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Spectral Over-Bounding of Frequency Data for Modeling Product Variability in Hard Disk Drive Actuators

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***Abstract*— This paper presents a method of finding a control relevant, low-order weighting filter that over bounds the uncertainty variations in measured frequency domain data. Frequency response measurements of hard disk drive actuators are used to model manufacturing and mounting variation. A polynomial positivity condition is use to guarantee that a minimal phase representation of the uncertainty that is found and linear constraints in the frequency domain are used to bound the magnitude of uncertainty model. The positivity condition can be formulated as a linear matrix inequality and linear constraints are used to shape the filter. Manufacturing and mounting variations in hard disk drives are modeled as a multiplicative uncertainty and a low-order over bound is found from data on a finite grid. This application compares a linear programming based methods for finding uncertainty models with the purpose of designing a robust controller.**

I. INTRODUCTION

Models suitable for robust control [1] typically consist of a nominal model and a model in the form of a weighting filter that describes the associated uncertainty. Several method exist to this end including prediction error methods [2] and bounded error estimation or set membership identification [3]. Due to the presence of modeling error, a model of the uncertainty weighting filter may be of high order to sufficiently describe the variations in dynamics. Typical robust control design methods (based upon H_∞ control), will result in an optimal controller with order that is dependent upon the order of the nominal model and uncertainty weighting filter. Therefore, low order weighting filters are often desired for the controller synthesis.

When dealing with frequency domain data on a finite grid, the uncertainty model (that typically includes a nominal model and weighting filter) can be obtained directly from the available frequency domain data. The nominal model can be found via a least squares curve fitting procedure [4], but the weighting filter must satisfy additional constraints. Minimal phase filters are typically used and this condition can be enforced via a positivity condition. In addition to this condition, the weighting filter must satisfy constraints on the spectrum so that the robust performance condition can be satisfied.

The spectrum constraints can be written as an over bounding optimization problem. In [5] a frequency grid is used to transform a over-bounding constraint and the positivity condition into a linear programming (LP) problem. One drawback of this method is the inconsistency of the gridded constraints with the positivity of the spectrum. That is, solutions to the gridded frequency domain problem may result in a spectrum that cannot be factored into a real finite-dimensional system. Changing the positivity condition can eliminate this problem, which is done in this paper. Additionally, the intended use of the filter is emphasized by the appropriate choice of frequency dependent weighting functions during the optimization.

The upshot is a convex optimization problem that seeks a fixed-order, control-relevant weighting filter that can be used in the design of robust controllers. Previous applications of the type of positivity condition used here have been focused on filter design [6], [7], [8] and system identification [8] in the frequency domain. These positivity conditions have not been applied to estimate control relevant uncertainty models for robust control. A linear programming based method was used in [5] to find an uncertainty model for a large flexible structure based upon frequency domain information on a finite grid. However, this method only constrains the positivity of the polynomials (and therefore the spectrum) at the grid points only and did not seek a control relevant model. This is in contrast to the method used here. In this paper, we seek fixed-order, control-relevant weighting filter that can be used in an uncertainty model. The technique developed in this paper is applied to model manufacturing and mounting variations in hard disk drives on the basis of frequency domain data.

II. DESCRIPTION OF PZT-ACTUATED SUSPENSION

The experimental results presented in this paper are based on a dual-stage servo system that utilizes PZT-actuated suspensions manufactured by Hutchinson as the second-stage actuator [9]. In the HTI Magnum 5E PZT-actuated suspension, two PZT bars are used to move the slider in a direction perpendicular to the track by a push-pull configuration. In a hard disk with multiple disks, several suspensions are mounted on a single E-block, each carrying on its gimbal a slider with the read-write head. The E-block connects the suspensions to a radial Voice Coil Motor (VCM) for the gross movements of all of the read/write heads. The PZT-actuated suspensions are used as a secondary actuator for the fine movements of each of the read/write heads. Generally, the servo system performs track-following control of a single read-write head at the time.

To perform the uncertainty modeling of the dual-stage servo system, experiments were conducted to observe variations in its dynamic behavior, when several PZT-actuated suspensions were installed on the E-block. The variations are primarily caused by manufacturing and E-block mounting variations of the different suspensions. Both conditions result

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in (uncertain) variations in the dynamic response of the dual-stage actuators, even when the PZT-suspensions are mounted on the same E-block. As multiple suspension are controlled by a single servo controller, these uncertainties have to be taken into account when designing a high-performance robust dual-stage servo controller for track following.

The variations in the dynamical behavior of the dual-stage actuator were examined by measuring the frequency response of several units. To generate experimental data, several E-blocks with 4 suspension connection points and 9 different suspensions mounted at a fixed distance from the hard disks were used to measure 36 frequency responses. The slider position was measured by the Laser Doppler Vibrometer (LDV) and fed back to a Digital Signal Processor (DSP) on which a simple Proportional Derivative control algorithm was implemented to stabilize the VCM and slider position for the LDV measurements.

III. Background from Robust Control

Results from robust control will be used to create a model of the manufacturing and mounting variations in the PZT-actuated suspensions. A brief review of the robust performance problem for multiplicative uncertainty follows. Consider the Single Input, Single Output (SISO) system depicted in Fig. 1 and the associated *robust performance*

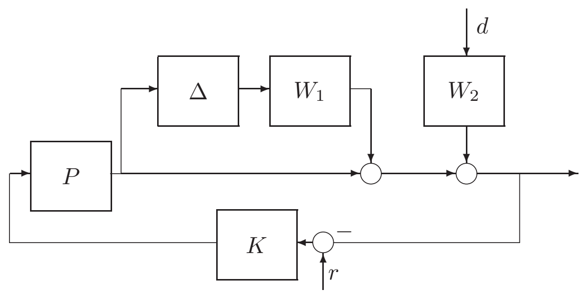

Fig. 1. Uncertain system described with multiplicative uncertainty and used in the design of a robust controller.

control problem: find a controller K such that

$$|W_2(e^{j\omega})||S(e^{j\omega})|+|W_1(e^{j\omega})||T(e^{j\omega})| \le 1, \quad \forall\omega \in [-\pi,\ \pi], \tag{1}$$

where

$$T = \frac{KP}{1+KP}, \quad S = \frac{1}{1+KP}, \quad W_1, W_2 \in \mathcal{RH}_\infty. \tag{2}$$

Condition (1) is a necessary and sufficient condition(see [1]) for robust performance of the SISO uncertain plant $P_\Delta \in \Pi$ where

$$\Pi := \{(1+W_1\Delta)P : \Delta \in \mathcal{RH}_\infty \, \|\Delta\|_\infty < 1\}. \tag{3}$$

In addition to these requirements it is often desirable to have $W_1^{-1}, W_2^{-1} \in \mathcal{RH}_\infty$. The order of the weighting filters W_1 and W_2 and the order of the plant P dictate the resulting order of the controller and the numerical ability to compute the actual controller. Thus, in order to obtain a low order controller, low order models of W_1 and W_2 are desired. On the other hand, W_1 will typically be of high order to sufficiently describe the uncertainty that is due to the dynamics of the system that are not modeled. In order to create a low order approximation that is relevant for robust performance, one might think of replacing the condition (1) by

$$|W_2(e^{j\omega})||S(e^{j\omega})|+|\hat{W}_1(e^{j\omega})||T(e^{j\omega})| \le 1,\ \forall\omega \in [-\pi,\ \pi], \tag{4}$$

where $\hat{W}_1$ is a low-order transfer function that over bounds the spectrum of W_1, that is,

$$|W_1(e^{j\omega})| \le |\hat{W}_1(e^{j\omega})|, \quad \forall\omega \in [-\pi, \pi]. \tag{5}$$

This allows for the design of a low order controller and provides a sufficient condition for robust performance.

The methods developed in this paper can deal with general optimization problems with constraints involving the square of spectrum of transfer functions. For instance, one such problem, of which the above problem is a particular case, is the one of finding lower and upper bounds on the spectrum of a low order transfer function $\hat{W}$ such that

$$W_l(\omega) \le |\hat{W}(e^{j\omega})| \le W_u(\omega), \quad \forall\omega \in [-\pi,\ \pi]. \tag{6}$$

For example, various choices for lower and upper bound functions can be used in robust control, such as the ones listed below:

1) For $W_l = |W_1|$ and $W_u = |T|^{-1}(\gamma - |W_2||S|)$ one obtains a robust design that aims to satisfy robust performance.
2) For $W_l = |W_1|$ and $W_u = \gamma|T|^{-1}$ robust stability is emphasized with a robustness level γ for all frequencies. When $\gamma \le 1$ robust stability is guaranteed.
3) With $W_u = \sqrt{\gamma|F|^2 + |W_l|^2}$, the general overbounding problem suggested in [5] is achieved. Here $F = F(\omega)$ is a weighting function that penalizes the optimization.

IV. Working with Data

In the previous section, problems were posed where all constraints should be satisfied at all values of $\omega \in \mathbb{R}$. It is our interest to pose and solve problems that can be formulated directly upon experimental data, from which frequency information may only be available upon a discrete grid $\omega \in \{\omega_1, \dots, \omega_N\}$. From this point on we focus on the problem of finding $\hat{W}$ such that

$$W_l(\omega_k) \le |\hat{W}(e^{j\omega_k})| \le W_u(\omega_k), \quad k = 1, \dots, N, \tag{7}$$

$$\hat{W}, \hat{W}^{-1} \in \mathcal{RH}_\infty. \tag{8}$$

Note that the constraint (7) is on the grid and that (8) is for all $\omega \in \mathbb{R}$. Requiring that $\hat{W} \in \mathcal{RH}_\infty$ will ensure that the bounds obtained in (7) can be attained by a stable transfer function. The constraint (7) imposes no conditions on the phase of $\hat{W}$, and we should expect that a minimum phase solution, i.e. $\hat{W}^{-1} \in \mathcal{RH}_\infty$, should exist.

V. Characterization of Multiplicative Uncertainty

Let $F_l(\omega) \in \mathbb{C}$ $(l = 1, \dots, p)$ represent the l^{th} frequency response function (FRF) measured at a finite number of frequency points $\omega \in \Omega$. Fig. 2 (Top) shows how the 36 different FRF differ from one another. The resonance modes shift in frequency which can make the control design problematic.

The worst case upper bound for a multiplicative model error is given by

$$\delta_f(\omega_k) = \max_{l=1,\dots,p} \left\| \frac{F_l(\omega_k) - F_{nom}(\omega_k)}{F_{nom}(\omega_k)} \right\|, \ \forall\ k \qquad (9)$$

where $F_{nom}(\omega_k)$ denotes the nominal FRF to be determined. In order to find a nominal $F_{nom}(\omega_k) \in \mathbb{C}$ with the smallest multiplicative error $\delta_f(\omega_k)$, a convex optimization has to be solved for each k [9]. The nominal FRF is used to find a discrete time linear time invariant system that has a similar frequency response.

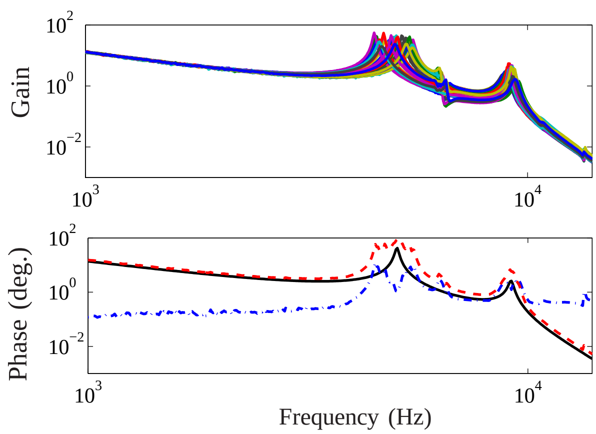

Fig. 2. Top: Frequency response functions of hard disk drives $F_l(\omega)$ that characterizes manufacturing variations. Bottom: (Solid) Nominal Model $P_o(\omega)$. (Dash-dotted) Approximation of multiplicative uncertainty $\delta_m(\omega)$. (Dashed) Characterization of uncertainty set described with nominal model $P_o(\omega)$ and multiplcative uncertainty $\delta_m(\omega)$.

A nominal model $P_o(e^{j\omega})$ that fits $F_{nom}(\omega_k)$ is found with the method described in [9]. This model is used to approximate the FRF of the multiplicative uncertainty weighting filter that is given by

$$\delta_m(\omega_k) = \max_{l} \left\| \frac{F_l(\omega_k) - P_o(e^{j\omega_k})}{P_o(e^{j\omega_k})} \right\| \quad \forall\ k. \qquad (10)$$

This approximation of the multiplicative uncertainty weighting filter is shown in Fig. 2 along with the nominal model $P_0(e^{j\omega})$ and is used in the following section as a lower bound for the model $\hat{W}$. From this figure it should be clear that all of the FRF's (Top) are contained in the set

$$\hat{\Pi} = \{(1+\delta_m(\omega_k)\Delta(e^{j\omega_k})P_o(e^{j\omega_k}) : \\ \Delta \in \mathcal{RH}_\infty \, \|\Delta\|_\infty < 1\} \quad \forall k$$

which validates the use of $\delta_m(\omega_k)$ as the multiplicative uncertainty weighting.

VI. Control Relevant Approximation

A. Minimal Phase Constraint

Three conditions must be satisfied to find a suitable uncertainty model. The first condition that we will impose is a minimal phase constraint. To impose this constraint a representation for the spectrum must be used. Several different representations of discrete-time spectra have been considered in the literature. Most of these representation rely upon the Markov-Lukás theorem [7], [6] for positivity of polynomial on a line segment, the positive real lemma [10] from systems theory, or discrete transformations like the discrete Fourier transform and cosine transform. The end result is a convex feasibility test over the cone of autocorrelation coefficients (or equivalently the cone of non-negative trigonometric polynomials) or the cone of positive definite matrices. In both cases the dual cone is used to find computationally efficient routines to solve the convex optimization problems. Furthermore, the use of the barrier function as applied to these problems has been extensively studied in the literature. While these methods result in necessary and sufficient conditions for the representation, they are not unique and each have their advantages and drawbacks especially from a numerical point of view.

In [11] the necessary and sufficient condition for pseudopolynomial matrix to be nonnegative on the unit circle, real axis, and imaginary axis is obtained by using the well-known positive real lemma. Thus, the paper [11] treats MIMO continuous-time and discrete-time formulations of the representation presented in this paper and should be consulted for further information regarding this area. The computational aspects are also presented in full detail and exploit the dual cone. Alkire and Vandenberghe [12], [13], [8] studied the discrete time formulation of this problem and using convex duality were able to write efficient optimization routines that reduce the floating-point operations from $\mathcal{O}(n^6)$ to $\mathcal{O}(n^3)$. The connection between the LMI formulation obtained from the positive real lemma and the representation given in [11] is proven in [13]. Mclean and Woerdeman use a version of the Markov-Lukás theorem to find an expression for the sum-of-squares representation of matrix-valued trigonometric polynomials. An LMI formulation is obtained and a cholesky factorization is used to find the spectral factor. Roh and Vandenberghe [7] presented a new formulation of the sum-of-squares representation of nonnegative univariate polynomials, cosine polynomials, and trigonometric polynomials of a single variable by using discrete Fourier, cosine, and polynomial transformations. The resulting structure is simple and numerically well-conditioned.

All of the aforementioned works can be consulted to obtain a necessary and sufficient condition for a trigonometric polynomial to be nonnegative on the unit circle. For the purposes of this paper the representation obtained from the positive real lemma will be used although this is not the most efficient choice. The following theorem characterizes the set of nonnegative trigonometric polynomials on the unit circle and will be used in the following sections.

Theorem 1: Consider the polynomial

$$p(z) = y_0 + y_1 z + \dots + y_n z^n.$$

Let the modulus squared of $p(z)$ evaluated on the unit circle be given by

$$\begin{aligned} \Phi_p(e^{j\omega}) &= p(e^{-j\omega})p(e^{j\omega}) \\ &= (1 + y_1 e^{-j\omega} + \dots + y_n e^{-j\omega n})(1 + y_1 e^{j\omega} + \dots + y_n e^{j\omega n}) \\ &= x_n e^{-j\omega n} + x_{n-1} e^{j\omega(-n+1)} + \dots + x_0 + \\ &\quad + x_1 e^{j\omega} + \dots + x_n e^{j\omega n} \qquad (11) \\ &= 2x_n \cos(n\omega) + 2x_{n-1}\cos((n-1)\omega) + \dots + x_0 \qquad (12) \end{aligned}$$

where the coefficients x_k are given by

$$x_k = \sum_{i=0}^{n-k} y_i y_{i+k}.$$

Then $\Phi_p(e^{j\omega}) \geq 0 \; \forall \omega \in [-\pi, \pi]$ if and only if there exists a matrix $P \in \mathbb{S}^n$ such that

$$\begin{bmatrix} P & \tilde{x} \\ \tilde{x}^T & x_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix} \geq 0 \tag{13}$$

where $\tilde{x} = [x_n \; x_{n-1} \dots x_1]^T$.

Proof: The proof is obtained directly from the results in [13] and others, and it based upon the KYP lemma from systems theory. ■

Now consider a discrete time system

$$\hat{W}(z) = \frac{b_0 + b_1 z + \dots + b_n z^n}{1 + a_1 z + \dots + a_n z^n}.$$

Let the spectrum of $\hat{W}(z)$ be given by

$$\begin{aligned} &\Phi_{\hat{W}}(e^{j\omega}) = \hat{W}(e^{-j\omega})\hat{W}(e^{j\omega}) \\ &= \frac{b_0 + b_1 e^{-j\omega} + \dots + b_n e^{-j\omega n}}{1 + a_1 e^{-j\omega} + \dots + a_n e^{-j\omega n}} \frac{b_0 + b_1 e^{j\omega} + \dots + b_n e^{j\omega n}}{1 + a_1 e^{j\omega} + \dots + a_n e^{j\omega n}} \\ &= \frac{B_n e^{-j\omega n} + B_{n-1} e^{j\omega(-n+1)} + \dots + B_0 + \dots + B_n e^{j\omega n}}{A_n e^{-j\omega n} + A_{n-1} e^{j\omega(-n+1)} + \dots + A_0 + \dots + A_n e^{j\omega n}} && (14) \\ &= \frac{2B_n \cos(n\omega) + 2B_{n-1}\cos((n-1)\omega) + \dots + B_0}{2A_n \cos(n\omega) + 2A_{n-1}\cos((n-1)\omega) + \dots + A_0} && (15) \\ &= \frac{\beta(\omega)}{\alpha(\omega)} && (16) \end{aligned}$$

where the coefficients B_k are given by

$$B_k = \sum_{i=0}^{n-k} b_i b_{i+k},$$

and the coefficients A_k are obtained similarly. Then $\Phi_{\hat{W}}(e^{j\omega}) \geq 0 \; \forall \omega \in [-\pi, \pi]$ if and only if there exists matrices $P_A, P_B \in \mathbb{S}^n$ such that

$$\begin{bmatrix} P_A & \tilde{A} \\ \tilde{A}^T & A_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_A \end{bmatrix} \geq 0 \tag{17}$$

$$\begin{bmatrix} P_B & \tilde{B} \\ \tilde{B}^T & B_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_B \end{bmatrix} \geq 0, \tag{18}$$

where $\tilde{A} = [A_n \; A_{n-1} \dots A_1]^T$ and $\tilde{B} = [B_n \; B_{n-1} \dots B_1]^T$.

To constrain the numerator and denominator polynomials to be strictly positive on the unit circle one can set

$$\begin{aligned} B_n e^{-j\omega n} + B_{n-1} e^{j\omega(-n+1)} + \dots B_0 + \dots + B_n e^{j\omega n} \geq \epsilon \\ A_n e^{-j\omega n} + A_{n-1} e^{j\omega(-n+1)} + \dots A_0 + \dots + A_n e^{j\omega n} \geq \epsilon, \end{aligned} \tag{19}$$

where ϵ is a small positive constant. Simple algebra changes (17) into

$$\begin{bmatrix} P_A & \tilde{A} \\ \tilde{A}^T & A_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_A \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} P_B & \tilde{B} \\ \tilde{B}^T & B_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_B \end{bmatrix} \geq 0.$$

B. Frequency Dependent Bounds

The remaining constraints that are imposed upon the uncertainty model are an upper and lower bound that are frequency dependent. In this section we incorporate these bounds into the design and show that the general frequency gridded spectral problem (7)-(8) is a convex problem, for which a feasible solution can be computed. More specifically, we show that it can be posed and solved in polynomial time as a semidefinite program. We start by squaring the constraints (7)

$$W_l^2(\omega_k) \leq \Phi_{\hat{W}}(\omega_k) \leq W_u^2(\omega_k), \quad k = 1, \dots, N. \tag{20}$$

Now, according to the discussion in the previous section, if $\hat{W}, \hat{W}^{-1} \in \mathcal{RH}_\infty$ then there is a LMI formulation for the numerator and denominator coefficients of $\Phi_{\hat{W}}(\omega)$ that is valid for $\omega \in [-\pi, \pi]$. Therefore, by using the notation from (16) we get

$$\alpha(\omega_k) W_l^2(\omega_k) \leq \beta(\omega_k) \leq \alpha(\omega_k) W_u^2(\omega_k), \quad k = 1, \dots, N.$$

Note that the above constraints are clearly linear on the coefficients of $\alpha(\omega)$ and $\beta(\omega)$. In fact, these linear constraints are related to the ones used in [5] to produce a linear program which attempts to solve the constraints (7)-(8) by approximating (8) only at the grid points. Ignoring (8) one risks to produce a spectrum $\Phi_{\hat{W}}$ that is nonnegative *only at the grid points*. In order to avoid such pitfall, one has to largely increase the number of grid points and/or the order of the polynomials (see [5] and the numerical example in Section VI-C).

As Theorem 1 in the previous section shows, using the constraints in (19) we completely eliminate this difficulty. In fact, this is equivalent to using a sum-of-squares representation [14], [7] for the spectrum, using the Nesterov parameterization for scalar polynomials [11] or constraining the numerator and denominator coefficients to be in the cone of finite autocorrelation coefficients [13].

Finally, note that $\Phi_P(e^{j\omega})$ is not uniquely defined in terms of $\alpha(\omega)$ and $\beta(\omega)$. In fact one can scale $\alpha(\omega)$ and $\beta(\omega)$ by an arbitrary positive scalar without changing $\Phi_{\hat{W}}$. For this reason, we add to the above constraints the *normalization constraint*

$$A_0 = 1. \tag{21}$$

This condition is similar to constraining the denominator of $\hat{W}$ to be monic so that an over parameterization of the transfer function is avoided and a unique minimum is found.

Once the optimization problem has stopped, one is left with a representation for spectrum $\Phi_{\hat{W}}(e^{j\omega})$. To find the minimal phase model $\hat{W}(z)$ the spectrum $\Phi_{\hat{W}}(e^{j\omega})$ must be factored into stable and unstable parts. Factoring a squared polynomial (hence a spectrum) is called spectral factorization and many methods exist to this end. Spectral factorization methods include the Bauer method [15], Schur Algorithm, Levinson-Durbin Algorithm, CKMS filter [10], Wilson Method, and Riccati method [10]. Most of these methods are reviewed and compared in [16].

C. Minimizing the Relative Error

An important particular case of the general spectral bounding problem presented (7)-(8) is the case when $W_u^2 = W_l^2 + \gamma F(\omega)$ and we want to find the smallest γ for which this holds such that the solution can be factored. The user specified nonnegative weighting function $F(\omega)$ is used to weight the optimization problem. A typical choice for $F(\omega)$ is $W_l^2(\omega)$, and with this choice of weighting function one minimizes the relative error. Under this formulation the optimization problem can be written as

$$\begin{aligned} \min_{\hat{W}} \quad & \max_{k=1,\cdots,N} \{(|\hat{W}(\omega_k)|^2 - W_l(\omega_k)^2)F(\omega_k)^{-1}\}, \\ \text{s.t.} \quad & |\hat{W}(\omega_k)| \geq W_l(\omega_k), \quad k = 1,\ldots,N, \\ & \hat{W}, \hat{W}^{-1} \in \mathcal{RH}_\infty. \end{aligned}$$

In [5] a similar problem posed in discrete-time was solved using a LP approach. However, as we mentioned before, the LP version does not enforce the positivity of the polynomials over all frequencies but only at the grid points. As a results, solutions to the LP version may fail to produce a rational transfer function solution associated with the spectrum at the $k = 1,\ldots,N$ points.

Using the ideas discussed in the previous section it is not difficult to reformulate the above min-max problem into the form

$$\begin{aligned} \min_{\gamma, A_k, B_k, P_A, P_B} \quad & \gamma \\ \text{s.t.} \quad & \alpha(\omega_k)\left[W_l^2(\omega_k) + \gamma\, F(\omega_k)\right] \geq \beta(\omega_k) \\ & \qquad \geq \alpha(\omega_k)\delta_m^2(\omega_k), \\ & \qquad k = 1,\ldots,N, \\ & \begin{bmatrix} P_A & \tilde{A} \\ \tilde{A}^T & A_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_A \end{bmatrix} \geq 0, \\ & \begin{bmatrix} P_B & \tilde{B} \\ \tilde{B}^T & B_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_B \end{bmatrix} \geq 0, \\ & A_0 = 1, \qquad P_A = P_A^T, \qquad P_B = P_B^T, \end{aligned}$$

where $\tilde{A}$ and $\tilde{B}$ are defined as in (17) and $\alpha(\omega)$ and $\beta(\omega)$ are defined in (16).

This problem is convex for each fixed γ, hence quasi-convex, and the global optimal solution can be found via a line search method on γ such as bisection. The first $2k$ constraints are in place so that the upper and lower bounds are guaranteed. The remaining constraints are the necessary and sufficient conditions for the spectral factorization to exist and the normalization constraint.

Setting $W_l(\omega) = \delta_m(\omega)$ and $F(\omega) = \delta_m(\omega)^2$ and minimizing γ will result in an over bound that minimizes the relative error. That is, an over bound that minimizes the relative error between $\delta_m(\omega)$ and $\hat{W}(e^{j\omega})$. Fig. 3 show the result of this optimization along with a linear programming based approach [5] that aims to minimize the same cost. One should notice that there are several breaks in the graph. These points indicate that the linear programming based approach became negative in between grid points. The semidefinite based approach will never become negative since we are searching over the set of positive polynomials. Also notice that both methods bound the weighting filter at the grid points only and therefore it is possible to violate the bounds in between them.

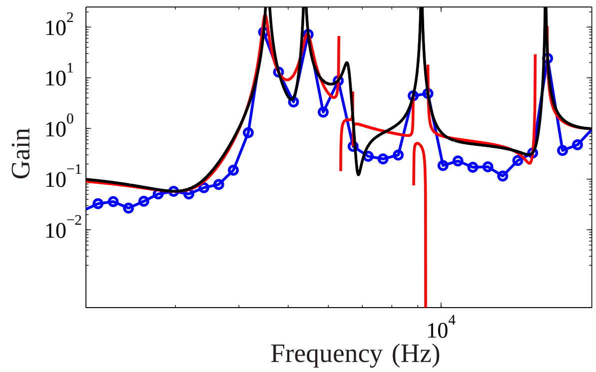

Fig. 3. (o) Lower bound $\delta_m(\omega)$ that approximates the multiplicative uncertainty weighting filter.( ) Linear programming based over bound that minimizes the relative error. ( ) Semidefinite programming based over bound that minimizes the relative error. There are several breaks in the graph. These points indicate that the linear programming based approach became negative in between grid points.

D. Control Relevant Approximation

In order for a control-relevant over bound of the uncertainty to be found it is important to recall the discussion of Section III. That is, different choices for W_l and W_u will result in optimization problems that emphasis the frequency domain fit to suit different design needs. For this example problem, we seek to emphasis robust stability and therefore set $W_l = |\delta_m(\omega)|$ and $W_u = |T|^{-1}\gamma$. Thus, by minimizing γ the fit of $\hat{W}$ onto δ_m will emphasis robust stability and it is easy to see that for $\gamma \leq 1$ the current controller satisfies robust performance for the designed weighting filter $\hat{W}$ which over bounds the estimated multiplicative error δ_m. The design of a subsequent robust performing controller is guaranteed in this case.

T must be measured (implying a current controller exists) for this method to work. However, for comparison purposes T was generated by designing a robustly stabilizing controller for the uncertainty weighting filter found in the previous section. The magnitude of the complementary sensitivity function T and weighted complementary sensitivity function $W_1 T$ are shown in Fig. 4. Here it can be seen that the current controller satisfies robust stability.

It is not difficult to recast the general bounding problem of (7) and (8) into the form

$$\begin{aligned} \min_{\gamma, A_k, B_k, P_A, P_B} \quad & \gamma \\ \text{s.t.} \quad & \alpha(\omega_k)\left[|T(\omega_k)|^{-1}\gamma\right]^2 \geq \beta(\omega_k) \geq \alpha(\omega_k)\delta_m^2(\omega_k), \\ & \qquad k = 1,\ldots,N, \\ & \begin{bmatrix} P_A & \tilde{A} \\ \tilde{A}^T & A_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_A \end{bmatrix} \geq 0, \\ & \begin{bmatrix} P_B & \tilde{B} \\ \tilde{B}^T & B_0 - \epsilon \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & P_B \end{bmatrix} \geq 0, \\ & A_0 = 1, \qquad P_A = P_A^T, \qquad P_B = P_B^T, \end{aligned}$$

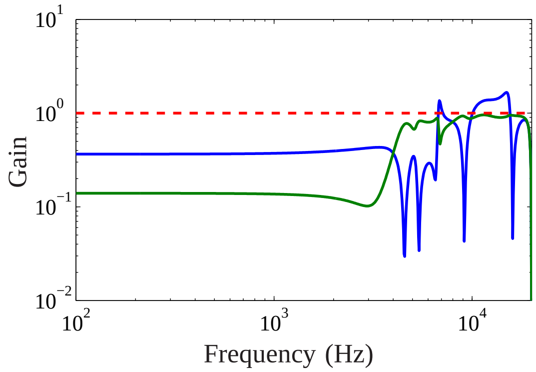

Fig. 4. ( )The magnitude of the complementary sensitivity function $|T|$. ( )The magnitude of the weighted complementary sensitivity function $|W_1 T|$.

where $\tilde{A}$ and $\tilde{B}$ are defined as in (17) and $\alpha(\omega)$ and $\beta(\omega)$ are defined in (16).

This problem is convex for each fixed γ, hence quasi-convex, and again the global optimal solution can be found via a line search method like bisection. The results of this optimization are shown in Fig. 5. This figure depicts the difference between the relative error over bound and the control relevant over bound. The control relevant over bound is very accurate at the higher frequencies and not very accurate at low frequencies. Conversely, the relative error over bound is more accurate at low frequencies where it is not needed to satisfy robust stability. This can be verified by looking at Fig. 4, where at high frequencies $|T|$ is near unity indicating the need for a better fit.

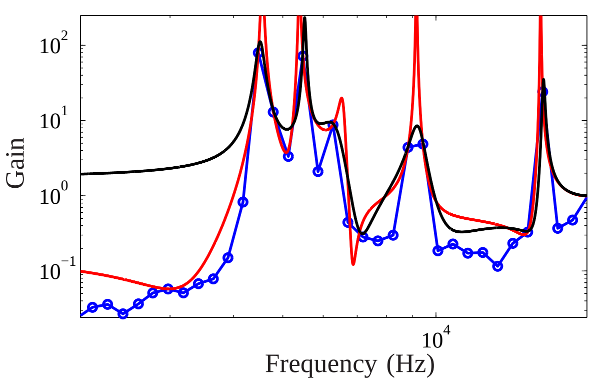

Fig. 5. (**o**) Lower bound $\delta_m(\omega)$ that approximates the multiplicative uncertainty weighting filter.( ) Semidefinite programming based over bound that minimizes the relative error.( ) Control relevant over bound based upon semidefinite programming.

VII. Conclusions

In this paper, it has been shown that manufacturing and mounting variations in hard disk drives can be represented with a multiplicative uncertainty where a weighting filter is used to represent the frequency dependent size of the uncertainty. The weighting filter for the uncertainty and nominal model can be found with via convex optimizations. In particular, the weighting filter design used a linear matrix inequality to constrain the spectrum to be positive and linear constraints to enforce lower and upper bounds that shape the weighting filter. Two different choices for the upper bound were compared. One upper bound resulted in a weighting filter that minimizes the relative error and the other resulted in an emphasis in robust performance. This comparison indicated that a better fit for the purposes of robust control of the frequency domain data can be achieved as compared to standard methods. Additionally, the linear matrix inequality constraints guaranteed that a minimal phase representation can be found which is a improvement over other methods of finding a weighting filters for robust control that rely upon linear programming.

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