ESTIMATING PARAMETERS IN A LUMPED PARAMETER SYSTEM WITH FIRST PRINCIPLE MODELING AND DYNAMIC EXPERIMENTS

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Abstract: Commercially available mechanical systems are available to teach and demonstrate the principles behind dynamics and control. A single system can be used for basic dynamic analysis in an undergraduate class to teaching and applying sophisticated identification techniques in a graduate class. In this paper it shown how a commercial system is used at the undergraduate level to estimate lumped parameter coefficients using multiple step responses and first principle modeling. At the graduate level, the same commercial system is used to teach concepts of system identification for the estimation of models for a multi-degree of freedom mechanical system.

Keywords: identification; education; lumped parameter models

1. INTRODUCTION

Dynamic models are important to illustrate the main concepts in dynamic system analysis and linear control. With a mathematical description of a linear dynamic system, either in the form of a differential equation, a transfer function or a state space model, main concepts such as dynamic response, stability and feedback control can be taught and demonstrated (Dorf and Bishop 2000, Stefani *et al.* 2001, Franklin *et al.* 2001). Consequently, derivation of a dynamic model should be an integral part of a course on dynamic system analysis and control system design.

A fundamental step in constructing a dynamic model is based on first principle modeling. Especially in undergraduate engineering education where students develop a background in analyzing equations of motion, thermodynamics and circuit theory, dynamic models are based on governing equations obtained from the various disciplines (Bryson 1994, Morari and Zafiriou 1997, Franklin *et al.* 2001). With applications and students coming from various disciplines, a challenge in teaching control system design is to demonstrate that dynamic models arising from different disciplines have a similar dynamic structure and can be subjected to the same dynamic system analysis needed in control system design.

Important components in dynamic system analysis are the transient response and the frequency response of a model. Time and frequency domain analysis illustrate the main dynamic concepts of a model, but also illuminate the similarities between dynamic models from various disciplines. Unfortunately, in most undergraduate courses on dynamic system analysis and control system design, time and frequency domain analysis is used only to illustrate the dynamic behavior of a system. It is beneficial to include a reversal of this information and use dynamic responses to *derive* the dynamic behavior of the system.

Estimation of models on the basis of data can be integrated in undergraduate education by providing dynamic analysis of time and frequency response data with the purpose of characterizing dynamic model properties. In this paper it is shown how this can be done for a flexible mechanical system on the basis of step response experiments.

2. ESTIMATION OF MODELS FROM DATA

Estimating models from data in general requires a substantial background in the field of system identification. This is one of the primary reason not to include data based modeling techniques in basic undergraduate courses on dynamic system analysis and control system design. However, without a background in system identification, estimation of models from data can still be done by using relatively simple experiments. Such experiments may include step responses that provide an intuitive understanding of the dynamic behavior. Experiments of this nature are particular useful in a laboratory environment where students are asked to develop models for control system design on the basis of measured data. When combined with first principle modeling, data based modeling illustrates the possibility to obtain parameters of a system from dynamic experiments.

In this paper it is illustrated how a commercially available mechanical lumped parameter system is used to teach the basic concepts of model parameter estimation from experimental data. By combining first principle modeling and standard vibration analysis, mass, damping and spring parameters are estimated on the basis of simple step experiments that are carried out in an undergraduate laboratory course. At the graduate level, the same commercial system is used to teach concepts of system identification by estimating of models for a multi-degree of freedom mechanical system.

The objective of this paper is to illustrate the use of a commercially available mechanical lumped parameter system to estimate system parameters using multiple step responses and first principle modeling. The application of simple step experiments in a laboratory environment enables students to derive models from data, whereas the lumped parameter system can also be used to derive a model from first principles. The experiment is currently used in an undergraduate laboratory course on control system design at the Mechanical and Aerospace Engineering department at the University of California, San Diego.

3. LABORATORY EXPERIMENT

3.1 Mechanical system

The rectilinear (and torsional) system used for the laboratory experiments discussed in this paper are mechanical systems with multiple mass, spring and damper element with one-dimensional motion. The systems consist of mass, spring and damper components and a picture of the rectilinear system is depicted in Figure 1. The systems are commercially produced by Educational Control Products (ECPsystems.com) and are equipped with a hardware interface and a user-friendly software environment for data acquisition and controller implementation purposes.

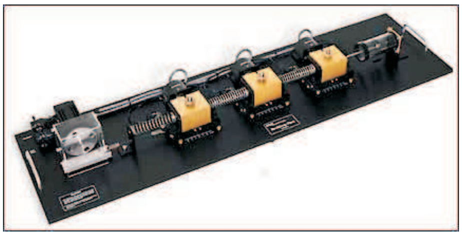

Fig. 1. Rectilinear 3 mass system used for dynamic experiments

The mechanical system depicted in Figure 1 consists of several carts with adjustable weights connected by spring elements. Optional air-restriction dampening devices can be added to increase the damping coefficients of the overall mechanical system. To simplify the analysis, the mechanical system in Figure 1 is equipped with only 2 charts connected via a spring element. As a result, a 2 mass mechanical system is created where each mass or inertia has a positioning freedom. A schematic diagram of the 2 mass system is depicted in Figure 2.

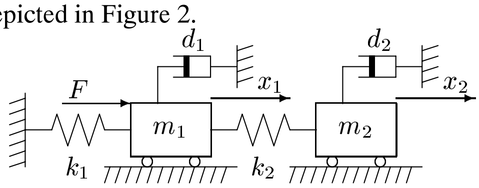

Fig. 2. Schematic view of 2 mass rectilinear system

A force F can be applied to the first mass m_1 via a linear DC-motor. The purpose of the laboratory experiment is to model and control the vibrations of the 2 mass system. The DC-motor is chosen such that its dynamics is negligible compared to the dynamics of the mechanical vibrations of the system. A dynamic model is required to develop a controller to reduce the residual vibrations and to change the position x_1 of a mass/inertia m_1 as fast as possible by means of a controlled force F.

3.2 First principle modeling

Using standard analysis based on 2nd Newton's law, the equations of motion for the lumped parameter system in Figure 2 can be derived. The equations of motion are given by

$$m_1\ddot{x}_1 = -k_1x_1 - d_1\dot{x}_1 - k_2(x_1 - x_2) + F \quad (1)$$
$$m_2\ddot{x}_2 = k_2(x_1 - x_2) - d_2\dot{x}_2 \quad (2)$$

where m_1, m_2 represent the mass or inertia, while x_1, x_2 represent displacement of the masses. The coefficients d_1, d_2 and k_1, k_2 represent respectively damping and stiffness parameters of the mechanical system and F denotes the applied control force.

In order to find the dynamical relation between control force F and the displacement x_1 of the first mass m_1, x_2, $\dot{x}_2$ and $\ddot{x}_2$ are eliminated from (1) and (2). By means of a Laplace transformation, the equation of motions (1) and (2) for this 2 mass system can be written in the matrix representation

$$T(s)\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

with

$$T(s) = \begin{bmatrix} m_1s^2 + d_1s + (k_1 + k_2) & -k_2 \\ -k_2 & m_2s^2 + d_2s + k_2 \end{bmatrix} \quad (3)$$

From (3) it can be seen that the computation of a solution of x_1 (and x_2) involves the inversion of a matrix $T(s)$. For a 2 mass model, $T(s)$ is a 2×2 matrix and the computation of the inverse only requires the computation of the determinant of $T(s)$. With the determinant $d(s)$ of $T(s)$ given by

$$(m_1s^2 + d_1s + k_1 + k_2)(m_2s^2 + d_2s + k_2) - k_2^2$$

it can be concluded that $T^{-1}(s)$ is given by

$$\frac{1}{d(s)}\begin{bmatrix} m_2s^2 + d_2s + k_2 & k_2 \\ k_2 & m_1s^2 + d_1s + (k_1 + k_2) \end{bmatrix}.$$

With this analysis it follows that the resulting transfer function $G(s)$, that relates the control effort $F(s)$ to the position $x_1(s)$, is given by

$$G(s) = \frac{b_2s^2 + b_1s + b_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \quad (4)$$

where the coefficients are

$$\begin{array}{lll} & & a_4 = m_1m_2 \\ b_2 = m_2 & & a_3 = (m_1d_2 + m_2d_1) \\ b_1 = d_2 & & a_2 = (k_2m_1 + (k_1 + k_2)m_2 + d_1d_2) \\ b_0 = k_2 & & a_1 = ((k_1 + k_2)d_2 + k_2d_1 \\ & & a_0 = k_1k_2 \end{array}$$

For the accurate prediction of the flexibilities in the 2 mass lumped parameter model, the mass m_1, m_2, the stiffness k_1, k_2 and the damping d_1, d_2 have to be determined. Estimation of these parameters on the basis of (dynamic) laboratory experiments provides valuable insight in the basic concepts of model parameter estimation in an undergraduate course.

4. EXPERIMENTS FOR PARAMETER ESTIMATION

4.1 Single mass experiments

To facilitate the estimation of the parameters of the 2 mass mechanical system, experiments with only a single mass system are used. Performing the experiments on a single mass is possible, due to the nature of the lumped parameter system. By either physically disconnecting the masses or restricting the displacement of one of the masses, a single mass system is obtained.

The rationale behind the usage of single mass experiments is the ability to isolate the various resonance modes in the lumped parameter system. For a standard 2nd order mass/spring/damper system, the relation between force input F and displacement y is given by the transfer function

$$G(s) = \frac{1}{ms^2 + ds + k} = C\cdot\frac{\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \quad (5)$$

where

$$\begin{array}{lll} C & = \frac{1}{k} & \text{steady state gain} \\ \omega_n & = \sqrt{\frac{k}{m}} & \text{undamped resonance frequency} \\ \beta & = \frac{d}{2\sqrt{km}} & \text{damping ratio} \end{array}$$

The relationship between mass m, damping d,stiffness k, the (undamped) resonance frequency ω_n and the damping ratio β is taught at the undergraduate level in standard vibration analysis. This knowledge can be exploited to estimate the model parameters by a sequence of well-planned experiments, where in each experiment only one degree of freedom of the mechanical system is analyzed.

Following the schematic diagram in depicted in Figure 2, the following three experimental conditions can be constructed:

- Disconnect mass m_1 and m_2 by removing spring k_2, resulting in a single mass system with mass m_1, stiffness k_1 and damping d_1.
- Connect mass m_1 and m_2 by spring k_2, but restrict motion of mass m_2. This results in a single mass system with a mass m_1 and stiffness $k_1 + k_2$.
- Restrict motion of mass m_1. This results in a single mass system with mass m_2, stiffness k_2 and damping d_2. It should be noted that no control force F can be applied to mass m_2 in this configuration.

The first experiment can be used to gather information about the the parameters m_1, d_1 and k_1 by observing the steady stage gain $C = 1/k_1$, the undamped resonance frequency $w_n = \sqrt{k_1/m_1}$ and the damping ratio $\beta = \frac{d_1}{2\sqrt{k_1m_1}}$. As no control force can be applied to m_2, the second experiment is used to estimate the sum of the stiffness from which k_2 can be computed by using the knowledge of k_1 obtained from the first experiment. With the knowledge of the stiffness k_2, the last experiment is used to estimate the mass m_2 and damping d_2 parameters by again observing the undamped resonance frequency $w_n = \sqrt{k_2/m_2}$ and the damping ratio $\beta = \frac{d_2}{2\sqrt{k_2m_2}}$.

4.2 Step response experiments

The dynamic behavior of the lumped 2 mass system in Figure 2 can be determined by performing relatively simple single mass experiments. The parameters estimated in each experiment are combined to form the complete model of the mechanical system.

For the estimation of the parameters C, ω_n and β in each experiment based on a single mass system, a step experiment will be used. The response to a step input signal can be computed analytically for a 2nd order system and gives rise to a straightforward estimation of the parameters of a single mass system from the observed data.

The displacement $y(t)$ due to a step input $F(t) = U,\ t \geq 0$ is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{C \cdot \frac{\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \cdot \frac{U}{s}\right\} \\ &= CU\left[1 - \mathrm{e}^{-\beta\omega_n t}(\cos\omega_d t + \phi\sin\omega_d t)\right] \end{aligned} \tag{6}$$

where

$$\begin{aligned} \omega_d &= \omega_n\sqrt{1-\beta^2} \\ \phi &= \frac{\beta}{\sqrt{1-\beta^2}} \end{aligned}$$

The step response of a single mass system with damping $0 < \beta < 1$ is an exponentially decaying sinusoidal function. As a result, the damped resonance frequency ω_d, the damping β and the static gain or DC-gain $\frac{1}{k}$ can be estimated from an observed step response.

5. ESTIMATION OF MODEL PARAMETERS

5.1 Direct estimation

For the estimation of the parameter, consider the step response depicted in Figure 3. From the observed step response the following parameters can be estimated.

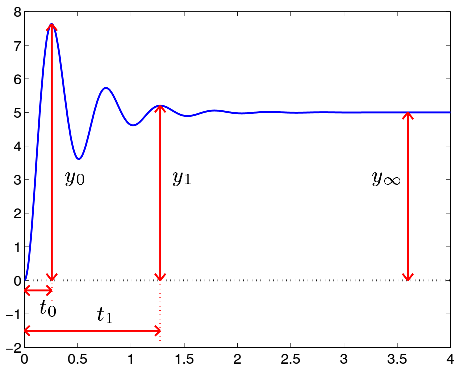

Fig. 3. Typical step response of single mass system

- The steady state behavior is given by
$$\lim_{t\to\infty} y(t) := y_\infty = CU$$
and with U known as the step size of the input signal
$$\hat{C} = \frac{y_\infty}{U} \tag{7}$$
is an estimation of the parameter C.
- From the oscillation in the step response $y(t)$, the damped resonance frequency ω_d can be estimated. For that purpose, consider the time measurements t_0 and t_1 to distinguish two subsequent maximum values of oscillations in the output $y(t)$, then
$$\hat{\omega}_d = 2\pi\frac{n}{t_1 - t_0} \tag{8}$$
gives an estimate for the damped resonance frequency ω_d of the single mass system, where n is the number of oscillations between the two subsequent maximum values of output $y(t)$.
- From the decay of the oscillation in the step response $y(t)$, the damping coefficient β can be estimated. For that purpose, consider the difference of the steady state value y_∞ with two subsequent maximum values y_0 and y_1 of oscillations in the output $y(t)$. With the analytic solution of the step response given in (6) it can be verified that
$$\hat{\beta\omega_n} = \frac{1}{t_1 - t_0}\ln\left(\frac{y_0 - y_\infty}{y_1 - y_\infty}\right) \tag{9}$$
is an estimate for the product of the damping ration β and the (undamped) resonance frequency ω_n.

Combining the estimates in (8) and (9) yields

$$\begin{aligned} \hat{\omega}_n &= \sqrt{\hat{\omega}_d^2 + (\hat{\beta\omega_n})^2} \\ \hat{\beta} &= \frac{\hat{\beta\omega_n}}{\omega_n} \end{aligned}$$

and gives estimates for the undamped resonance frequency ω_n and the damping ratio β. With the values of the estimates $\hat{C}$, $\hat{\omega}_n$ and $\hat{\beta}$, the values of the mass m, damping d and stiffness k in (5) (up to a scaling constant) are computed via

$$\begin{aligned} \hat{k} &= \frac{1}{\hat{C}} && \text{(stiffness constant)} \\ \hat{m} &= \frac{1}{\hat{C}\,\hat{\omega}_n^2} && \text{(mass/inertia)} \\ \hat{d} &= \frac{2\hat{\beta}}{\hat{C}\,\hat{\omega}_n} && \text{(damping constant)} \end{aligned} \tag{10}$$

which concludes the estimation of the system parameters from a single step experiment. The parameter estimates in (10) are unbiased estimates of the system parameters, provided step experiment are used. To improve the variance properties of the estimates in case of noise experiments, averaging of the estimates over multiple step experiments can be performed. The simple step experiments provide means to estimate the unknown parameters in the 2 mass system. The parameters are estimated by a sequence of step experiments that only require the knowledge of the dynamic behavior of a standard 2nd order system.

5.2 Estimation via step response realization

It can be observed that the estimation of the parameters in (7)-(9) is based on only three discrete data points: y_0, y_1 and y_∞ with the corresponding time elements t_0, t_1 and t_∞. With the help of the (continuous time) analytic solution of the step response in (6), an explicit expression for the parameters estimates of the single mass/spring/damper can be obtained. Although the parameter estimation gives insight in determining system parameters from dynamic experiments, the parameter estimation is highly influenced by noise. An extra level of complexity can be added to the estimation by including all the data points of the step response in the parameter estimation.

One possible solution would be to pose an optimization using a parametrized continuous model (Ljung and Glad 1994, Ljung 1999) to find the optimal values for the mass m, stiffness k and damping d parameters. Although this is a viable solution, non-linear optimization techniques would be required to solve the parameter estimation technique. An alternative would be to construct a model using realization techniques (Ho and Kalman 1966, Kung 1978), which only requires standard matrix manipulations to estimate a model (Vandewalle and de Moor 1991). Matrix manipulations such as singular value decompositions are taught at the undergraduate level. Therefore, realization based techniques can be easily adapted to the discrete time step response measurements obtained from the laboratory experiments.

For the estimation via the step response realization, consider the step response data $y(t)$. Since $y(t)$ is (discrete time) step response data,

$$y(t) = \sum_{\tau=0}^{t} g(\tau)$$

where $g(\tau)$ indicates the (unmeasured) discrete time impulse response data. By forming the matrix

$$S := \begin{bmatrix} y(1) & y(2) & \cdots & y(n) \\ y(2) & y(3) & \cdots & y(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ y(n) & y(n+1) & \cdots & y(2n-1) \end{bmatrix} - \begin{bmatrix} y(0) & y(0) & \cdots & y(0) \\ y(1) & y(1) & \cdots & y(1) \\ \vdots & \vdots & \vdots & \vdots \\ y(n-1) & y(n-1) & \cdots & y(n-1) \end{bmatrix}$$

from the step response measurement $y(t)$, $t = 1, \ldots, 2n-1$, it can be observed that

$$S = HI_u \tag{11}$$

where H is the standard impulse response based Hankel matrix

$$H = \begin{bmatrix} g(1) & g(2) & \cdots & g(n) \\ g(2) & g(3) & \cdots & g(n+1) \\ \vdots & \vdots & \vdots & \vdots \\ g(n) & g(n+1) & \cdots & g(2n-1) \end{bmatrix}$$

and I_u is an upper triangular matrix with ones on the diagonal and in the upper triangular part. Due to the structure of H and I_u in (11) it can be verified that $rank(S) = rank(H)$. With $rank(S) = r$, a decomposition

$$\begin{gathered} S = S_1 S_2 \\ S_1^T, S_2 \in R^{r \times n},\ rank(S_1^T) = rank(S_2) = r \end{gathered} \tag{12}$$

can be computed via a singular value decomposition of the matrix S similar to the realization algorithm of Kung (1978).

For a discrete time system the impulse response $g(\tau)$ satisfies $g(\tau) = CA^{\tau-1}B$, where (A, B, C) denote the matrices of a state space realization of the system. With the elements of $g(\tau)$ present in H and the structure of I_u in (11) it can also be verified that

$$\bar{S} = S_1 A S_2$$

where

$$\bar{S} := \begin{bmatrix} y(2) & y(3) & \cdots & y(n+1) \\ y(3) & y(4) & \cdots & y(n+2) \\ \vdots & \vdots & \vdots & \vdots \\ y(n+1) & y(n+2) & \cdots & y(2n) \end{bmatrix} - \begin{bmatrix} y(1) & y(1) & \cdots & y(1) \\ y(2) & y(2) & \cdots & y(2) \\ \vdots & \vdots & \vdots & \vdots \\ y(n) & y(n) & \cdots & y(n) \end{bmatrix}$$

As a result, the A-matrix of the state space realization can be computed by

$$A = \bar{S}_1 \bar{S} \bar{S}_2 \tag{13}$$

where $\bar{S}_1$ and $\bar{S}_2$ indicate respectively the left and right inverse of the matrices S_1 and S_2 in the decomposition (12).

The step response realization technique can be applied to any step response measurement to find a state space model based on standard realization techniques (Kung 1978). For the step response experiment of a single mass system considered in Section 4 of this paper, the rank of S in the decomposition (12) will be 2. As a result, a 2×2 matrix A will be estimated on the basis of the step response experiments from which the (continuous time) system parameter m, k and d have to be computed.

For a lightly damped single mass system, the eigenvalues of the matrix A will appear in a complex conjugate pair. Since the matrix A is the discrete time equivalent of the continuous time system, the eigenvalue $\lambda(A)$ can be used to estimate the equivalent continuous time natural frequency ω_n and damping ratio β via

$$\begin{aligned} \hat{\omega}_n &= \left| \frac{\ln \lambda(A)}{\Delta T} \right| \\ \hat{\beta} &= -\cos \tan^{-1} \frac{imag \ln \lambda(A)}{real \ln \lambda(A)} \end{aligned} \tag{14}$$

by assuming a zero order old discrete time equivalent model with a sampling time of ΔT seconds. With an estimate of $\bar{C}$ based on the steady state value of the step response $y(t)$, the estimates in (14) can be used to compute the estimate of the mass m, spring k and damping d parameters in (10).

6. ILLUSTRATION OF PARAMETER ESTIMATION

Consider the measured step response of a one of the single mass experiments depicted in Figure 4. The data is obtained from the rectilinear ECP system by applying a 0.8 Volt step input on the DC-motor and measuring the position of the mass in encoder counts with a sampling time of 9 msec.

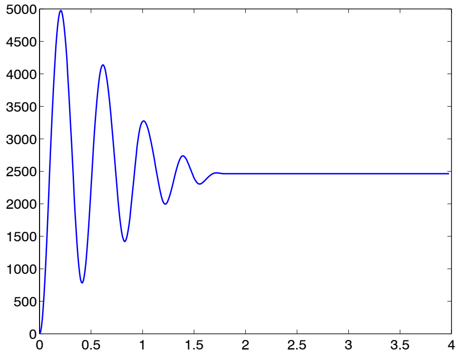

Fig. 4. Measured step response of a single mass using the ECP rectilinear system

Using the first and third peak in the step response oscillations for estimation purposes one finds

$$\hat{\omega}_d = = 2\pi\frac{2}{1.05-0.21} \approx 4.76\pi$$
$$\hat{\beta\omega_n} = \frac{1}{0.84}\ln\left(\frac{4975-2464}{3276-2464}\right) \approx 1.34$$

giving the estimated values:

$$\begin{aligned}\hat{\omega}_n &= \sqrt{\hat{\omega}_d^2 + (\hat{\beta\omega_n})^2} \approx 4.78\pi \\ \hat{\beta} &= \frac{\hat{\beta\omega_n}}{\hat{\omega}_n} \approx 0.0895\end{aligned} \tag{15}$$

Estimation of a 2×2 state space matrix A using the step response realization yields a step response matrix S with 2 singular values significantly larger than the remaining singular values. Computation of a rank 2 decomposition S_1 and S_2 via a singular value decomposition gives a A-matrix via (13) that is given by

$$A = \begin{bmatrix} 0.9744 & -0.0710 \\ 0.2428 & 0.9900 \end{bmatrix}$$

Computation of the complex conjugate eigenvalue pair $\lambda(A)$ and the equivalent continuous time natural frequency ω_n and damping ratio β via (14) yields the estimates

$$\begin{aligned}\hat{\omega}_n &\approx 4.7014\pi \\ \hat{\beta} &\approx 0.0686\end{aligned} \tag{16}$$

The differences between (15) and (16) can be attributed to the difference in information used from the data to estimate the parameters. In case of noise free data obtained from an actual 2nd order system, both estimation results would be similar. However, due to the friction present in the mechanical system, the damping is not consistent throughout the experiment.

7. CONCLUSIONS

Without a comprehensive background in system identification, estimation of lumped parameters in a mechanical system from experimental data can be done by using relatively simple experiments. Such experiments may include step responses that provide an intuitive understanding of the dynamic behavior. In this paper it shown how a commercial educational lumped parameter system with two degrees of freedom is used at the undergraduate level to estimate mechanical parameters using multiple step responses and first principle modeling. By a series of well thought experiments, students find the parameters in a first principle model and compare these results to simulations based on the first principle model.

REFERENCES

Bryson, A. (1994). *Control of Spacecraft and Aircraft*. Princeton University Press.

Dorf, R.C. and R.H. Bishop (2000). *Modern Control Systems*. Pearson Education.

Franklin, G.F., J.D. Powel and A. Emami-Naeini (2001). *Feedback Control of Dynamic Systems*. Prentice Hall, Upper Saddle River, NJ, USA.

Ho, B.L. and R.E. Kalman (1966). Effective construction of linear state-variable models from input/output functions. *Regelungstechnik* **14**, 545–548.

Kung, S.Y. (1978). A new identification and model reduction algorithm via singular value decomposition. In: *Proc. 12th Asilomar Conference on Circuits, Systems and Computers*. Pacific Grove, USA. pp. 705–714.

Ljung, L. (1999). *System Identification: Theory for the User (second edition)*. Prentice-Hall, Englewood Cliffs, New Jersey, USA.

Ljung, L. and T. Glad (1994). *Modeling of Dynamic Systems*. Prentice-Hall, Englewood Cliffs, New Jersey, USA.

Morari, M. and E. Zafiriou (1997). *Robust process Control*. Prentice Hall.

Stefani, R.T., G. Hostetter, B. Shahian and C.J. Savant (2001). *Design of Feedback Control Systems*. Oxford University Press.

Vandewalle, J. and B. de Moor (1991). On the use of the singular value decomposition in identification and signal processing. *Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms* pp. 321–360.