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Coprime Factor Based Closed-Loop Model Validation

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1 The model validation problem

1.1 Problem definition

In order to provide a formal definition of the model validation problem, it is necessary to provide more details on an uncertainty model, denoted by $\mathcal{P}$, that consists of a nominal model $\hat{P}$ and an additional model uncertainty or allowable model perturbation Δ. A general description of an uncertainty model $\mathcal{P}$ can be given via an upper fractional transformation $\mathcal{F}_u(Q,\Delta)$:

$$\begin{array}{c}\mathcal{P} = \{P \mid P = \mathcal{F}_u(Q,\Delta)\} \text{ with} \\ \mathcal{F}_u(Q,\Delta) := Q_{22} + Q_{21}\Delta(I - Q_{11}\Delta)^{-1}Q_{12}\end{array} \tag{1}$$

where the entries of Q contain the information on the nominal model $\hat{P}$ and how the uncertainty Δ is characterized around the nominal model. Typically, $Q_{22} = \hat{P}$ as for $\Delta = 0$ the nominal model should be obtained.
The entries of Q in (1) are assumed to be LTI discrete-time systems. Furthermore, Δ is assumed to be a member of a class of perturbations with $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty < 1$. Given these assumptions, the discrete time experimental data $\{u(t), y(t)\}$, coming from the plant and consisting of N observations, is described by

$$y(t) = \mathcal{F}_u(Q,\Delta)u(t) + Hd(t) \tag{2}$$

where d denotes an unknown but bounded noise $d \in l_2$ with $\|d\|_2 < 1$ and H is a stable and stably invertible noise filter. The model validation problem [1] can be stated as follows.
Problem: Given the Q of the uncertainty model $\mathcal{P}$ in (1), the noise model H and observations $\{u(t), y(t)\}$ for $t = 0, 1, \ldots, N-1$, does there exists a discrete time signal d with $\|d\|_2 < 1$ and an uncertainty Δ with $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\| < 1$ such that (2) holds.

1.2 Model validation via optimization

The general approach taken in model validation is an optimization either in time domain [2, 3] or frequency domain [4, 5, 6], where the value of the smallest model perturbation Δ and the noise signal $d(t)$ is found that could have produced the experimental data in (2). In case the smallest model perturbation or noise signal exceeds the assumed bounds, the model is invalidated by the experimental data.
Posing the condition $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\| < \alpha$ is a key step in solving the model validation problem by optimizing the value of the smallest model perturbation Δ. This condition has been solved in [2] for the LTI case and can be summarized as follows.

Lemma 1.1 *Given the signals $v(t)$ and $z(t)$ for $t = 0, 1, \ldots, N-1$ with $v = \Delta z$. Then there exists a $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < \alpha$ if and only if*

$$V^TV < \alpha^2 Z^T Z$$

where V and Z are block Toeplitz matrices derived from $v(t)$ and $z(t)$.

With this result, the model validation can be solved via a convex optimization. Crucial in this convex optimization is the fact that the uncertainty Δ appears linearly in (2). With the general LFT form of the uncertainty model in (1), it can be seen that this is the case for $Q_{11} = 0$ and holds for example for an uncertainty model with a multiplicative uncertainty description.
With an the uncertainty Δ appearing linearly in (2), finding the smallest model perturbation Δ and the noise signal $d(t)$ is simply an additional linear constraint added to the convex minimization. With $Q_{11} = 0$ and $z(t) = Q_{12}u(t)$, the following convex optimization needs to be solved.

$$\begin{array}{c}\min_\alpha, \text{ subjected to} \\ V^TV < \alpha^2 Z^T Z,\ d^T d < 1 \\ z(t) = Q_{12}u(t) \\ y(t) - Q_{22}u(t) = Q_{21}v(t) + Hd(t)\end{array} \tag{3}$$

In case $\alpha \geq 1$, the model is invalidated by the experimental data $\{u(t), y(t)\}$
Although the optimization mentioned above addresses the model validation problem, the question arises whether or not there are alternative uncertainty descriptions that still allow the use of a convex optimization. Moreover, it is preferable to perform the model validation in closed-loop, in case a model needs to be validated for control purposes. In case a feedback connection is created around the signals $u(t)$ and $y(t)$, the resulting optimization involved with the model validation may not be convex.

2 Uncertainty models based on coprime factor perturbations

For the purpose of the closed-loop model validation using coprime factor uncertainty models, the nominal model $\hat{P}$ is represented in a coprime factor representation $\hat{P} = \hat{N}\hat{D}^{-1}$ where $(\hat{N}, \hat{D})$ denotes a right coprime factorization (*rcf*) of the model $\hat{P}$. The accompanying uncertainty on the model is assumed to be modeled as a perturbations in a dual-Youla parametrization [7]. This perturbation uses the knowledge of the feedback controller to characterize the model uncertainty and yields a uncertainty model $\mathcal{P}$ that can be characterized as follows.

Proposition 2.1 *Let a nominal model P with a* rcf $(\hat{N}, \hat{D})$ *and a controller C with a* rcf (N_c, D_c) *form an internally stable feedback connection. Then an uncertainty model $\mathcal{P}$ is constructed by*

$$\begin{array}{c} \mathcal{P}(\hat{N}, \hat{D}, N_c, D_c, \hat{V}, \hat{W}) := \\ \{P \mid P = (\hat{N} + D_c\Delta_R)(\hat{D} - N_c\Delta_R)^{-1} with \\ \Delta_R \in \mathcal{RH}_\infty,\ \Delta := \hat{V}\Delta_R\hat{W} \text{ and } \|\Delta\|_\infty < 1\} \end{array} \quad (4)$$

and $\hat{V}$, $\hat{W}$ are stable and stably invertible weighting functions.

As indicated in the proposition, the uncertainty model $\mathcal{P}$ essentially depends on the factorization $(\hat{N}, \hat{D})$ of the nominal model $\hat{P}$, the factorization (N_c, D_c) of the known controller C and the weighting functions $\hat{V}$, $\hat{W}$ that take into account the shape and scaling of the uncertainty.

3 Closed-loop Model validation using coprime factorizations

3.1 Favorable properties

Although the uncertainty model $\mathcal{P}$ in (??) looks more complicated than a standard additive or multiplicative uncertainty, the only knowledge needed to construct the uncertainty model $\mathcal{P}$ is a coprime factorization of the nominal model $\hat{P}$ and the controller C. With this information, the construction of the model uncertainty Δ_R is not more complicated than a standard uncertainty description and due to the specific structure of the uncertainty model, the uncertainty Δ_R in (??) will appear in an affine way in any closed-loop transfer function. For an explanation of this property, consider the transfer function matrix

$$T(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} \begin{bmatrix} C & I \end{bmatrix} \quad (5)$$

which captures all possible closed-loop transfer functions. With the general closed-loop transfer function matrix $T(P,C)$ given in (5) and the uncertainty model given in (??), the following result can be given.

Lemma 3.1 *Consider the uncertainty model $\mathcal{P}$ of* (??) *and the controller C used in the construction of $\mathcal{P}$. With the definition of the closed-loop transfer function matrix $T(P,C)$ given in* (5), *the uncertainty model $\mathcal{P}$ satisfies*

$$\begin{array}{c} \mathcal{P} = \{P \mid T(P,C) = \mathcal{F}_u(M, \Delta) \\ with\ \Delta \in \mathcal{RH}_\infty,\ \|\Delta\|_\infty < 1\} \end{array}$$

where the entries of M are given by

$$\begin{array}{l} M_{11} = 0,\ M_{12} = \hat{W}^{-1}(\hat{D} + C\hat{N})^{-1} \begin{bmatrix} CI \end{bmatrix} \\ M_{21} = -\begin{bmatrix} I \\ -C \end{bmatrix} D_c\hat{V}^{-1},\ M_{22} = T(\hat{P}, C) \end{array} \quad (6)$$

The entries of M in (6) are all known quantities. It can be verified that all these entries are stable *if and only if* the controller C internally stabilizes the nominal model $\hat{P}$. It can also be observed that the uncertainty Δ appears affinely in all possible (weighted) closed-loop transfer functions of $T(P,C)$.

3.2 Closed-loop model validation

On the basis of the closed-loop representation in Lemma 3.1, a closed-loop model validation problem can be formulated where the affine appearance of the model uncertainty Δ in (6) is used. The discrete time experimental data $\{u(t), y(t)\}$, coming from the closed-loop plant, consisting of N observations is described by

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \mathcal{F}_u(M, \Delta) \begin{bmatrix} r_2(t) \\ r_1(t) \end{bmatrix} + Hd(t) \quad (7)$$

where r_1, r_2 denote reference signals and d denotes an unknown but bounded noise $d \in l_2$ with $\|d\|_2 < 1$ and H is a stable and stably invertible noise filter. The closed-loop model validation problem can now be formalized as follows.

Problem: Given the M of the uncertainty model $\mathcal{P}$ in (6), the noise model H and observations $\{r_2(t), r_1(t), y(t), u(t)\}$ for $t = 0, 1, \ldots, N-1$, does there exists a discrete time signal d with $\|d\|_2 < 1$ and an uncertainty Δ with $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\| < 1$ such that (7) holds.

Along the lines of Section 1.2, the closed-loop model validation problem can be solved via a convex optimization, as $M_{11} = 0$. With the uncertainty Δ appearing linearly in (6), finding the smallest model perturbation Δ and the noise signal $d(t)$ is simply an additional linear constraint added to the convex minimization:

$$\begin{array}{c} \min_\alpha, \text{ subjected to} \\ V^TV < \alpha^2 Z^T Z,\ d^T d < 1 \\ z(t) = M_{12} \begin{bmatrix} r_2(t) \\ r_1(t) \end{bmatrix} \\ \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} - M_{22} \begin{bmatrix} r_2(t) \\ r_1(t) \end{bmatrix} = M_{21}v(t) + Hd(t) \end{array}$$

In case $\alpha \geq 1$, the model is closed-loop invalidated by the closed-loop experimental data $\{r_2(t), r_1(t), y(t), u(t)\}$.

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