Contribution to the Mechatronics Handbook to be published by CRC Press

Section 4.5: RESPONSE OF DYNAMIC SYSTEMS

Raymond de Callafon University of California, San Diego
Dept. of Mechanical and Aerospace Engineering
9500 Gilman Drive
La Jolla, CA 92093-0411, U.S.A
callafon@ucsd.edu
tel 858-534 3166

August 1, 2001

4.5 RESPONSE OF DYNAMIC SYSTEMS

4.5.1 System and signal analysis

In dynamic system design and analysis it is important to predict and understand the dynamic behavior of the system. Examining the dynamic behavior can be done by using a mathematical model that describes the relevant dynamic behavior of the system in which we are interested. Typically, a model is formulated to describe either continuous or discrete time behavior of a system. The corresponding equations that govern the model are used to predict and understand the dynamic behavior of the system.

A rigorous analysis can be done for relatively simple models of a dynamic system by actually computing solutions to the equations of the model. Usually, this analysis is limited to linear first and second order models. Although limited to small order models, the solutions tend to give insight in the typical responses of a dynamic system. For more complicated, higher order and possibly non-linear models, numerical simulation tools provide an alternative for the dynamic system analysis.

In the following we review the analysis of linear models of discrete and continuous time dynamic systems. The equations that describe and relate continuous and discrete time behavior are presented. For the analysis of continuous time systems extensive use is made of the Laplace transform that converts linear differential equations in algebraic expressions. For similar purposes, a z-transform is used for discrete time systems.

Continuous time systems

Models that describe the linear continuous time dynamical behavior of a system are usually given in the form of differential equations that relate an input signal $u(t)$ to an output signal $y(t)$. The differential equation of a time invariant linear continuous time model has the general format

$$\sum_{j=0}^{n_a} a_j \frac{d^j}{dt^j} y(t) = \sum_{j=0}^{n_b} b_j \frac{d^k}{dt^j} u(t) \tag{4.1}$$

in which a linear combination is taken using the jth order time derivatives $\frac{d^j}{dt^j}$ of a single output $y(t)$ and a single input $u(t)$. In (4.1), the scalar real valued numbers a_j for $j = 0, \ldots, n_a$, $a_{n_a} \neq 0$ and b_j for $j = 0, \ldots, n_b$, $b_{n_b} \neq 0$ respectively are called the denominator and numerator coefficients. The input $u(t)$ is distinguished from the output $y(t)$ in (4.1) by requiring $n_a \geq n_b$. As a result, the n_ath derivative is the highest derivative of the output $y(t)$ and n_a is used to indicate the order of the differential equation.

An alternative representation of a model of a continuous time system can be obtained by rewriting the n_ath order differential equation in (4.1) into a set of (coupled) first order differential equations. This can be done by introducing a state variable $x(t)$ and rewriting the higher order differential equation into

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{4.2}$$

where A, B, C and D are real valued matrices. The set of first order differential equations given in (4.2) is referred to as a state space representation. The state variable $x(t)$ is a column vector and contains n_a variables, where n_a is the order of the differential equation.

The size of the matrices in (4.2) corresponds to the order of differential equation from which the state space realization is derived. For generalization purposes, consider multiple inputs and outputs rearranged in $m \times 1$ input column vector $u(t)$ and a $p \times 1$ output column vector $y(t)$. Given the $n_a \times 1$ size of the state vector, the state matrix A has size $n_a \times n_a$, the input matrix has size $n_a \times m$, the output matrix C has size $p \times n_a$ and the feedthrough matrix D has size $m \times p$. From these size considerations it can be observed that the state space realization in (4.2) easily generalizes the model description of multi input multi output systems.

To illustrate the concepts, consider the differential equation

$$m\frac{d^2}{dt^2}y(t) + c\frac{d}{dt}y(t) + ky(t) = u(t) \tag{4.3}$$

that describes the dynamical behavior of the one cart system given in Figure 4.1. The differential equation (4.3) is found by writing Newton's second law for the cart mass m with position output $y(t)$, spring force $ky(t)$, damper force $c\frac{d}{dt}y(t)$ and force input $u(t)$. Comparing with (4.1) it can be seen that $n_a = 2 \geq n_b = 0$, making (4.3) a second order differential equation. The differential equation can be rewritten into a state space representation (4.2) by defining the a state variable

$$x(t) := \begin{bmatrix} y(t) \\ \frac{d}{dt}y(t) \end{bmatrix}$$

that consists the position and velocity of the mass. With this state variable (4.3) can be rewritten into

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0u(t) \end{aligned}$$

which yields a state space model similar to (4.2). In this case, the size of the state matrix A is 2×2, the input matrix B is 2×1, the output matrix C is 1×2 and the feedthrough matrix $D = 0$ is scalar.

Discrete time systems

Discrete time models approximate and describe the sampled data behavior of a continuous time dynamical system. In some applications, such as digital control, the dynamical control system is inherently discrete time. In these situation, analysis with discrete time equivalent models in necessary.

For analysis purposes, both input $u(t)$ and output $y(t)$ are assumed to be sampled on a regular discrete time interval

$$t = k\Delta T,\ k = 0, 1, 2, \ldots$$

where ΔT indicates the sampling time. To maintain uniform notation throughout the analysis, the sampling time ΔT is normalized to $\Delta T = 1$ and the time dependency t is assumed to be discrete with $t = k = 0, 1, 2, \ldots$.

Given sampled or discrete time input/output data, a linear discrete time model can be formulated in the form of a difference equation

$$\sum_{j=0}^{n_c} c_k y(k+j) = \sum_{j=0}^{n_d} d_k u(k+j). \tag{4.4}$$

in which a linear combination is taken of positive time shifted inputs $u(k)$ and outputs $y(k)$. To distinguish the differential equation from the differential equation (4.1), different scalar real valued numbers c_j for $j = 0, \ldots, n_c$, $c_{n_c} \neq 0$ and d_j for $j = 0, \ldots, n_d$, $d_{n_d} \neq 0$ are used. The input $u(k)$ is distinguished from the output $y(k)$ in (4.1) by requiring $n_c \geq n_d$ for causality purposes. As a result, the n_c is the largest time shift of the output $y(k)$ and n_c is used to indicate the order of the difference equation.

The simplicity with which the difference equation can be represented also allows an algebraic representation of (4.4). Introducing the time shift operator

$$qu(k) := u(k+1) \tag{4.5}$$

allows (4.4) to be rewritten into the algebraic expression

$$y(k)\sum_{j=0}^{n_c} c_j q^j = u(k)\sum_{j=0}^{n_d} d_j q^j$$

Following this analysis, the discrete time output $y(k)$ can be represented by the difference model

$$y(k) = G(q)u(k), \text{ with } G(q) = \frac{\sum_{j=0}^{n_d} d_j q^j}{\sum_{j=0}^{n_c} c_j q^j} \tag{4.6}$$

where the scalar real valued numbers c_j for $j = 0, \ldots, n_c$, $c_{n_c} \neq 0$ and d_j for $j = 0, \ldots, n_d$, $d_{n_d} \neq 0$ respectively indicate the the denominator and numerator coefficients.

Similar to the continuous time system representation, the higher order difference equation (4.4) can also be rewritten into a set of (coupled) first order difference equations for analysis purposes. This can be done by introducing a state variable $x(k)$ and rewriting the higher order difference equation into

$$\begin{aligned} qx(k) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) + Ju(k) \end{aligned} \tag{4.7}$$

where $qx(k) = x(k+1)$ according to (4.5). The state variable $x(k)$ is a column vector and contains n_c variables, where n_c is the order of the difference equation. The state space matrices in (4.7) are labeled differently to distinguish them form the continuous time state space model.

Laplace and z transform

An important mathematical concept for the analysis of models described by linear differential equations such as (4.1) and (4.2) is the Laplace transform. As indicated before, the Laplace transform converts linear differential equations into algebraic expressions. With this conversion, proper algebraic manipulation can be used to recover solutions of the differential equation. In a similar manner, the z-transform is used for discrete time models described by difference equations. Although it was shown in (4.6) that a difference equation can be written as an algebraic expression, the z-transform allows complex analysis of the discrete time models.

The Laplace transform of a signal $u(t)$ is defined to be

$$L\{u(t)\} := u(s) = \int_{t=0}^{\infty} u(t)e^{-st}\, dt \tag{4.8}$$

where the integration over t eliminates the time dependency and the transform $u(s)$ is a function of the Laplace variable only. This is indicated in the transform $u(s)$ where the dependency of t has been dropped and $u(s)$ is a function of the (complex valued) Laplace variable s only.

The integral (4.8) exists for most commonly used signals $u(t)$, provided certain condition on s are imposed. To illustrate the transform, consider a (unity) step signal

$$u(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

where the shape of $u(t)$ resembles a step wise change of an input signal. With the definition of the Laplace transform in (4.8) the transform of the step signal becomes

$$u(s) = \int_{t=0}^{\infty} u(t) e^{-st}\, dt = \int_{t=0}^{\infty} e^{-st}\, dt = -\frac{e^{-st}}{s}\bigg|_{0}^{\infty} = \frac{1}{s} \tag{4.9}$$

where it is assumed that the real part of s is greater than zero so that $\lim_{t \to \infty} e^{-st} = 0$.

If a signal $u(k)$ is given at discrete time samples $k = 0, 1, 2, \ldots$ the integral expression of (4.8) cannot be applied. Instead, a transform similar to the Laplace transform can be used and denoted by the z-transform. The z-transform of a discrete time signal $u(k)$ is defined as:

$$L\{u(k)\} := u(z) = \sum_{k=0}^{\infty} u(k) z^{-k}. \tag{4.10}$$

The series (4.10) converges if assumed that there exists values r_l and r_u with $r_l < |z| < r_u$ as bounds on the magnitude of the complex variable z.

The z-transform has the same role in discrete time systems that the Laplace transform has in continuous time systems. In case of sampling, the complex variable z of the z-transform is related to the complex variables s in the Laplace transform via

$$z = e^{s\Delta T} \tag{4.11}$$

where ΔT is the sampling time used for sampling. Both the Laplace and z-transform are linear operators and satisfies

$$L\{\alpha u(t) + \beta y(t)\} = \alpha L\{u(t)\} + \beta\{y(t)\}. \tag{4.12}$$

Using the definition in (4.8) and the linearity property in (4.12), the transform of most commonly used functions has been precalculated and tabulated.

Of particular interest for the analysis of linear differential equations such as (4.1) and (4.2) is the Laplace transform of a derivative:

$$\begin{aligned} L\{\frac{d}{dt}u(t)\} &= \int_{t=0}^{\infty} \frac{d}{dt}u(t)e^{-st}\,dt = u(t)e^{-st}\big|_0^{\infty} + s\int_{t=0}^{\infty} u(t)e^{-st}\,dt \\ &= su(s) - u(0) \end{aligned}$$

With $u(0) = 0$ it can be seen that the Laplace transform of the derivative of $u(t)$ is simply s times the Laplace transform of $u(s)$. This result can be extended to higher order derivatives and the result for the nth derivative is given by

$$L\{\frac{d^n}{dt^n}u(t)\} = s^n u(s) - \sum_{j=1}^{n} s^{n-j} \left.\frac{d^{j-1}}{dt^{j-1}}u(t)\right|_{t=0}.$$

In case the signal $u(t)$ satisfies the initial zero conditions $\left.\frac{d^{j-1}}{dt^{j-1}}u(t)\right|_{t=0} = 0$ for $j = 1, \ldots, n$, the formula reduces to

$$L\{\frac{d^n}{dt^n}u(t)\} = s^n u(s)$$

and the Laplace transform of a nth order derivative is simply s^n times the transform $u(s)$.

For discrete time systems the interest lies in the z-transform of a time shifted signal. Similar to the Laplace transform, the z-transform of a n time shifted signal can be computed and is given by

$$L\{q^n u(k)\} = z^n u(z) - \sum_{j=0}^{n-1} z^{n-j} u(j).$$

In case the discrete time signal $u(k)$ satisfies the initial zero conditions $u(j) = 0$ for $j = 0, \ldots, n-1$, the formula reduces to

$$L\{q^n u(k)\} = z^n u(z)$$

and the z-transform of a n time shifted discrete time signal is simply z^n times the transform $u(z)$.

Transfer function models

The result on Laplace and z-transform can be used to reduce linear differential equations (4.1) and difference equation (4.4) to algebraic expression. Starting with the differential equations for continuous time models and assuming zero initial conditions for both the input $u(t)$ and output signal $y(t)$, the Laplace transform of (4.1) yields

$$y(s)\sum_{j=0}^{n_a} a_j s^j = u(s)\sum_{j=0}^{n_b} b_j s^j$$

which can be written in transfer function format

$$y(s) = G(s)u(s), \text{ with } G(s) = \frac{\sum_{j=0}^{n_b} b_j s^j}{\sum_{j=0}^{n_a} a_j s^j}. \tag{4.13}$$

In (4.13), the transfer function $G(s)$ is the ratio of the numerator polynomial $\sum_{j=0}^{n_b} b_j s^j$ and the denominator polynomial $\sum_{j=0}^{n_a} a_j s^j$. As indicated before, the scalar real valued numbers a_j for $j = 0, \ldots, n_a$, $a_{n_a} \neq 0$ and b_j for $j = 0, \ldots, n_b$, $b_{n_b} \neq 0$ respectively are called the denominator and numerator coefficients.

Similarly for the discrete time model, assuming zero initial conditions for both the input $u(k)$ and output signal $y(k)$, the z-transform of (4.4) yields

$$y(z) \sum_{j=0}^{n_c} c_j z^j = u(z) \sum_{j=0}^{n_d} b_j z^j$$

which can be written in transfer function format

$$y(z) = G(z)u(z), \text{ with } G(z) = \frac{\sum_{j=0}^{n_c} c_j z^j}{\sum_{j=0}^{n_a} a_j z^j}. \tag{4.14}$$

From the transfer function representations, poles and zeros of the dynamic system can be computed for dynamic system analysis. The poles of the system are defined as the roots of the denominator polynomial. The zeros of the system are defined as the roots of the numerator polynomial.

The Laplace and z-transform can also be used to reduce the state space representation to a set of algebraic expressions that consists of (coupled) first order polynomials. Assuming zero initial conditions for the state vector $x(t)$, application of the Laplace transform to (4.2) yields

$$\begin{aligned} sx(s) &= Ax(s) + Bu(s) \\ y(s) &= Cx(s) + Du(s) \end{aligned}$$

in which the state vector $x(s)$ can be eliminated. Solving for $x(s)$ gives $x(s) = (sI - A)^{-1}Bu(s)$ and the above transform can be rewritten into a transfer function representation

$$y(s) = G(s)u(s), \text{ with } G(s) = D + C(sI - A)^{-1}B. \tag{4.15}$$

Under mild technical conditions involving controllability and observability of the state space model, the transfer function representations in (4.13) and (4.15) are similar in case the state space model in (4.2) is derived from the differential equation (4.1) and visa versa.

4.5.2 Dynamic response

The Laplace and z-transform offer the possibility to compute the dynamic response of a dynamic system by means of algebraic manipulations. The analysis of the dynamic response gives insight in the dynamic behavior of the system by addressing the response to typical test signals such as impulse, step and sinusoid excitation of the system.

The response can be computed for relatively simple continuous or discrete dynamical systems given by low order differential or difference equations. Both the state space model and the transfer function descriptions provide helpful representations in the analysis of a dynamic system. The result are presented in the following.

Pulse and step response

A possible way to evaluate the response of a dynamic system is by means of pulse and step based test signals. For continuous time systems an input impulse signal is defined as a δ function

$$u_{imp}(t) := \delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

with the property

$$\int_{t=-\infty}^{\infty} f(t)\delta(t) = f(0)$$

where $f(t)$ is an integrable function over $(-\infty, \infty)$. Although an impulse signals is not practical from an experiment point of view, the computation or simulation of the impulse response gives insight in the transient behavior of the dynamical system.

With the properties of the impulse function $\delta(t)$ mentioned above, the Laplace transform of the impulse function is given by

$$L\{\delta(t)\} = \delta(s) = \int_{t=0}^{\infty} \delta(t)e^{-st}\, dt = e^{-s0} = 1.$$

Hence the output $y(s)$ due to an impulse input is given by $y_{imp}(s) = G(s)u_{imp}(s) = G(s)\delta(s) = G(s)$. As a result, an immediate inverse Laplace transform of the continuous time transfer function $G(s)$

$$y_{imp}(t) = L^{-1}\{G(s)\}$$

gives the dynamic response $y_{imp}(t)$ of the system to an impulse input response.

The computation of the step response is done in a similar way. In (4.9), the Laplace transform of the step signal

$$u_{step}(t) := \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

is given as $u_{step}(s) = \frac{1}{s}$. Consequently, with $y_{step}(s) = G(s)u_{step}(s) = \frac{G(s)}{s}$, the inverse Laplace transform of $\frac{G(s)}{s}$

$$y_{step}(t) = L^{-1}\{\frac{G(s)}{s}\}$$

will yield the dynamic response $y_{step}(t)$ of the system to an step input response.

From a practical point of view, the computation of an inverse Laplace transform is limited to low order models of 1st or 2nd order. However, the results give insight in the dominant behavior of most dynamic systems. This is illustrated in the following examples.

- Consider a first order continuous model given by the transfer function

  $$G(s) = \frac{K}{\tau s + 1}$$

  where K and τ indicates respectively the static gain and the time constant of the system. Such a transfer function may arise from a simple RC network with $\tau = RC$. In order to compute the step response of the system, the inverse Laplace transform of $\frac{G(s)}{s}$ needs to be computed. This inverse Laplace transform is given by

  $$y_{step}(t) = L^{-1}\{\frac{G(s)}{s}\} = \frac{K}{\tau}(1 - e^{-t/\tau})$$

  and it can be seen that the step response is an exponential function. For stability the time constant τ needs to satisfy $\tau > 0$. It can also be observed that the smaller the time constant, the faster the response.

- Consider a second order continuous time model given by the transfer function

  $$G(s) = \frac{\omega_n^2}{s^2 + 2\beta\omega_n s + \omega_n^2} \tag{4.16}$$

  where ω_n and β respectively indicate the undamped resonance frequency and the damping coefficient of the system. This model can be derived from the dynamical behavior of the one cart system depicted in Figure 4.1 and given in (4.3). For $\beta < 1$ (underdamped), the inverse Laplace transform of $G(s)$ is given by

  $$y_{imp}(t) = \frac{\omega_m}{\sqrt{1-\beta^2}} e^{-\beta\omega_n t} \sin \omega_n \sqrt{1-\beta^2} t.$$

From this expression it can be observed that the response is a decaying sinusoid with a resonance frequency of $\omega_n\sqrt{1-\beta^2}$. For stability, both $\omega_n > 0$ and $\beta > 0$ and the larger $omega_n$, the faster the decay of the sinusoid and the higher is the frequency of the response $y_imp(t)$. Illustration of the impulse response of this second order system have been depicted in Figure 4.2 and Figure 4.3 where variations in the undamped resonance frequency ω_n and the damping coefficient β illustrate the dynamic behavior of the system.

For discrete systems, the analysis of the pulse response is based on the discrete time pulse function

$$u_{imp}(k) := \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

which has a value of 1 at $k = 0$ and zero anywhere else. The step signal is similar to the continuous time signal and is given by

$$u_{step}(k) := \begin{cases} 0, & k < 0 \\ 1, & k \geq 1 \end{cases}$$

In order to characterize the discrete time pulse and step response a similar procedure as for the continuous time model can be followed by using the z-transform. It is easy to show that the z-transform $u_{imp}(z) = 1$ and the z-transform of the step signal equals $u_{step}z = \frac{z}{z-1}$. Hence, the response of the discrete time system to a pulse or step signal can be computed with

$$y_{imp}(k) = L^{-1}\{G(z)\},\ y_{step}(k) = L^{-1}\{\frac{G(z)z}{z-1}\}$$

In addition to the approach using a z-transform, the ratio of the polynomials in the difference model (4.6) can be written in an series expansion

$$G(q) = \frac{\sum_{j=0}^{n_d} d_j q^j}{\sum_{j=0}^{n_c} c_j q^j} = \sum_{j=0}^{\infty} g_k q^{-k}.$$

With the discrete time pulse function $u_{imp}(k)$ as an input, it can be observed that

$$y_{imp}(k) = \sum_{j=0}^{\infty} g_k q^{-k}\delta(k) = g_k$$

and it can be concluded that the pulse response $y_{imp}(k)$ equals the coefficients in the series expansion of the difference equation. Similarly, with the discrete time step function $u_{step}(k)$

as an input, it can be observed that

$$y_{imp}(k) = \sum_{j=0}^{\infty} g_k q^{-k} u_{step}(k) = \sum_{j=0}^{k} g_k$$

and it can be concluded that the step response $y_{step}(k)$ values are computed as a finite sum of the coefficients in the series expansion of the difference equation. The computation of a discrete time pulse response for a first order discrete time model is given in the following example.

- Consider a first order discrete model given by the difference model

  $$G(q) = \frac{1}{q+d}$$

  where d indicates the discrete time constant of the system. The series expansion of the difference model can be computed as follows

  $$G(q) = \frac{1}{q-d} = \sum_{j=0}^{\infty} d^j$$

  and it can be seen that the discrete time pulse response

  $$y_{imp}(k) = d^k$$

  is an exponential function. For stability the discrete constant d needs to satisfy $|d| < 1$. Similar as in the continuous time model it can be observed that the smaller the time constant, the faster the response. Additionally, the first order discrete time model may exhibit an oscillation in case $-1 < d < 0$.

Sinusoid and frequency response

So far we have considered transient effects caused by step, pulse and impulse inputs to investigate the dynamic properties of a dynamical system. However, periodic inputs occur frequently in practical situations and the analysis of a dynamic system to periodic inputs and especially sinusoidal inputs can yield more insight in the behavior of the system.

The response of a linear system to a sinusoidal input is referred to as the frequency response of the system. An input signal $u(t)$

$$u(t) = U \sin \omega t$$

that is a sine wave with amplitude U and frequency ω_j has a Laplace transform

$$u(s) = \frac{U\omega}{s^2 + \omega^2}.$$

Consequently, the response of the system is given by

$$y(s) = G(s)\frac{U\omega}{s^2 + \omega^2}$$

and a partial fraction expansion of $y(s)$ will result in terms that represent the (stable) transient behavior of $y(s)$ and the term associated to the sinusoidal input $u(s)$. Elimination of the transient effects and performing an inverse Laplace transform will yield a periodic time response $y(t)$ of the same frequency ω_j given by

$$y(t) = AU \sin(\omega t + \phi)$$

where the amplitude magnification A and the phase shift ϕ are given by

$$\begin{aligned} A &= G(s)|_{s=i\omega} \\ \phi &= \angle\, G(s)|_{s=i\omega} \end{aligned} \tag{4.17}$$

By evaluating the transfer function $G(s)$ along the imaginary axis $s = i\omega,\ \omega \geq 0$, the magnitude $|G(i\omega)|$ gives information on the relative amplification of the sinusoidal input, whereas the phase $\angle G(i\omega)$ gives information on the relative phase shift between input and output.

This analysis can be easily extended to discrete time systems by employing the relation between the Laplace variable s and the z transform variable in (4.11) to obtain to discrete time sinusoidal response

$$y(k) = AU \sin(\omega k + \phi)$$

where the amplitude magnification A and the phase shift ϕ are given by

$$\begin{aligned} A &= G(z)|_{z=e^{i\Delta T\omega}} \\ \phi &= \angle\, G(z)|_{z=e^{i\Delta T\omega}} \end{aligned} \tag{4.18}$$

Due to the sampling nature of the discrete time system, the transfer function $G(z)$ is now evaluated on the unit circle

$$e^{i\Delta T\omega},\ 0 \leq \omega < \frac{\pi}{\Delta T}$$

to attain information of the magnitude and phase shift of the sinusoidal response.

Plotting the frequency response of a dynamical system gives insight in the pole locations (resonance modes) and zero locations of the dynamical system. As an example, the frequency response of the second order system given in (4.16) has been depicted in Figure 4.4. It can be seen from the figure that, as expected, the second order system is less damped for smaller damping coefficients β and this results to a larger amplitude response of the second order system at the resonance frequency $\omega_n = 6$ rad/s. It can also be observed that the phase change at the resonance frequency becomes more abrupt for smaller damping coefficients.

4.5.3 Performance indicators for dynamic systems

Step response parameters

Specifications for dynamic systems often involve requirements on the transient behavior of the system. Transient behavior requirements can be formulated on the basis of a step response and the most significant parameters have been summarized below and illustrated in Figure 4.5.

- Steady state or DC value y_s of step response output
- The steady state error y_{se} is the error between steady state value y_s and desired DC value of step response output.
- The maximum overshoot A_m is the maximum deviation of the step response output above its steady state value y_s.
- The peak time t_p is the time at which the maximum overshoot occurs.
- Settling time t_s is the time at which the step response input stays within some small percentage rage of the steady state value y_s. Typically, a percentage of 2% or 5% is chosen to determine the settling time.
- The rise time t_r is usually defined as the time required for the step response output to rise from 10% to 90% of the steady state value y_s.
- The delay time t_d is defined as the time required to reach 50% of the steady state value y_s.

Most of the above value can be obtained from an experimentally determined step response. In general, they cannot be obtained in an analytical form, except for low order models. For the second order model of the one mass system given in (4.3), some analytical results can be obtained. For a second order model of (4.3), the maximum overshoot A_m is determined by

$$A_m = 100e^{-\pi\beta}/sqrt1 - \xi^2, \text{ where}\xi = \frac{A}{\sqrt{\pi^2 + A^2}}, \; A = \ln \frac{100}{A_m}$$

The peak time t_p can be computed by

$$t_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}$$

whereas the delay time t_d can be approximated by

$$t_d \approx \frac{1 + 0.7\xi}{\omega_n}.$$

As the maximum overshoot increases with a smaller damping coefficient β in the system, the maximum overshoot is often used to indicate the relative stability of the system.

Frequency domain parameters

With the frequency domain analysis of dynamic systems, specifications for the dynamic properties of a system can also be stated in the frequency domain. Frequency domain specifications in filter design often address ripple, bandwidth, roll-off and phase lag parameters. Similar characteristics can also be specified for dynamic systems in case the model of the system is analyzed in the frequency domain. The most significant parameters have been summarized below and illustrated in Figure 4.6.

- The bandwidth ω_b is a notion for the maximum frequency at which the output will track a sinusoidal input in a satisfactory manner. By convention, the bandwidth is defined as the frequency at which the output is attenuated -3dB (0.707).

- The resonant frequency ω_r is the first frequency at which a significant resonance mode with low damping occurs. The resonance mode can, if uncontrolled, negatively influence the settling time of the dynamic system and plays an important role in characterization of performance.

- The resonant peak M_r is the height of a resonance mode. The resonant peak is a measure for the damping. As illustrated in Figure 4.2 for a second order model, the resonance mode increases at lower damping coefficients.

- Steady state errors M_e can also be analyzed in the frequency response of a system. Using the final value theorem for continuous time systems

$$\lim_{t\to\infty} y(t) = y_s = \lim_{s\to 0} sy(s)$$

the presence of steady state errors can be inspected in the frequency domain by evaluation $|G(s)|$ at $s = i\omega = 0$ or for small values of the frequency vector ω. This can be seen as follows. As the Laplace transform $u_{step}(s)$ of a step input signal $u_{step}(t)$ is $u_{step}(s) = \frac{1}{s}$

$$\lim_{t\to\infty} y_{step}(t) = \lim_{s\to 0} sy_{step}(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s).$$

By evaluating $|G(i\omega)|$ for small frequencies ω, the steady state behavior of $G(s)$ can be studied.

A similar result exist for discrete time systems, where the final value theorem reads as follows. If $u(z)$ converges for $|z| > 1$ and all poles of $(z-1)u(z)$ are inside the unit circle, then

$$\lim_{k\to\infty} u(k) = \lim_{z\to 1}(z-1)u(z).$$

Hence, for discrete time systems the steady state behavior of a transfer function $G(z)$ can be studied by evaluating $|G(e^{i\omega\Delta T})|$ for small frequencies ω.

- Roll-off R_d at high frequencies is the defined as the negative slope of the frequency response at higher frequencies. The roll-off determines the performance of the dynamic system as high frequent disturbances can be amplified if a dynamic system does not have enough high frequent roll-off.

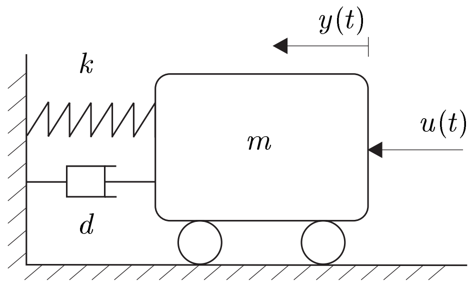

Figure 4.1: One cart system representing a single mass dynamical system with cart mass m, spring constant k and damping constant c.

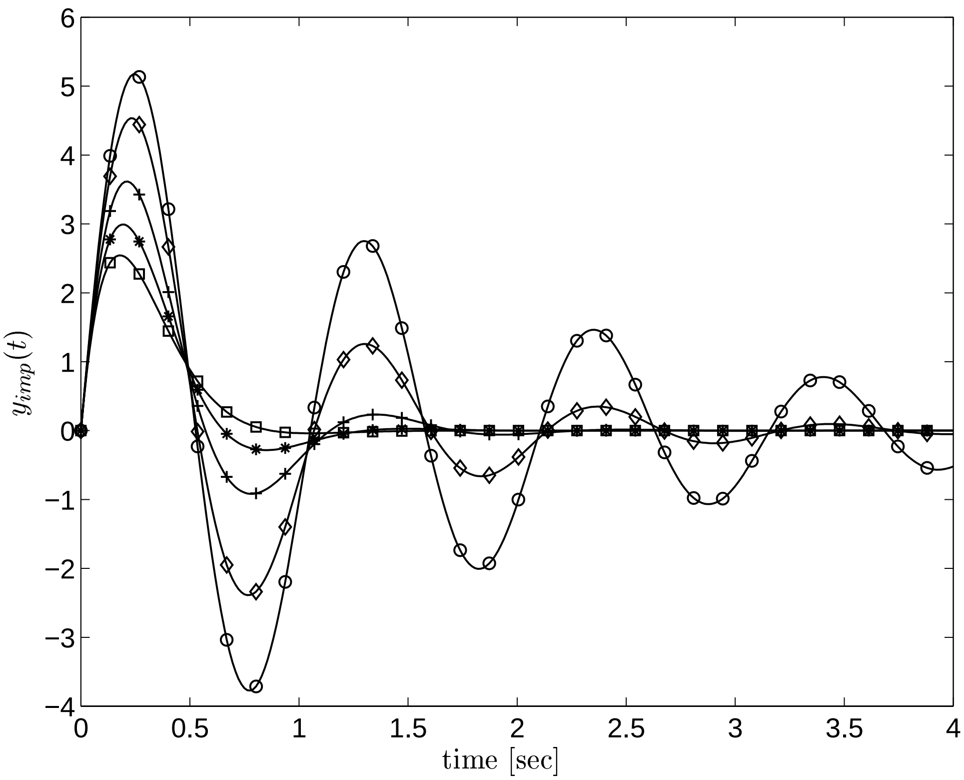

Figure 4.2: Variations in impulse response $y_{imp}(t)$ of second order system with $\omega_n = 6$ and $\beta = 0.1$ o, $\beta = 0.2$ $\diamond$, $\beta = 0.4$ $+$, $\beta = 0.6$ $\star$ and $\beta = 0.8$ $\Box$

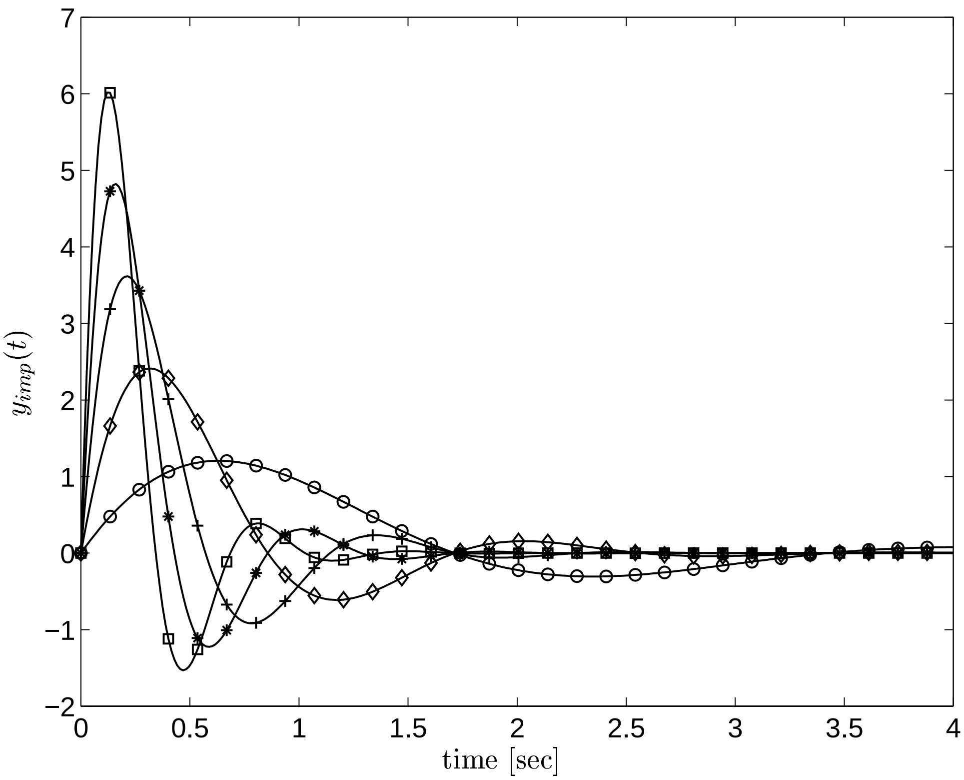

Figure 4.3: Variations in impulse response $y_{imp}(t)$ of second order system with $\beta = 0.4$ and $\omega_n = 2$ o, $\omega_n = 4$ $\diamond$, $\omega_n = 6$ $+$, $\omega_n = 8$ $\star$ and $\omega_n = 10$ $\square$

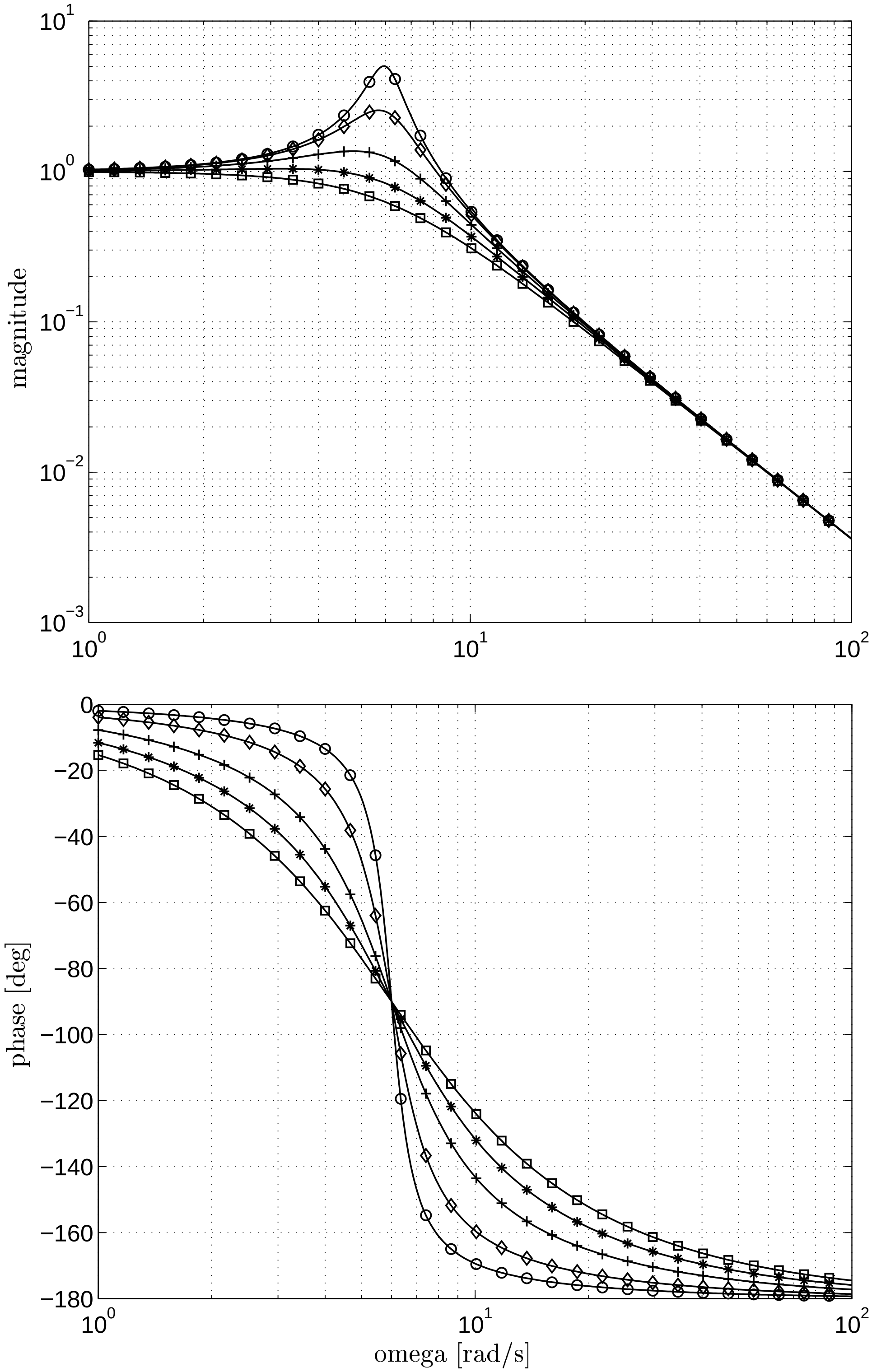

Figure 4.4: Variations in frequency response of second order system $G(s)$ with $\omega_n = 6$ and $\beta = 0.1$ o, $\beta = 0.2$ $\diamond$, $\beta = 0.4$ +, $\beta = 0.6$ $\star$ and $\beta = 0.8$ $\Box$

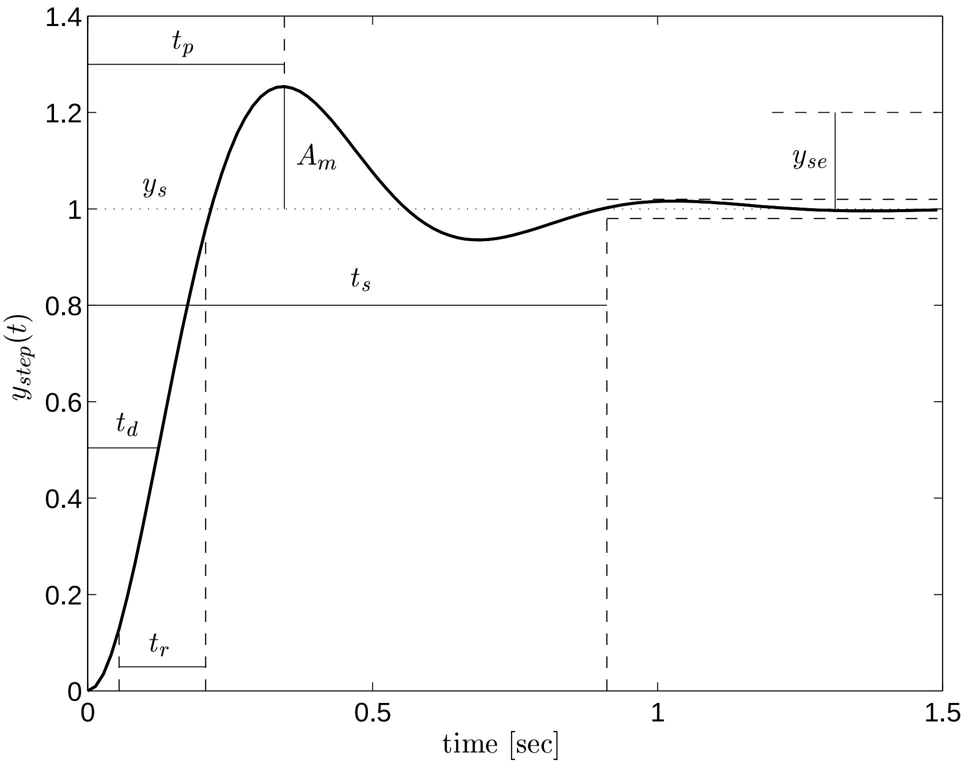

Figure 4.5: Parameters for step-response behavior: steady state value y_s, steady state error y_{se}, maximum overshoot A_m, peak time t_p, settling time t_s, rise time t_r and delay time t_d

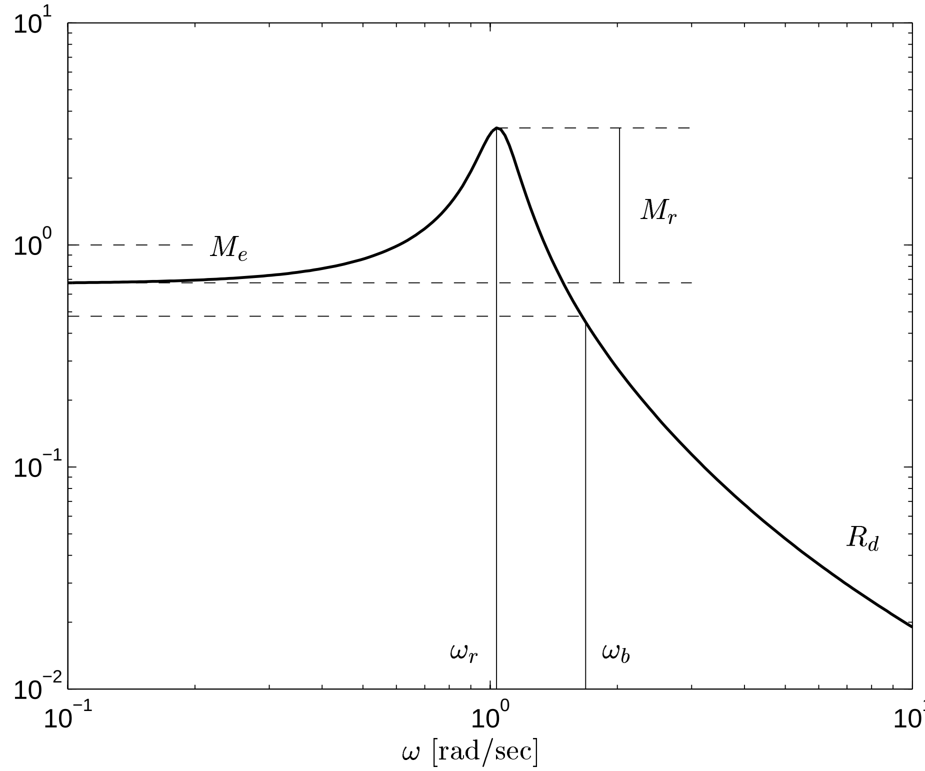

Figure 4.6: Parameters for frequency response behavior: bandwidth ω_b, resonance frequency ω_r, resonant peak M_r, steady state error M_e and roll-off R_d