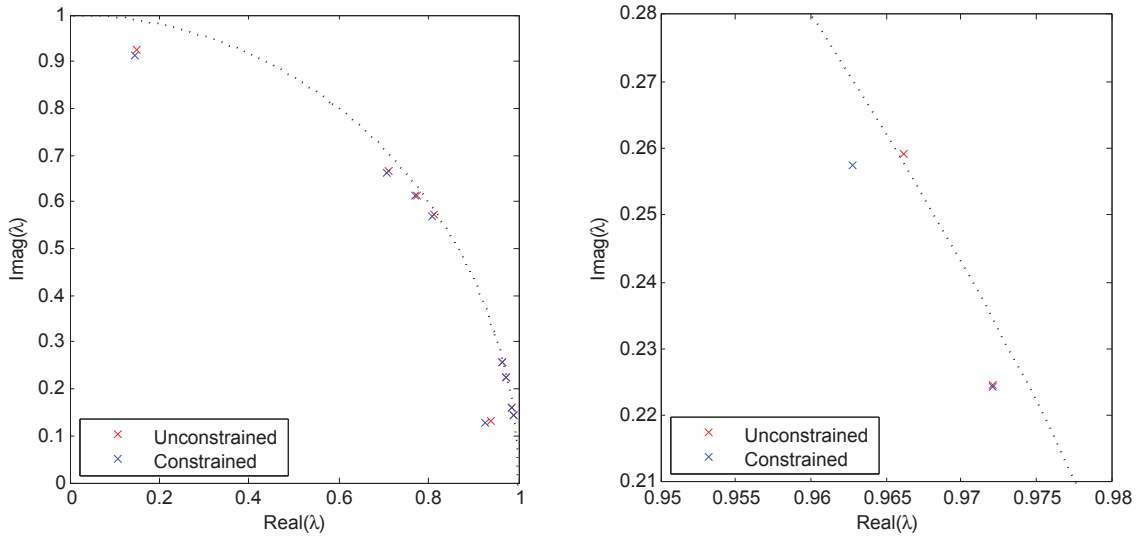

Figure 3. Poles Locations for Unconstrained and Constrained Estimates

Maximum singular values in frequency for the transfer functions of the true (discretized) system and the estimated system are shown in Figure 4. Note that only results from the constrained method are shown, since it is not possible to compute a regressor to estimate B in (1) if A is unstable. The constrained method successfully captures the dominant dynamics of the model.

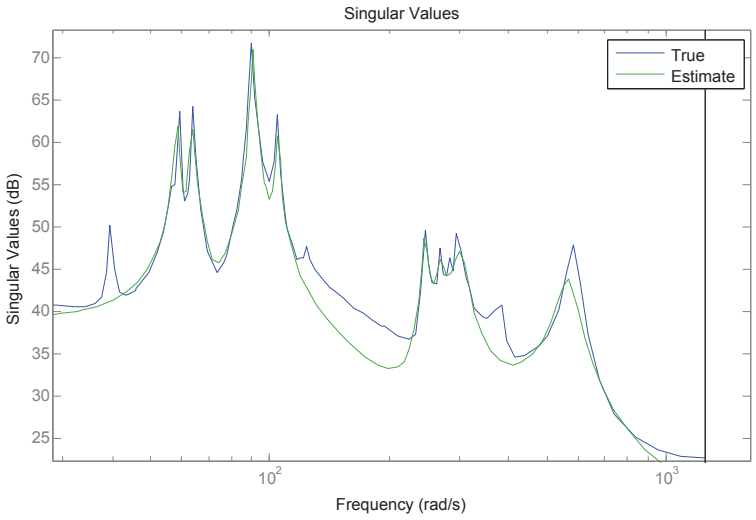

Figure 4. Maximum Singular Value of True and Estimated Systems

VI. Conclusion

We have presented a unifying method for the incorporation of eigenvalue constraints into realization-based identification methods, which includes the ERA and subspace methods. The method incorporates the idea of LMI regions into the identification procedure by restating the realization-based identification problem as the minimization of a Frobenius-norm cost function. The method was demonstrated to enforce stability when estimating the aeroelastic dynamics of the NASA F/A-18.

The most significant challenge to application of the theory so far has been numerical conditioning issues resulting from system estimates with high condition numbers. The current implementation also relies on the YALMIP tool to auto-generate the linear program needed for SeDuMi, and it is possible that explicitly formulating the problem in primal-dual form could result in a better conditioned problem that is also capable of identifying models from larger data sets.

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