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Brief paper

An alternative Kalman–Yakubovich–Popov lemma and some extensions☆

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ABSTRACT

This paper introduces an alternative formulation of the Kalman–Yakubovich–Popov (KYP) Lemma, relating an infinite dimensional Frequency Domain Inequality (FDI) to a pair of finite dimensional Linear Matrix Inequalities (LMI). It is shown that this new formulation encompasses previous generalizations of the KYP Lemma which hold in the case the coefficient matrix of the FDI does not depend on frequency. In addition, it allows the coefficient matrix of the frequency domain inequality to vary affinely with the frequency parameter. One application of this results is illustrated in an example of computing upper bounds to the structured singular value with frequency-dependent scalings.

0. Notation

The scalar $\mathrm{j} = \sqrt{-1}$. We denote by $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) the space of rectangular complex (real) matrices of dimension $m \times n$, and by $\mathbb{HC}^n$ the space of $\mathbb{C}^{n\times n}$ Hermitian matrices. For a matrix $X \in \mathbb{C}^{m\times n}$: $\overline{X}$, X^*, $X_\perp$ are, respectively, the complex-conjugate, the complex-conjugate transpose, and a basis for the null space of X, i.e., a full column rank matrix such that $XX_\perp = 0$ and $\begin{bmatrix} X^T & X_\perp \end{bmatrix}$ has also full column rank. $\mathrm{He}\{X\}$ is short-hand notation for $X + X^*$. We use $X \succ 0$ ($X \succeq 0$) to denote that $X \in \mathbb{HC}^n$ is positive (semi)definite. $X \otimes Y$ is the Kronecker product of X and Y.

1. Introduction and motivation

It is common practice in systems and control to specify performance and robustness of dynamical systems using the frequency domain. Frequency domain plots, such as Bode plots or, more recently, singular value plots (Zhou, Doyle, & Glover, 1996), convey lots of information about the system under consideration and are popular tools among the systems and control community. In the last decades, thanks mostly to the result known as Kalman–Yakubovich–Popov (KYP) Lemma (see Anderson (1967) and Ranstzer (1996) and the survey paper (Gusev & Likhtarnikov, 2006)), Frequency Domain Inequalities (FDIs) became a major tool in the analysis of dynamic systems. The KYP Lemma establishes the equivalence between the FDI

$$\begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0, \tag{1}$$

for all $\omega \in \mathbb{R}$ where matrices A, B and the Hermitian coefficient matrix Θ of appropriate finite dimensions are given, and the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \tag{2}$$

which should hold for some Hermitian matrix P. The main role of the KYP Lemma is to convert the infinite dimensional inequality (1) into the finite dimensional inequality (2) where appropriate choices for the coefficient matrix Θ represent the analysis of various system properties. Being an LMI, the set of feasible solutions to inequality (2) is convex and P can be computed efficiently.

An extension of the KYP Lemma, first proposed in Iwasaki, Meinsma, and Fu (2000), established the equivalence between the FDI (1) which should now hold for all $\omega_1 \le \omega \le \omega_2$, i.e. on a finite frequency interval, with the LMI

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + \mathrm{j}\omega_c Q \\ P - \mathrm{j}\omega_c Q & -\omega_1\omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \prec 0, \tag{3}$$

where $\omega_c := (\omega_1 + \omega_2)/2$. The above LMI should hold for some Hermitian matrix P and some positive semidefinite matrix Q. The readers are referred to Bai and Freund (2000), Chu and Tan (2008), Gusev and Likhtarnikov (2006), Iwasaki and Hara (2003a,b, 2005) and Scherer (2005) for extensive discussions of the features and applications of this and related results. From a practical

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perspective, it allows one to pose and check frequency domain specifications within a certain frequency range which might be the most relevant to a specific application. Furthermore, by combining ranges one can pose frequency specifications in different ranges without augmenting the plant with frequency dependent scalings or weights.

One contribution of this paper is to provide an alternative condition for the LMI (3) specified in terms of a pair of inequalities

$$\mathrm{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -\mathrm{j}\omega_i I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\}+\Theta\prec 0,\quad i=\{1,2\},\tag{4}$$

which should hold for some matrices F and G. We will show that a projection (which is obtained through Finsler's Lemma) can prove that feasibility of the pair of LMI (4) implies feasibility of the FDI (1). Conversely, we will construct matrices F and G that can make the pair of LMI (4) feasible whenever there exists P and $Q\succ 0$ that make (3) feasible. This will ensure that condition (4) is, therefore, necessary and sufficient for checking feasibility of inequality (1).

Furthermore, the pair of inequalities (4) can be extended to allow the matrix Θ to depend on the frequency. More specifically, the pair of LMI

$$\mathrm{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -\mathrm{j}\omega_i I\end{bmatrix}\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\right\}+\Theta_i\prec 0,\quad i=\{1,2\}\tag{5}$$

hold if and only if (1) holds with

$$\Theta(\omega)=\frac{\omega_2-\omega}{\omega_2-\omega_1}\Theta_1+\frac{\omega-\omega_1}{\omega_2-\omega_1}\Theta_2\tag{6}$$

for all $\omega_1\le\omega\le\omega_2$. One can recognize in the above formulation concepts from robust analysis of uncertain polytopic systems (de Oliveira, Bernussou, & Geromel, 1999; de Oliveira & Skelton, 2001), in which the frequency ω is treated as a real uncertain parameter. In the present paper, condition (5) is shown to be sufficient. Proving the necessity of (5) requires an entirely different construction for which results have been presented in Graham and de Oliveira (2008).

The KYP Lemma has been further generalized in Iwasaki and Hara (2005), establishing the equivalence between the LMI

$$H^*(\Phi\otimes P+\Psi\otimes Q)H+\Theta\prec 0,\tag{7}$$

which should hold for some Hermitian matrix P and some positive semidefinite matrix Q, and the FDI

$$\left(\begin{bmatrix}I & -\xi I\end{bmatrix}H\right)_\perp^*\Theta\left(\begin{bmatrix}I & -\xi I\end{bmatrix}H\right)_\perp\prec 0,\tag{8}$$

for all $\xi\in\mathbf{\Lambda}(\Phi,\Psi)$, where $H\in\mathbb{C}^{2n\times(n+m)}$ and

$$\mathbf{\Lambda}(\Phi,\Psi):=\{\xi\in\mathbb{C}:\sigma(\xi,\Phi)=0,\sigma(\xi,\Psi)\ge 0\},\tag{9}$$

where $\sigma(s,\Pi):=\begin{pmatrix}s\\1\end{pmatrix}^*\Pi\begin{pmatrix}s\\1\end{pmatrix}$. In the above, $\Phi,\Psi\in\mathbb{HC}^2$ are given matrices satisfying certain conditions (see Iwasaki and Hara (2005) and Section 3) that generalize the LMI condition of (3). For instance, by setting

$$H=\begin{bmatrix}A & B\\ I & 0\end{bmatrix},\qquad \Phi=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$\Psi=\begin{bmatrix}-1 & \mathrm{j}\omega_c\\ -\mathrm{j}\omega_c & -\omega_1\omega_2\end{bmatrix},\tag{10}$$

one recovers the LMI (3). As we will see later in this paper, all curves of the form $\mathbf{\Lambda}(\Phi,\Psi)$ correspond to segments of the imaginary axis (finite or infinite) transformed by the bilinear transformation.

Another contribution of this paper is an extension of the LMI of the form (4) and (5) for the general class of FDI (8). We show that if the pair of LMI

$$\mathrm{He}\left\{\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}I & -\mathrm{j}\tilde\omega_i I\end{bmatrix}(T\otimes I)H\right\}+\Theta_i\prec 0,\quad i=\{1,2\}\tag{11}$$

hold for some matrices F and G then (8) holds with $\Theta=\Theta(\omega(\xi))$ as given in (6). The precise relationship between $\tilde\omega_1\le\omega\le\tilde\omega_2$ and $\xi\in\mathbf{\Lambda}(\Phi,\Psi)$ will be developed in Section 3, where we will show how to determine, among other things, the extreme points ω_i, $i=1,2$ and the constant matrix $T\in\mathbb{C}^{2\times 2}$ to be used in (11). In case Θ is constant, i.e., $\Theta=\Theta_1=\Theta_2$, the pair of inequalities (11) are equivalent to (7). As with our previous results, when $\Theta_1\neq\Theta_2$, the pair of inequalities (11) is equivalent to a generalized FDI that is introduced later, see (20), and for which sufficiency is proved in the present paper. The proof of necessity is possible but requires a complex construction along the lines of Graham and de Oliveira (2008), which will be presented in a future paper.

Section 4 presents several interesting extensions and applications of LMI (5) and (11). For instance, Section 4.1 considers piecewise affine coefficient matrices $\Theta(\omega)$. Section 4.2 shows how the finite frequency LMI (5) can be modified to handle infinite frequency intervals which do not reduce to the standard KYP Lemma. Application of the LMI (11) to discrete-time systems and positive pseudo-polynomial matrices (Genin, Hachez, Nesterov, & Van Dooren, 2003) can be found in Graham (2007). Section 4.3 shows how to apply LMI (5) and (11) to the problem of robustness analysis in the context of the *structured singular value* (μ analysis) (Fan, Tits, & Doyle, 1991; Packard & Doyle, 1993). By allowing Θ to contain frequency dependent scalings (weights), in the spirit of Chou, Tits, and Balakrishnan (1999), Balakrishnan, Huang, Packard, and Doyle (1994) and Ly, Safonov, and Chiang (1994), we are able to obtain tighter upper bounds for μ (see Graham, de Oliveira, and de Callafon (2006) and Graham, de Oliveira, and de Callafon (2007)). Different from the existing methods, where the original system realization needs to be augmented with a set of fixed poles of the scalings, we take advantage of the special form of the proposed conditions to produce a test that has roughly the same complexity as the LMI from the standard KYP Lemma.

As a final note in this introduction, it is important to discuss the role played by the low complexity of the conditions (5) and (11), since recently developed polynomial techniques can be used to produce robust stability tests with little or no conservatism (see Chesi, Garulli, Tesi, and Vicino (2005), Lasserre (2001), Parrilo (2003) and Scherer (2005, 2003)). Indeed, arbitrary polynomial dependence of Θ on ω could be obtained, of course, at the expense of an exponential growth in the number of variables and size of inequalities. A remarkable feature of our results is that there is no extra cost associated with solving the inequalities (5) as compared with (4) while still enlarging the class of matrices $\Theta(\omega)$ being tested from constant to affine in ω.

2. KYP lemma on finite frequency intervals

In order to gain an understanding of the methods used in this paper for generalizing the KYP Lemma, this section presents an extension on the results of finite frequency KYP Lemma (Iwasaki & Hara, 2005; Iwasaki et al., 2000). The next theorem establishes the equivalence between the finite frequency KYP Lemma, that is with FDI (1) and LMI (3), and an *extended* condition in the lines of the work (de Oliveira & Skelton, 2001).

Theorem 1. *Let matrices $A\in\mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B\in\mathbb{C}^{n\times m}$ and $\Theta\in\mathbb{HC}^{n+m}$ be given. Let scalars ω_1, $\omega_2\in\mathbb{R}$ be also given, then the following statements are equivalent.*

(i) *The FDI (1) holds for all $\omega_1\le\omega\le\omega_2$.*
(ii) *There exist matrices $P,Q\in\mathbb{HC}^n$, $Q\succeq 0$ such that the LMI (3) holds where $\omega_c:=(\omega_1+\omega_2)/2$.*
(iii) *There exist matrices $F\in\mathbb{C}^{n\times n}$ and $G\in\mathbb{C}^{m\times n}$ such that the pair of LMI (4) holds*

Equivalence of (i) and (ii) is from Iwasaki et al. (2000) and Iwasaki and Hara (2005). Equivalence between (i) and (iii) will be proved in Section 3.1.

Remark 1. The importance of the above result is in reducing the infinite number of inequalities to be checked in the FDI (1) to two finite dimensional LMI, as in (3) with $Q \succeq 0$ or as in (4), both involving two matrix variables, P and Q, or F and G, respectively. LMI can be efficiently solved using Convex Programming (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994).

Remark 2. In (ii), one has to solve inequality (3) of dimension $n+m$ and inequality $Q \succeq 0$ of dimension n in $2n^2$ real optimization variables, the matrices $P, Q \in \mathbb{HC}^n$. In contrast, in (iii), one has to solve two inequalities (4) of dimension $n+m$ in $2n(n+m)$ real optimization variables in the matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$.

One can recognize concepts from the analysis of polytopic systems (de Oliveira & Skelton, 2001) that the frequency is treated as a real uncertain parameter. Indeed, at no extra cost, it is possible to use form (iii) of the above theorem to handle the particular form of a frequency-dependent matrix $\Theta(\omega)$ as defined in (6) for $\omega_1 \le \omega \le \omega_2$, where $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$. Note that $\Theta(\omega)$ is not a proper rational function of ω, which means that the Θ cannot be realized as a proper rational transfer function of ω. In Section 4.1 we will extend our results to piecewise affine functions $\Theta(\omega)$. In the next theorem $\Theta(\omega)$ is as in (6).

Theorem 2. *Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. Let scalars $\omega_1, \omega_2 \in \mathbb{R}$ be also given. If there exist matrices $F \in \mathbb{C}^{n\times n}$ and $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI (5) hold then the FDI*

$$\begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta(\omega) \begin{bmatrix} (\mathrm{j}\omega I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \tag{12}$$

holds for all $\omega_1 \le \omega \le \omega_2$ with $\Theta(\omega)$ as given in (6).

The above theorem is proved in Section 3.1. The result is proved necessary and sufficient in Graham and de Oliveira (2008). The technical developments in the next section are devoted to constructing generalized versions of Theorems 1 and 2.

3. The generalized KYP lemma

In this section we focus on the classes of curves on the complex plane considered in Iwasaki and Hara (2005). Let $\Phi, \Psi \in \mathbb{HC}^2$ be given and define sets $\mathbf{\Lambda}(\Phi, \Psi)$ as in (9). In Iwasaki and Hara (2005), conditions on Φ and Ψ have been presented for which the above set represents a curve. We need some of these results here.

Lemma 3. *Let $\Phi, \Psi \in \mathbb{HC}^2$ be given. Suppose $\det(\Phi) < 0$. Then there exists a common congruence transformation $T \in \mathbb{C}^{2\times 2}$ such that $\Phi = T^*\Phi_0 T$, $\Psi = T^*\Psi_0 T$ where*

$$\Phi_0 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 := \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \tag{13}$$

and $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha \le \gamma$.

Proposition 4. *Let $\Phi, \Psi \in \mathbb{HC}^2$ be given and define the set $\mathbf{\Lambda}(\Phi, \Psi)$ by (9). Then $\mathbf{\Lambda}(\Phi, \Psi)$ represents curves on the complex plane if and only if $\det(\Phi) < 0$ and either $0 \le \alpha \le \gamma$ or $\alpha < 0 < \gamma$, where α and γ are defined by the factorization (13).*

Proofs of Lemma 3 and Proposition 4 can be found in Iwasaki and Hara (2005). The importance of these results is that all curves parametrized by (9) are equivalent to the curve

$$\mathbf{\Lambda}(\Phi_0, \Psi_0) := \{s = \mathrm{j}\omega, \omega \in \mathbb{R} : \alpha\omega^2 + \gamma \ge 0\}, \tag{14}$$

since $\sigma(s, \Phi_0) = s^* + s = 0 \Rightarrow s = \mathrm{j}\omega, \omega \in \mathbb{R}$, and $\sigma(\mathrm{j}\omega, \Psi_0) = \alpha\omega^2 + \gamma \ge 0$. In fact, this relationship is made more precise with the following lemma.

Lemma 5. *Let Φ_0, Ψ_0 in (13) and a nonsingular matrix*

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2\times 2} \tag{15}$$

be given. Define the linear-fractional transformation[1]

$$\psi : \mathbb{C} \to \mathbb{C}, \qquad \psi(s) = \frac{b - ds}{cs - a}. \tag{16}$$

Then $\{\xi \in \mathbb{C} : \xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T),\ c\xi + d \ne 0\} = \{\psi(s) \in \mathbb{C} : s \in \mathbf{\Lambda}(\Phi_0, \Psi_0),\ cs \ne a\}$.

For a proof of Lemma 5 see Iwasaki and Hara (2005), Lemma 3. In the context of the present paper, a slightly modified version of the above lemma is also enlightening. We give it as the next corollary, whose proof is omitted for brevity.

Corollary 6. *Let Φ_0, Ψ_0 in (13) and a nonsingular matrix T as in (15) be given. Define the inverse linear-fractional transformation*

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \qquad \psi^{-1}(\xi) = \frac{a\xi + b}{c\xi + d}. \tag{17}$$

Then $\{\omega \in \mathbb{R} : \mathrm{j}\omega \in \mathbf{\Lambda}(\Phi_0, \Psi_0), \mathrm{j}c\omega \ne a\} = \{\psi^{-1}(\xi) \in \mathrm{j}\mathbb{R} : \xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T),\ c\xi + d \ne 0\}$.

Lemma 5 and Corollary 6 are equivalent, however, Corollary 6 highlights the fact that any curve given by sets $\mathbf{\Lambda}(\Phi, \Psi)$ can be indeed parametrized by a transformation of a segment of or the entire imaginary axis. This motivates the developments to follow.

Before proceeding, notice that nonsingularity of T, i.e. $ad \ne bc$, excludes the possibility of the image of the mapping ψ be reduced to a single point in $\{-d/c, -\mathrm{j}b/a, \infty, -\infty\}$, depending on whether the letters a, b, c and d are not zero. Conversely, nonsingularity of T excludes the possibility of the image of the mapping ψ^{-1} be reduced to a single point in $\{-\mathrm{j}b/d, -\mathrm{j}a/c, \mathrm{j}\infty, -\mathrm{j}\infty\}$.

The next result is a version of Theorem 2 in the context of the generalized FDI (8), where Θ is frequency dependent.

Theorem 7. *Let $a, b, c, d \in \mathbb{C}$ with $ad \ne bc$, $T \in \mathbb{C}^{2\times 2}$ as in (15), the inverse linear-fractional mapping $\psi^{-1} : \mathbb{C} \to \mathbb{C}$ as in (17), and Φ_0, Ψ_0 as in (13) be given. Assume that $\alpha < 0 < \gamma < \infty$ and define*

$$\tilde{\omega}_1 = -|\gamma/\alpha|^{1/2}, \qquad \tilde{\omega}_2 = |\gamma/\alpha|^{1/2}. \tag{18}$$

Assume that $\mathrm{j}c\tilde{\omega} \ne a$ for all $\tilde{\omega}_1 \le \tilde{\omega} \le \tilde{\omega}_2$. Let $H \in \mathbb{C}^{2n\times(n+m)}$ be given. The following statements are true.

(i) *Let $\Theta_1 = \Theta_2 = \Theta \in \mathbb{HC}^{n+m}$ be given. There exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI (11) have feasible solutions if and only if the FDI*

$$\left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_\perp^* \Theta \left(\begin{bmatrix} I & -\xi I \end{bmatrix} H\right)_\perp \prec 0, \tag{19}$$

holds for all $\xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$.

[1] Also known as the *bilinear transformation* in standard complex analysis textbooks (Churchill, 1960).

(ii) *Let $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI (11) have feasible solutions then the FDI*

$$\left(\begin{bmatrix} I & -\xi I\end{bmatrix} H\right)_\perp^* \Theta(-\mathrm{j}\psi^{-1}(\xi)) \left(\begin{bmatrix} I & -\xi I\end{bmatrix} H\right)_\perp \prec 0, \tag{20}$$

holds for all $\xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$ with $\Theta(\cdot)$ given by (6).

Proof. We start by proving (ii), which also covers the sufficiency part of (i) by making $\Theta_1 = \Theta_2 = \Theta$. Assume that the pair of inequalities (11) have feasible solutions. The sum of (11) for $i = 1$ multiplied by $\lambda(\tilde\omega) = (\tilde\omega_2 - \tilde\omega)/(\tilde\omega_2 - \tilde\omega_1) \in [0, 1]$ and of (11) for $i = 2$ multiplied by $(1 - \lambda(\tilde\omega))$ produces

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G\end{bmatrix}\begin{bmatrix} I & -\mathrm{j}\tilde\omega I\end{bmatrix}(T\otimes I)H\right\} + \Theta(\tilde\omega) \prec 0,$$

for all $\tilde\omega_1 \le \tilde\omega \le \tilde\omega_2$ where $\Theta(\tilde\omega)$ is defined by (6). Note that for any $s \in \mathbb{C}$ such that $cs \ne a$ we have

$$\begin{aligned}\begin{bmatrix} I & -sI\end{bmatrix}(T\otimes I)H &= (a-cs)\begin{bmatrix} I & -\dfrac{b-ds}{cs-a} I\end{bmatrix} H \\ &= (a-cs)\begin{bmatrix} I & -\psi(s)I\end{bmatrix} H.\end{aligned}$$

Since $a \ne \mathrm{j}c\tilde\omega$ for all $\tilde\omega_1 \le \tilde\omega \le \tilde\omega_2$, then multiply the inequality above on the right by

$$\tilde N(\tilde\omega) := \left(\begin{bmatrix} I & -\mathrm{j}\tilde\omega I\end{bmatrix}(T\otimes I)H\right)_\perp = \left(\begin{bmatrix} I & -\psi(\mathrm{j}\tilde\omega)I\end{bmatrix} H\right)_\perp$$

and on the left by its transpose conjugate to obtain the frequency domain inequality

$$\left(\begin{bmatrix} I & -\psi(\mathrm{j}\tilde\omega)I\end{bmatrix} H\right)_\perp^* \Theta(\tilde\omega)\left(\begin{bmatrix} I & -\psi(\mathrm{j}\tilde\omega)I\end{bmatrix} H\right)_\perp \prec 0,$$

which should hold for all $\tilde\omega_1 \le \tilde\omega \le \tilde\omega_2$.

Since $\alpha < 0 < \gamma < \infty$ the set $\{s = \mathrm{j}\tilde\omega, \tilde\omega \in \mathbb{R} : \tilde\omega_1 \le \tilde\omega \le \tilde\omega_2\}$ with (18) is equivalent to $\mathbf{\Lambda}(\Phi_0, \Psi_0)$ as given in (14). Therefore, we can use Lemma 5, Corollary 6 and the correspondences $\xi = \psi(\mathrm{j}\tilde\omega)$ and $\tilde\omega = -\mathrm{j}\psi^{-1}(\xi)$, which hold for all $\xi \in \mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$ to establish (20).

We now prove the necessity part of item (i). Equivalence between the FDI (19) and the existence of $P, Q \in \mathbb{HC}^n$, $Q \succeq 0$ such that (7) is feasible has been established in Iwasaki and Hara (2005). Assume therefore that some $P, Q \succeq 0$ satisfying (7) exist. Define the matrix

$$\tilde X(\tilde\omega) := \begin{bmatrix} -\alpha Q & \mathrm{j}\tilde\omega\alpha Q \\ -\mathrm{j}\tilde\omega\alpha Q & \gamma Q\end{bmatrix} = \begin{bmatrix} -\alpha & \mathrm{j}\tilde\omega\alpha \\ -\mathrm{j}\tilde\omega\alpha & \gamma\end{bmatrix}\otimes Q.$$

Note that since $\alpha < 0$, then $\tilde X(\tilde\omega) \succeq 0$ for all $s = \mathrm{j}\tilde\omega \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$ because using Schur complement

$$Q \succeq 0, \qquad \alpha\tilde\omega^2 + \gamma \ge 0, \qquad \alpha < 0, \quad \Longrightarrow \quad \tilde X(\tilde\omega) \succeq 0.$$

Now add the matrix $H^*(T\otimes I)^*\tilde X(\tilde\omega)(T\otimes I)H \succeq 0$, which is positive semidefinite for all $s = \mathrm{j}\tilde\omega \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$, to the right hand side of (7) so that for all $s = \mathrm{j}\tilde\omega \in \mathbf{\Lambda}(\Phi_0, \Psi_0)$ we have

$$\begin{aligned}\Theta &\prec H^*(T\otimes I)^*\left[-\Phi_0\otimes P - \Psi_0\otimes Q + \tilde X(\tilde\omega)\right](T\otimes I)H, \\ &= H^*(T\otimes I)^*\left(\begin{bmatrix} -2\alpha Q & \mathrm{j}\tilde\omega\alpha Q \\ -\mathrm{j}\tilde\omega\alpha Q & 0\end{bmatrix}\right. \\ &\quad \left. + \begin{bmatrix} 0 & -(P+\beta Q) \\ -(P+\beta Q) & 0\end{bmatrix}\right)(T\otimes I)H, \\ &= H^*(T\otimes I)^*\left(\begin{bmatrix} -\alpha Q \\ -(P+\beta Q)\end{bmatrix}\begin{bmatrix} I & -\mathrm{j}\tilde\omega I\end{bmatrix}\right. \\ &\quad \left. + \begin{bmatrix} I \\ \mathrm{j}\tilde\omega I\end{bmatrix}\begin{bmatrix} -\alpha Q & -(P+\beta Q)\end{bmatrix}\right)(T\otimes I)H.\end{aligned}$$

The above inequality provides the key technical step that enables one to choose

$$\begin{bmatrix} F \\ G\end{bmatrix} = H^*(T\otimes I)^*\begin{bmatrix} \alpha Q \\ P+\beta Q\end{bmatrix},$$

so that

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G\end{bmatrix}\begin{bmatrix} I & -\mathrm{j}\tilde\omega I\end{bmatrix}(T\otimes I)H\right\} + \Theta \prec 0,$$

for any $\tilde\omega_1 \le \tilde\omega \le \tilde\omega_2$, where $\tilde\omega_1$ and $\tilde\omega_2$ are given by (18), in particular, for $\tilde\omega = \tilde\omega_1$ and $\tilde\omega = \tilde\omega_2$ which imply that the pair of inequalities (11) are feasible. □

Note that the only case included in the Generalized KYP Lemma that is not covered by Theorem 7 is the case when Ψ_0 is such that $0 \le \alpha \le \gamma$. However, as noticed in Iwasaki and Hara (2005), this implies $\Psi_0 \succeq 0$ which means that Q can be set to zero in (7), reducing the Generalized KYP Lemma to a standard frequency-independent KYP Lemma. Indeed, for any choice of $0 \le \alpha \le \gamma$, the curve associated to $\mathbf{\Lambda}(\Phi_0, \Psi_0)$ is the entire imaginary axis. For this reason, there is no need to treat such a case separately. However, by exploring the properties of linear-fractional mappings we are able to derive conditions that hold for the entire imaginary axis in which the associated matrix Ψ_0 is not positive semidefinite. As we shall see in Section 4.2, such conditions do not reduce to the standard KYP Lemma.

3.1. *Proof of Theorems 1 and 2*

In this section we will use Theorem 7 to prove Theorems 1 and 2. First notice that the segment of the imaginary axis $\mathrm{j}[\omega_1, \omega_2]$ can be described by the set $\mathbf{\Lambda}(\Phi, \Psi)$ with the choices Φ and Ψ as in (10) (see Iwasaki and Hara (2003a, 2005)). Recall that $\omega_c := (\omega_1 + \omega_2)/2$. From Lemma 3, there exists a nonsingular transformation matrix $T \in \mathbb{C}^{2\times 2}$ such that this curve can be represented by $\mathbf{\Lambda}(T^*\Phi_0 T, T^*\Psi_0 T)$, where Φ_0 and Ψ_0 are given in the form (13). One can verify that such transformation and matrices Φ_0 and Ψ_0 are

$$T = \begin{bmatrix} 1 & -\mathrm{j}\omega_c \\ 0 & 1\end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & \hat\omega^2\end{bmatrix},$$

where $\hat\omega := (\omega_2 - \omega_1)/2$. Note that $-1 = \alpha < 0 < \gamma = \hat\omega^2$ for any $\omega_1 \ne \omega_2$. Let H be as in (10) and $\tilde\omega_1 = -|\hat\omega| = (\omega_1 - \omega_2)/2$, $\tilde\omega_2 = |\hat\omega| = (\omega_2 - \omega_1)/2$, such that

$$\begin{bmatrix} I & -\mathrm{j}\tilde\omega_i I\end{bmatrix}(T\otimes I)H = \begin{bmatrix} I & -\mathrm{j}\omega_i I\end{bmatrix}\begin{bmatrix} A & B \\ I & 0\end{bmatrix},$$

after performing the change-of-variables $\omega := \tilde\omega + \omega_c$. Furthermore $\omega_1 = \tilde\omega_1 + \omega_c \le \omega \le \tilde\omega_2 + \omega_c = \omega_2$. That is, Theorems 1 and 2 are particular instances of items (i) and (ii) of Theorem 7, respectively.

4. Extensions and applications

In this section we discuss some extensions and applications of the results presented so far. Note that the many applications discussed in Iwasaki and Hara (2005) can be reinterpreted and extended in the light of our new results. We tried to avoid such repetition in the present paper, but the interested reader is referred to Graham (2007).

4.1. Piecewise affine multipliers

The results presented in previous sections can be extended to cope with piecewise affine frequency-dependent matrices $\Theta(\omega)$ by simply solving the pair of inequalities (5) or (11) for each sub-interval $\omega_{2\ell-1} \le \omega \le \omega_{2\ell}$, $\ell = 1, \dots, N$ of the form

$$\Theta_\ell(\omega) = \frac{\omega_{2\ell} - \omega}{\omega_{2\ell} - \omega_{2\ell-1}}\,\Theta_{2\ell-1} + \frac{\omega - \omega_{2\ell-1}}{\omega_{2\ell} - \omega_{2\ell-1}}\,\Theta_{2\ell}.$$

Note that $\Theta(\omega)$ is not a proper rational function of ω, which means that the Θ cannot be realized as a proper rational transfer function of ω. In fact, $\Theta(\omega)$ might not even be a continuous function of ω, or be defined in a contiguous interval. When the sub-intervals are contiguous, continuity of $\Theta(\omega)$ can be achieved by imposing $\Theta_{2\ell} = \Theta_{2\ell+1}$ for some or all $1 \le \ell < N$. Nevertheless, rational or other types of bounded functions of ω can be approximated by piecewise affine functions of ω, especially on finite frequency intervals.

One interesting application of piecewise affine matrices $\Theta(\omega)$ is to handle real-valued matrices A and B in Theorem 2, or real-valued H in Theorem 7. In such cases, as will be shown by the next lemma, there exists a piecewise affine matrix $\Theta(\omega)$ so that feasibility of the proposed test on the interval $\omega_1 \le \omega \le \omega_2$ also implies feasibility of the frequency domain inequality on the symmetric interval $-\omega_2 \le \omega \le -\omega_1$. A similar result can be obtained in the context of Theorem 7. The proof of this next lemma is omitted for brevity.

Lemma 8. *Let matrices $A \in \mathbb{R}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n\times m}$ and $\Theta_1, \Theta_2 \in \mathbb{HC}^{n+m}$ be given. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that the pair of LMI (5) has a feasible solution then the frequency domain inequality (12) holds for all $\omega_1 \le |\omega| \le \omega_2$ with*

$$\Theta(\omega) := \begin{cases} \dfrac{\omega_2 + \omega}{\omega_2 - \omega_1}\,\overline{\Theta}_1 - \dfrac{\omega_1 + \omega}{\omega_2 - \omega_1}\,\overline{\Theta}_2, & -\omega_2 \le \omega \le -\omega_1, \\ \dfrac{\omega_2 - \omega}{\omega_2 - \omega_1}\Theta_1 + \dfrac{\omega - \omega_1}{\omega_2 - \omega_1}\Theta_2, & \omega_1 \le \omega \le \omega_2, \end{cases} \tag{21}$$

being piecewise affine.

4.2. Handling infinite frequency extremes

Theorem 2 may have difficulties in handling unbounded frequency ranges. For instance, in the case $\omega_2 \to \infty$, one could conceptually search for limits on the problem variables as ω_2 increases by solving a sequence of pairs of inequalities (5). A more elegant solution is to transform the frequency variable and solve a modified problem on the transformed frequency that now has a finite limit.

Consider the high-frequency condition $|\omega| \ge |z|$. This inequality can be described by the curve $\mathbf{\Lambda}(\Phi, \Psi)$ with

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & -z^2 \end{bmatrix}.$$

One can verify that this curve is associated with

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi_0 = \begin{bmatrix} -z^2 & 0 \\ 0 & 1 \end{bmatrix},$$

Application of Theorem 7 yields the pair of inequalities

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix} -\mathrm{j}\tilde{\omega}_i I & I \end{bmatrix}\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\} + \Theta_i \prec 0, \quad i = \{1, 2\},$$

where $\tilde{\omega}_1 = -|1/z|$ and $\tilde{\omega}_2 = |1/z|$.

The above approach, which corresponds to the linear-fractional transformation $\psi(s) = s^{-1}$, handles the limit $\omega \to \infty$ at the expense of a creating a singularity at $\omega = 0$. In this way, it cannot be used to construct a generalization of the KYP Lemma, that is, a condition that holds for all frequencies, since the extreme points $|\tilde{\omega}_1| = |\tilde{\omega}_2| = |1/z|$ tend to infinity as $z \to 0$. This problem can be overcome by alternatively considering

$$\psi : \mathbb{C} \to \mathbb{C}, \qquad \psi(s) = \mathrm{j}z + \frac{\mathrm{j}y\,(s - \mathrm{j}x)}{(y - x)(\mathrm{j}y - s)}, \tag{22}$$

with $x, y, z \in \mathbb{R}$, $x < y$, $y > 0$, which maps the finite segment of the imaginary axis $s \in \mathrm{j}[x, y)$ onto the infinite segment of the imaginary axis $\xi \in \mathrm{j}[z, \infty)$. Mapping (22) can be used in Theorem 7 with

$$T = \begin{bmatrix} 1 & \frac{\mathrm{j}}{2}(x - y) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(y - x) & \mathrm{j}y[x + z(x - y)] \\ \mathrm{j}(x - y) & z(x - y) + y \end{bmatrix}. \tag{23}$$

We will not proceed with this general form, but rather focus on a particular choice in order to simplify the exposition. The next corollary is a version of Theorem 2 specialized to cover the segment $s \in \mathrm{j}[0, 1]$.

Corollary 9. *Let matrices $A \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{C}^{n\times m}$, and $\Theta_1, \Theta_2 \in \mathbb{HC}^{m+n}$ be given. Let $z \in \mathbb{R}$ be also given. If there exist matrices $F \in \mathbb{C}^{n\times n}$, $G \in \mathbb{C}^{m\times n}$ such that*

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix} (1 - \omega_i)I & -\mathrm{j}[z + \omega_i(1 - z)]I \end{bmatrix} \times \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\} + \Theta_i \prec 0, \tag{24}$$

for $i = \{1, 2\}$ where $\omega_1 = 0$ and $\omega_2 = 1$, then the following frequency-domain inequality holds

$$\begin{bmatrix} (\mathrm{j}\eta I - A)^{-1}B \\ I \end{bmatrix}^* \Theta\left(\frac{\eta - z}{1 - z + \eta}\right)\begin{bmatrix} (\mathrm{j}\eta I - A)^{-1}B \\ I \end{bmatrix} \prec 0 \tag{25}$$

for all $z \le \eta < \infty$.

Proof. Setting $x = 0$ and $y = 1$ in (23) we obtain

$$T = \begin{bmatrix} 1/2 & -\mathrm{j}(z + 1)/2 \\ -\mathrm{j} & 1 - z \end{bmatrix}, \qquad \Phi_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\Psi_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Invoking Theorem 7 with H as in (10) we obtain

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix} I & -\mathrm{j}\tilde{\omega}_i I \end{bmatrix}\left(\begin{bmatrix} 1/2 & -\mathrm{j}(z + 1)/2 \\ -\mathrm{j} & 1 - z \end{bmatrix} \otimes I\right) \times \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\} + \Theta_i \prec 0,$$

for $i = \{1, 2\}$. Inequalities (24) come after noticing that

$$\begin{bmatrix} I & -\mathrm{j}\tilde{\omega}_i I \end{bmatrix}\left(\begin{bmatrix} 1/2 & -\mathrm{j}(z + 1)/2 \\ -\mathrm{j} & 1 - z \end{bmatrix} \otimes I\right) = \begin{bmatrix} (1 - \omega)I & -\mathrm{j}[z + \omega(1 - z)]I \end{bmatrix},$$

where we have performed the change of variables $\omega = \tilde{\omega} + 1/2$. Note that

$$\psi^{-1} : \mathbb{C} \to \mathbb{C}, \qquad \psi^{-1}(\xi) = \frac{\xi - \mathrm{j}z}{1 - z - \mathrm{j}\xi},$$

so that for $\xi = \mathrm{j}\eta$ we have the relationship $\omega = -\mathrm{j}\psi^{-1}(\xi) = (\eta - z)/(1 - z + \eta)$, which appear in (25). □

The above corollary handles $\eta \to \infty$ by letting (23) attain a finite limit as $\omega_2 \to y = 1$, so that the resulting LMI can be solved without numerical complications. In fact, inequalities (24) involve only finite coefficients as long as z is finite, including zero. Combining Lemma 8 with Corollary 9 one obtains an interesting extension of the KYP Lemma, that is, an LMI condition that establishes an FDI for all $\eta \in \mathbb{R}$, while still allowing for a frequency dependent $\Theta(\omega)$. The particular pair of inequalities (24) associated with this case $z = 0$ is given by

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix}(1-\omega_i)I & -\mathrm{j}\omega_i I\end{bmatrix}\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\} + \Theta_i \prec 0, \tag{26}$$

where $\omega_1 = 0, \omega_2 = 1$.

The conditions of Corollary 9, in the general case z is arbitrary, are associated with the curve $\mathbf{\Lambda}(\Phi, \Psi)$ where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} 0 & \mathrm{j}/2 \\ -\mathrm{j}/2 & -z \end{bmatrix},$$

which is simply $\{s \in \mathbb{C} : s = \mathrm{j}\omega, \mathrm{j}(s^* - s)/2 \geq z\} = \{\omega \in \mathbb{R} : \omega \geq z\}$. It is interesting to note that in the particular case $z = 0$ the matrix Ψ is not positive semidefinite, which implies that one cannot set $Q = 0, F = P, G = 0$, in order to reduce these conditions to the original KYP Lemma for constant Θ. Application of (26) compared to the standard KYP Lemma will be illustrated in the numerical examples.

In the case $z = 0$, the transformation (22) with $x = 0$ and $y = 1$ implies that $\omega = -\mathrm{j}\psi^{-1}(\mathrm{j}\eta) = \eta/(1+\eta)$ which, evaluated at the limits of the interval $\eta \in [0, \infty)$, yields the approximation $\omega \approx \eta$, for $\eta \approx 0$, and $\omega \to 1$, for $\eta \to \infty$. This means that the frequency dependent scaling Θ should behave as $\Theta(\omega) \approx \Theta(\eta)$, for $\eta \approx 0$, and $\Theta(\omega) \to \Theta(1)$, for $\eta \to \infty$. That is, Θ is a linear function of η near the origin and it approaches a constant as $\eta \to \infty$. One could have arrived at the opposite scenario by choosing different constants on the mapping (22), a result that will not be pursued here for brevity.

Finally note that Theorem 2 and Corollary 9 are not equivalent and, in fact, they may produce different results for the very same frequency range. As seen above, the multiplier Θ is affine on $\omega = -\mathrm{j}\psi^{-1}(\mathrm{j}\eta)$, according to (6), but it is nonlinear on η.

4.3. μ-analysis

Consider the standard setup for robustness and performance μ analysis, i.e. the Linear Fractional Transformation (LFT) feedback connection of a nominal map M and an uncertainty or perturbation Δ. The nominal map $M(s)$ is assumed to be a rational function of the complex variable s, being a proper and square matrix that is analytic in the closed right-half plane. The unknown uncertainty is assumed to have the following structure

$$\mathbf{\Delta} := \Big\{\mathrm{diag}\left[\phi_1 I_{s_1}, \ldots, \phi_r I_{s_r}, \delta_1 I_{s_1}, \ldots, \delta_c I_{s_c},\right.$$
$$\left.\Delta_1, \ldots, \Delta_F\right] : \phi_i \in \mathbb{R}, \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\Big\}. \tag{27}$$

By choosing the number, size, and dynamic nature of the blocks of Δ, a variety of uncertainty structures can be translated into this standard form (see for instance Zhou et al. (1996)).

Let $\mathcal{T}(\Delta)$ denote the set of all block diagonal and stable rational transfer function matrices that have block structures such as $\mathbf{\Delta}$

$$\mathcal{T}(\Delta) := \{\Delta(\cdot) \in \mathcal{RH}_\infty : \Delta(s_0) \in \mathbf{\Delta} \forall s_0 \in \mathbb{C}_+\}. \tag{28}$$

The feedback connection of (M, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{T}(\Delta)$ with $\|\Delta\|_\infty < \beta^{-1}$ if and only if Zhou et al. (1996)

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}}(M(\mathrm{j}\omega)) \leq \beta, \tag{29}$$

where $\mu_{\mathbf{\Delta}}$ denotes the structured singular value of a matrix, which is defined as

$$\mu_{\mathbf{\Delta}}(M) := \left(\inf_{\Delta \in \mathbf{\Delta}} \{\|\Delta\| : \det(I - M\Delta) = 0\}\right)^{-1}.$$

In case no $\Delta \in \mathbf{\Delta}$ makes $(I - M\Delta)$ singular $\mu_{\mathbf{\Delta}}(M) := 0$.

In general, the structured singular value $\mu_{\mathbf{\Delta}}$ cannot be computed in polynomial time (NP-hard (Toker & Özbay, 1998)). In practice, the introduction of appropriate *scalings* or *multipliers* using duality theory is commonly used to provide computable upper bounds for $\mu_{\mathbf{\Delta}}$.

For instance, define the set of scaling matrices

$$\mathbf{Z} := \{\mathrm{diag}[Z_1, \cdots, Z_{s_r+s_c}, z_1 I_{m_1}, \ldots, z_F I_{m_F}] :$$
$$Z_i \in \mathbb{C}^{s_i \times s_i}, Z_i = Z_i^* > 0, z_j \in \mathbb{R}, z_j > 0\},$$
$$\mathbf{Y} := \{\mathrm{diag}[Y_1, \ldots, Y_{s_r}, 0, \ldots, 0] : Y_i = Y_i^* \in \mathbb{C}^{s_i \times s_i}\}.$$

Note that $\mathbf{Z}$ and $\mathbf{Y}$ commute with the matrices in $\mathbf{\Delta}$. Now define the matrix valued function

$$\Gamma_\beta(M, Z, Y) := M^* Z M - \mathrm{j}\left(M^* Y - Y M\right) - \beta^2 Z,$$

and the optimization problem

$$\rho_{\mathbf{\Delta}}(M) := \inf_{\beta \in \mathbb{R}, Z \in \mathbf{Z}, Y \in \mathbf{Y}} \sup_{\omega \in \Omega} \left\{\beta : \Gamma_\beta(M(\mathrm{j}\omega), Z(\omega), Y(\omega)) \prec 0\right\}.$$

Using duality (Fan et al., 1991) one has that $\sup_{\omega \in \Omega} \mu_{\mathbf{\Delta}}(M(\mathrm{j}\omega)) \leq \rho_{\mathbf{\Delta}}(M)$. The problem of computing $\rho_{\mathbf{\Delta}}(M)$ is simpler than the original problem (29). Yet it cannot be easily solved as well. Commonly found strategies for approaching this problem consider constant multipliers Z and Y for a single frequency $\Omega = \{\omega_1\}$, finite frequency range $\Omega \subset \mathbb{R}$ as in Iwasaki and Hara (2005), or the entire real axis $\Omega = \mathbb{R}$ as in the KYP lemma.

Theorem 2 can be used to produce upper bounds to $\rho_{\mathbf{\Delta}}$ which has as its main advantage the fact that scaling matrices Z and Y are allowed to be affine functions of ω. The next corollary was presented by the authors in Graham et al. (2006).

Corollary 10. *Let $A \in \mathbb{R}^{n \times n}$ with no eigenvalues on the imaginary axis, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$ be given. If there exist matrices $Z_1, Z_2 \in \mathbf{Z}$, $Y_1, Y_2 \in \mathbf{Y}$, $F \in \mathbb{C}^{n \times n}$, and $G \in \mathbb{C}^{m \times n}$ such that*

$$\mathrm{He}\left\{\begin{bmatrix} F \\ G \end{bmatrix}\begin{bmatrix} I & -\mathrm{j}\omega_i I\end{bmatrix}\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}\right\}$$
$$+ \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -\mathrm{j}Y_i \\ \mathrm{j}Y_i & -\beta^2 Z_i \end{bmatrix}\begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0, \tag{30}$$

for $i = \{1, 2\}$ has feasible solutions then $\rho_{\mathbf{\Delta}}(C(\mathrm{j}\omega I - A)^{-1}B + D, \Omega) \leq \beta$ for all $|\omega| \in \Omega = [\omega_1, \omega_2]$.

Proof. Follows from Lemma 8 noting that

$$\Theta_i = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} Z_i & -\mathrm{j}Y_i \\ \mathrm{j}Y_i & -\beta^2 Z_i \end{bmatrix}\begin{bmatrix} C & D \\ 0 & I \end{bmatrix}, \quad i = \{1, 2\},$$

and that the matrix variables Z_i, Y_i appear linearly in (30). Feasibility of (30) implies that $\rho_{\mathbf{\Delta}}(C(\mathrm{j}\omega I - A)^{-1}B + D, \Omega) \leq \beta$ for all $\omega_1 \leq |\omega| \leq \omega_2$. □

5. Numerical example

In this section, Corollaries 9 and 10 are used to illustrate the possible reduction in conservativeness when using the generalization proposed by Theorem 2 in the context of μ-analysis. This example explores the fact that Θ appears affinely in the LMI to synthesize frequency-dependent multipliers appearing in $\Theta(\omega)$.

Table 1
Upper bounds for $\rho_{\mathbf{\Delta}}$; $\sup_{w\in\mathbb{R}} \mu_{\Delta} = 0.291$.

Method	Upper bound ($\rho_{\mathbf{\Delta}}$)	
KYP Lemma (2)	0.458	$0 \le \lvert\omega\rvert < \infty$
Corollary 9	0.293	$0 \le \lvert\omega\rvert < \infty$

Table 2
Upper bounds on $\rho_{\mathbf{\Delta}}$ for Example 2; $\sup_{w\in\mathbb{R}} \mu_{\Delta} = 0.291$.

Method	Upper bound ($\rho_{\mathbf{\Delta}}$)	
	$0 \le \lvert\omega\rvert \le 1$	$1 \le \lvert\omega\rvert \le \infty$
Method of Helmersson (1995)[a]	$\gamma_1 = 0.111$	$\gamma_2 = 0.509$
Gen KYP Lemma (Iwasaki & Hara, 2005)	$\beta_1 = 0.115$	$\beta_2 = 0.458$
Corollary 10/ 9	$\eta_1 = 0.102$	$\eta_2 = 0.293$

[a] Computed using mu command from Matlab μ-toolbox.

Here we consider the same feedback connection as in de Gaston and Safonov (1988) and Helmersson (1995) where the generalized plant is

$$M = \left[\begin{array}{cccc|ccc} -4 & 0 & -800 & 6400 & 80 & -0.2 & 0 \\ 1 & -6 & 0 & 0 & 0 & 0 & -0.3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -10 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and the uncertainty structure is $\mathbf{\Delta} = \{\mathrm{diag}[\phi_1, \phi_2, \phi_3] : \phi_i \in \mathbb{R}, i = \{1, 2, 3\}\}$.

Constant scalings $Z \in \mathbf{Z}$, and $Y \in \mathbf{Y}$ are used with the standard KYP Lemma. Note the presence of the Y associated with the real uncertainty. We used $Z_i \in \mathbf{Z}$, $Y_i \in \mathbf{Y}$, $i = \{1, 2\}$ to search for *frequency-dependent scalings* using Corollary 9. The smallest upper bounds found by each method for the frequency range $\omega \in \mathbb{R}$ are listed in Table 1. These values should be compared against the exact value for $\sup_{w\in\mathbb{R}} \mu_{\mathbf{\Delta}}$ with real uncertainty which in this case is known to be 0.291 (see de Gaston and Safonov (1988)).

Next we attempt a rough comparison with the results of Helmersson (1995), where tighter bounds for $\rho_{\mathbf{\Delta}}$ are produced by successively bisecting the frequency range into smaller intervals. Each iteration produces one more real uncertain parameter for which a new (constant) multiplier must be computed. In order to compare our results with this approach we take only two steps in the algorithm of Helmersson (1995), in which the positive imaginary axis is split into two frequency intervals $\{0 \le \omega \le 1\} \cup \{1 \le \omega < \infty\}$.

The smallest upper bounds found by each method compared are listed in Table 2. The results of Helmersson (1995) are given in the first row of Table 2. Note that the values shown are taken directly from Helmersson (1995), where they have been computed using the Matlab μ-toolbox.

Constant scalings are used for the Generalized KYP Lemma (3) with $-\omega_1 = \omega_2 = 1$ to compute the first bound shown in the second column and second row of Table 2. In order to compute the "high-frequency" bound on the third column of the second row, we used the high-frequency version of the Generalized KYP Lemma from Iwasaki and Hara (2005) with constant scalings. We used frequency dependent scalings using Corollaries 9 and 10. The values on the third row of Table 2 have been computed using Corollary 10 (second column) and Corollary 9 (third column) with $x = 0$ and $y = z = 1$.

The upper bounds from Table 2 are compared with the greatest lower bound for $\rho_{\mathbf{\Delta}}$ obtained on a dense grid in Fig. 1. It is worth noticing that the largest peak on the plot is very sharp, and that the max value of $\rho_{\mathbf{\Delta}}$ obtained with 100 logarithmically-spaced frequency points between 10^{-1} and 10^{2} was only 0.223. In Fig. 1,

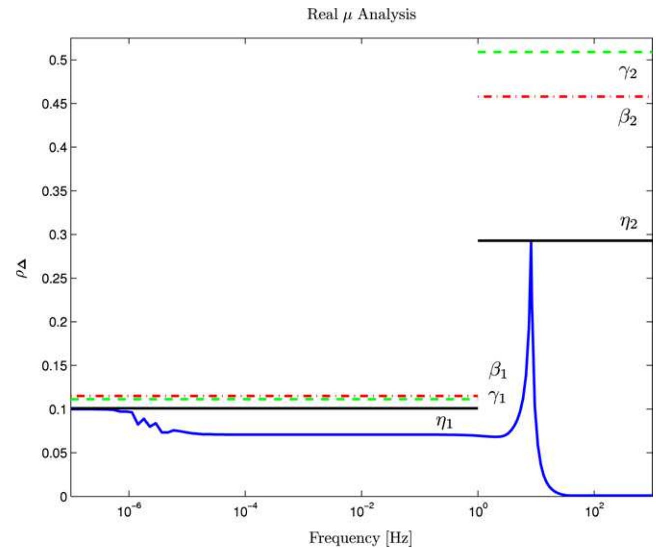

Fig. 1. Robust analysis for real structured uncertainty in Examples 2, comparing the bounds computed using the results of this paper with other results in the literature (high frequency). The labels have been defined in Table 2. The curved solid line is the greatest lower bound for $\rho_{\mathbf{\Delta}}$.

the known critical frequency $\omega = 8.22$ (see de Gaston and Safonov (1988)) was added to this grid in order to obtain the exact value of $\rho_{\mathbf{\Delta}} = \mu_{\mathbf{\Delta}} = 0.291$. Further splitting of the intervals $\{0 \le \omega \le 1\} \cup \{1 \le \omega < \infty\}$ is possible, however the upper bound η_2 computed for the range $\omega \in [1, \infty)$ using Corollary 9 already matches the best bounds obtained for all other finite frequency results. It remains open as to whether one can find an efficient splitting of frequency intervals that provides a good rough landscape for the upper bound on $\rho_{\mathbf{\Delta}}$.

6. Conclusions

This paper has provided an alternative formulation of the KYP Lemma that allows for the coefficient matrix of the frequency domain inequality to vary affinely with the frequency parameter. The result is shown to contain existing generalizations of the KYP Lemma in the particular case that the coefficient matrix does not depend on frequency. Applications of this result to the stability analysis of linear systems are illustrated in the paper including an effective new way to compute upper bounds for the structured singular value with frequency-dependent scalings. Regarding this application, the question of characterizing the class of uncertainties that is represented by the proposed robust stability test remains open.

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