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TIME-OPTIMAL INPUT SHAPING FOR DISCRETE-TIME LTI SYSTEMS WITH APPLICATION TO SEEK PROFILES OF A HDD SYSTEM

Oyvind N. Stamnes[1] **, Raymond A. De Callafon**[2]**.**

[1]Norwegian University of Science and Technology
Department of Engineering Cybernetics, Trondheim, Norway
oyvista@stud.ntnu.no

[2]University of California San Diego
Dept. of Mechanical and Aerospace Engineering, La Jolla, USA
callafon@ucsd.edu

Introduction

We introduce an numerical algorithm for calculating the time-optimal input sequence for a discrete-time Linear Time Invariant (LTI) plant with constraints. The algorithm takes into account constraints on the states and the input. Motivated by Receeding Horizon Control (RHC) [6] we formulate the problem as a set of linear inequalities and iteratively check feasibility of the constraints using a binary search to find the optimal number of samples. Similar results applied to a Hard Disk Drive (HDD) system can be found in [9] where a least-squares approach has been used. In [9] only constraints on the input where considered. A more general approach can be found in [3] where constraints on input and states are considered. Extension using a binary search algorithm to speed up the algorithm can be found in [7]. It has been proved that the time-optimal input for a LTI discrete-time system is not bang-bang, [4] and references therein. Our algorithm verifies these results. Non-uniqueness of the time-optimal input sequence is also explored through further optimization.

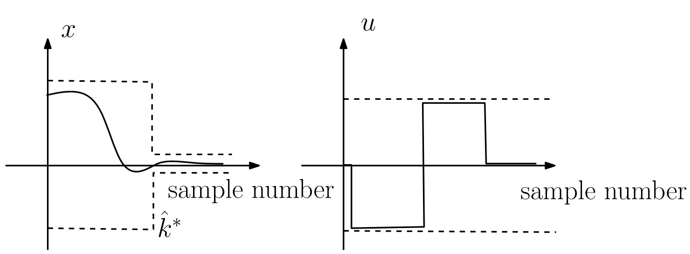

Figure 1. Constraints(dotted lines) on input u and states x , feasible trajectory solid line

1 Dynamics and Signal Constraints

To define the problem of optimal-input shaping, we first define the dynamics of the LTI system, along with constraints that need to be satisfied during the computation of time-optimal input profiles. For that purpose, consider the controllable state space representation of a discrete-time plant

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where the state vector $x(k) \in \mathbb{R}^n$, the input $u(k) \in \mathbb{R}^m$ and the matrices A and B are of appropriate dimensions. In addition, $k \in \{0\ 1\ \ldots\ N\}$ where N denotes the control horizon. Furthermore let the system (1) be subject to the following time-domain constraints:

$$\underline{u} \leq u(k) \leq \overline{u} \qquad 0 \leq k \leq N-1 \tag{2}$$

$$\underline{x_d} \leq x(k) \leq \overline{x_d} \qquad 0 \leq k < \hat{k}^* \tag{3}$$

$$\underline{x_f} \leq x(k) \leq \overline{x_f} \qquad \hat{k}^* \leq k \leq N \tag{4}$$

For simplicity, we only consider constant constraints here but results can be generalized to time varying constraints. We consider constraints typically associated to actuator limitation on the input $u(k)$ in (2). Performance constraints are defined by (3) as a restriction on the states during a transition while (4) defines the constraints on the states at the end of the input shaping. Figure 1 illustrates the natire of the different constraints. In addition, constraints on the rate of change of the input or constraints on any linear combination of the states can be included, but been left out for the sake of simplicity in this extended abstract.

Let k^* denote the smallest $\hat{k}^*$ such that all constraints are feasible. Our goal is to find this k^* and the accompanying input profile $u(k),\ k=0,\ldots,N-1$. Given an initial condition $x(0)$ and an initial estimate of $\hat{k}^*$, we pose the following question: *Does there exist an input sequence $u(0)$, $u(1)$, ... $u(N-1)$ so that (1)-(4) are satisfied?* Motivated by RHC [6], we can rewrite the dynamic system and its imposed constraints into a set of linear inequalities of the form

$$L\mathbf{u} \leq W \tag{5}$$

where $\mathbf{u} = [u(0)^T\ u(1)^T\ \ldots\ u(N-1)^T]^T$, and the matrices L and W are defined as:

$$L = \begin{bmatrix} I_{Nm} \\ -I_{Nm} \\ \Psi \\ -\Psi \end{bmatrix},\ W = \begin{bmatrix} \mathbf{u_{max}} \\ \mathbf{u_{min}} \\ \mathbf{x_{max}} \\ \mathbf{x_{min}} \end{bmatrix}. \tag{6}$$

In (6), the matrix I_{Nm} is an $Nm \times Nm$ identity matrix and corresponds to the inequality (2). Furthermore, Ψ is an $Nn \times Nm$ matrix corresponding to inequalities (3) and (4). Ψ is defined as:

$$\Psi = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}$$

For completeness, the corresponding vectors of W in (6) are defined as

$$\begin{aligned} \mathbf{u_{max}} &= \left[\overline{u} \cdots \overline{u}\right]^T \\ \mathbf{u_{min}} &= \left[\underline{u} \cdots \underline{u}\right]^T \\ \mathbf{x_{max}} &= \left[\overline{x_d} \cdots \overline{x_d}\ \overline{x_f} \cdots \overline{x_f}\right]^T - \Omega x(0) \\ \mathbf{x_{min}} &= \left[-\underline{x_d} \cdots -\underline{x_d}\ -\underline{x_f} \cdots -\underline{x_f}\right]^T + \Omega x(0) \\ \Omega &= \left[A^T\ (A^2)^T\ \cdots\ (A^N)^T\right]^T \end{aligned} \tag{7}$$

where the input at $k=-1$, $u(-1)$, will be assumed to be zero in our case.

The question we posed earlier can now be formulated as *Does there exist a vector* $\mathbf{u}$ *such that (5) is satisfied*? One way to solve this problem is to compute a vector $\mathbf{u}$. This is related to finding an initial feasible point for a linear or quadratic program and [8, p. 462] suggests the following feasibility linear program:

$$\begin{gathered} \min_{\mathbf{u},\mathbf{z}} \mathbf{e}^T\mathbf{z} \\ \text{s.t. } L\mathbf{u} - \mathbf{z} \leq W \\ \text{and } \mathbf{z} \geq 0 \end{gathered}$$

where $\mathbf{e} = [1\ \ldots\ 1]^T$. The initial feasible point for this program is $\mathbf{z_0} = \max(L\mathbf{u_0} - W, 0)$ for any $\mathbf{u_0}$. If the optimal solution is $\mathbf{z} = \mathbf{0}$, then there exists a feasible solution to inequality (5). If the optimal solution $\mathbf{z} \neq \mathbf{0}$ then inequality (5) is infeasible. The linear program can be solved with GLPK solver [5] in polynomial time, allowing fast computation of a feasible solution.

2 Binary Search

Given the possibility of checking if there exist a feasible input sequence for a given set of constraints, a time-optimal solution one can found by performing a binary search through the sample space. We use an approach to a bisection algorithm found in [1] to compute the smallest value of k^*. The algorithm is summarized below:

Binary Search Algorithm

input : $k_l < k^* \leq k_u$

output: Optimal time k^*

repeat

  $\hat{k}^* = \left\lceil \frac{k_u + k_l}{2} \right\rceil$

  solve feasibility problem, $L\mathbf{u} \leq W$

  if feasible **then** $k_u = \hat{k}^*$ **else** $k_l = \hat{k}^*$

until $k_u - k_l < 2$

$k^* = k_u$

The algorithm presented above finds the minimum number of samples, k^*, needed for the constraints (2)-(4) to be feasible. Since there might be more than one time-optimal trajectory one can further optimize the solution. One example would be to use quadratic programming to minimize the control energy used, as will be illustrated in the computational results presented in this paper.

3 Computational Results

For a discrete-time double integrator with velocity constraints, the well-known optimal time-solution is bang-bang input profile with a coasting interval, typically seen in Hard Disk Drive applications. To verify and compare our computational results with these known results, we first consider such a discretized zero-order-hold [2] double integrator system:

$$x(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u(k) \tag{8}$$

where x_1 denotes position, x_2 is velocity and T is the sampling time. The initial condition for this simulation is

$x_0 = 0$, the constraints are:

$$-1 \leq u(k) \leq 1 \qquad 0 \leq k \leq N-1 \tag{9}$$
$$-2 \leq x_1(k) \leq 2 \qquad 0 \leq k < \hat{k}^* \tag{10}$$
$$-0.75 \leq x_2(k) \leq 0.75 \qquad 0 \leq k < \hat{k}^* \tag{11}$$
$$1-\varepsilon \leq x_1(k) \leq 1+\varepsilon \qquad \hat{k}^* \leq k \leq N \tag{12}$$
$$-\varepsilon \leq x_2(k) \leq \varepsilon \qquad \hat{k}^* \leq k \leq N \tag{13}$$

with $N = 300$. These constraints states that the input is bounded by ± 1, the position during the move is bounded by ± 2, the velocity during the move is bounded by ± 0.75, the position at the end of the move is bounded by $1 \pm \varepsilon$ and the velocity at the end of the move is bounded by $\pm\varepsilon$, with $\varepsilon = 10^{-3}$. With these numbers, the binary search algorithm computes the minimum number of samples to be $k^* = 207$. Figure 2 shows the corresponding minimum-time solution that also minimizes the control energy. The input profile for $u(k)$ was found as the solution to $\min u^T u$ over the minimum-time solution space using quadratic programming. It can be verified that the solutions indeed resembles the well-known bang-bang input profile with a coasting interval.

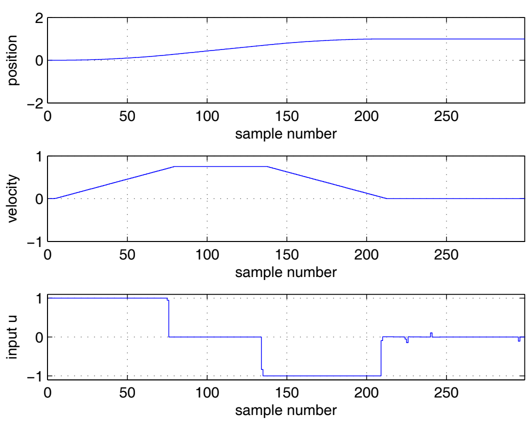

Figure 2. Simulation results for the discrete-time double integrator (8)

The algorithm can also handle systems with additional resonance modes. In this paper we also consider a system with one resonance mode for illustration purposes. The transfer function for the system is the ZOH equivalent of the continuous-time transfer function

$$G(s) = \frac{\omega_0^2}{s^2(s^2 + 2\zeta\omega_0 s + \omega_0^2)} \tag{14}$$

where $\zeta = 0.01$ and $\omega_0 = 20\pi$. For comparison purposes with the double-integrator system, the same constraints (9)-(13) are imposed. With these numbers, the binary search algorithm computes the minimum number of samples to be $k^* = 206$ which is even one sample faster than for the double-integrator system. Figure 3 shows the computed input profile as the solution to $\min u^T u$ over the minimum-time solution space using quadratic programming. It can be seen that the profile again resembles the conventional bang-bang with a coasting interval, but in addition small fluctuations in the input signal during transitions to minimize the settling time k^* can be observed.

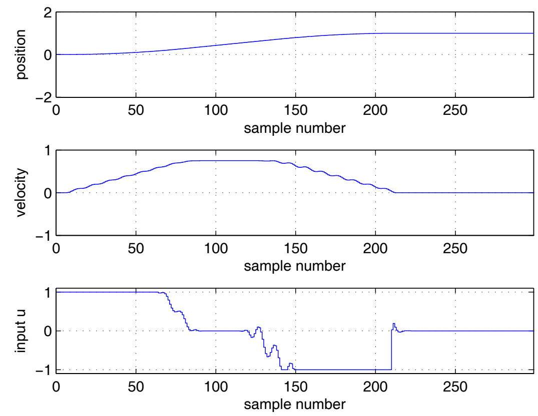

Figure 3. Minimum-time, minimum-control solution for the discrete-time ZOH equivalent of a double integrator with flexible mode defined in (14)

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