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Suboptimal Feedback Control by a Scheme of Iterative Identification and Control Design

R.A. DE CALLAFON[‡§] **P.M.J. VAN DEN HOF**[*]

Abstract

In this paper a framework for an iterative procedure of identification and robust control design is introduced wherein the robust performance is monitored during the subsequent steps of the iterative scheme. By monitoring the performance via a model-based approach, the possibility to guarantee performance improvement in the iterative scheme is being employed.

In order to monitor achieved performance (for a present controller) and to guarantee robust performance (for a future controller), an uncertainty set is used where the uncertainty structure is chosen in terms of model perturbations in the dual Youla parametrization. This uncertainty structure is shown to be particularly suitable for the control performance measure that is considered.

The model uncertainty set can be identified by an uncertainty estimation procedure on the basis of closed-loop experimental data. To obtain performance robustness, robust control design tools are used to synthesise controllers on the basis of the identified uncertainty set.

Keywords: coprime factorizations; robust control; system identification

1 Introduction

For a plant with unknown dynamics, the approach to obtain an enhanced and robust feedback controller with some predescribed optimality properties, is usually supported by the application of a system identification technique and a subsequent model-based control design. As the dynamical model applied in a model-based control design usually is an approximation of the unknown plant, there is a growing interest in investigating the interrelation between the problems of modelling by system identification and the design of feedback controllers. This interrelation has been the main motivation to develop methods to perform a so-called control-relevant identification, in which the quality of the dynamical

[‡]Mech. Engineering Systems and Control Group, Delft Univ. of Technology, Mekelweg 2, 2628 CD, Delft, The Netherlands.

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model with respect to the unknown plant needs to be evaluated in view of the control design application.

Several approaches that consist of iterative procedures of subsequent system identification and model-based control design have been proposed in the literature and one is referred to the survey papers [12] or [32] to obtain a comprehensive overview. In these approaches, the iterative minimization of an identification criterion and a model-based control design criterion is hoped to converge to a globally optimal feedback controller of restricted complexity that can be applied successfully to the unknown plant. As already reported by [30], both the identification and the model-based control design become interrelated and motivates the usage of a control-relevant system identification techniques.

Convergence of the iterative schemes in terms of performance improvement of the feedback controlled plant, using the controller designed during the subsequent steps, has not been proven (yet). Moreover, optimality of the feedback controller in case of undermodelling of the unknown plant even becomes questionable as reported in [17]. However, the motivation to apply an iterative procedure is induced by the fact that a simultaneous (off-line) optimization of both an identification and a model-based control design criterion can be highly non-linear [2]. Although convergence and optimality is not guaranteed, countless numerical simulation examples presented in the literature show promising results, see e.g. [1], [14], [19], [28] or [37]. Successful 'real life' implementations of iterative schemes of identification and model-based control design have been reported in e.g. [25] and [28].

Inevitably, it is important to have convergence of an iterative scheme in terms of a performance improvement of the feedback controlled plant. Debatable is the question, whether or not optimality of a restricted complexity controller applied to the (unknown) plant is the key issue. From a practical point of view, it is more valuable to have at least *guaranteed* performance improvement of the feedback controlled plant, while performing a step in an iterative scheme of subsequent identification and model-based control design. In this way, any effort put into a step of an iterative scheme is assured to give an improvement of the feedback controlled plant.

In most of the iterative schemes found in the references listed above, the attention is focused on iteratively trying to improve nominal performance specifications. However, the model-based controller should be implemented on the actual plant and robustness of the designed feedback controller should be incorporated. Compared to these approaches, the aim of this paper is to introduce a framework for an iterative scheme of identification and model-based control design wherein the *robust* performance of the feedback controlled plant will be monitored. By monitoring robust performance, it is possible to subsequently design model-based controllers in such a way, that the performance improvement of the feedback controlled plant can be *enforced*, during the iterative scheme of interrelated identification and model-based control design. Although this approach does not necessarily lead to a globally optimal restricted complexity controller for the unknown plant, control performance improvement is guaranteed by enforcing subsequent robust performance improvement in the iterative scheme.

To monitor and guarantee performance robustness, the estimation of a nominal model only, does not suffice. Instead, a set of models should be used that is guaranteed to contain the unknown plant. Such a set of models (or uncertainty set) can be obtained by the estimation of a nominal model along with an upper bound on an uncertainty characterization. Currently, system identification methods are available to estimate such a set and one is referred to e.g. [21] or [23] for a nice overview on the topic of worst-case system identification. This topic will not be discussed in this paper and the methods introduced in [13] or [36] will be employed to estimate such a set of models.

Clearly, the problem of obtaining control relevancy in system identification as pointed out above, emphasizes the need to estimate both a nominal model and an upper bound on an uncertainty characterization in a feedback relevant way. The feedback relevant aspects should not be used in the estimation of nominal models only, but should also address the characterization and estimation of the uncertainty set being used. Therefore, this paper focuses on the estimation of a control-relevant model, along with a characterization of the model mismatch with a specific structure that is particular useful for both system identification and control design purposes. It will be shown that the set of models obtained is well suited for (in)validation purposes wherein the closed loop application of the set can be taken into account. In this way, it is possible to subsequently design enhanced model-based controllers and to monitor the robust performance of the feedback controlled plant.

2 Problem description

2.1 General problem formulation

To formalize the problem formulations introduced in this section, first some basic notations will be introduced. The notation P will be used to denote any linear time invariant system that may represent the actual plant denoted by P_o or the model denoted by $\hat{P}$. Furthermore, let $\mathcal{P}$ be used to denote a set of models and C to represent a feedback controller. The subscript i that will be applied to the variables $\hat{P}$, $\mathcal{P}$ or C is used to indicate that the variable depends on the ith step in an iterative scheme of identification and model-based control design. Finally, a control objective function is denoted by $J(P,C)$ and the notion of performance cost will be characterized by the value of a norm $\|J(P,C)\|$: a smaller value of $\|J(P,C)\|$ indicates better performance.

Examples of commonly used control objective functions, as pointed out in [32], may include weighted or mixed sensitivity, as well as LQG and IMC type of control objectives. Throughout this paper, the control objective function $J(P,C) \in \mathbb{R}\mathcal{H}_\infty$ and is restricted to

H_∞-norm based performance specifications. A norm-based control design formulated by

$$C_{opt} := \arg\min_{C} \|J(P_o, C)\|_\infty \tag{1}$$

would lead to an optimal controller C_{opt} in the sense that the performance cost is being optimized. A way to find a feedback controller that will approach C_{opt}, is by performing subsequent steps of control design, wherein performance improvement is guaranteed in each step. Basically, this can be formulated as follows.

Problem 2.1 *Let the feedback connection of a plant P_o and a controller C_i satisfy the performance specification $\|J(P_o, C_i)\|_\infty \leq \gamma_i$. Design a controller C_{i+1} such that the performance $\|J(P_o, C_{i+1})\|_\infty$ satisfies*

$$\|J(P_o, C_{i+1})\|_\infty \leq \gamma_{i+1} < \gamma_i. \tag{2}$$

Although the subsequent design of a feedback controller as mentioned in problem 2.1 does not necessarily give rise to the optimal controller C_{opt} of (1), it may get arbitrary close during subsequent design. For this reason the controller found by the subsequent design mentioned in problem 2.1 is called sub-optimal. The same philosophy is used also in the general framework of H_∞ control design to compute sub-optimal controllers, see e.g. [10] or [38]. A controller that is guaranteed to satisfy the upper bound (2) can be computed and a sub-optimal controller is found by subsequently trying to lower the upper bound.

Clearly, the plant P_o is unknown, which makes problem 2.1 as formulated presently, intractable for a performance specification based on a general control objective function $J(P_o, C)$. To gain information on the control objective function, measurements can be taken from the plant P_o operating under closed loop conditions. If indeed the performance characterization can be accessed directly on the basis of (time domain) observations, the possibility to tune or optimize the controller directly, can be exploited to tackle problem 2.1. A similar idea is used in [18] to perform a model free tuning of a controller on the basis of a minimum variance performance specification.

Unfortunately, the restriction to performance specifications that can be accessed directly on the basis of observations from the plant P_o operating under closed loop conditions, will affect the general applicability of problem 2.1. Furthermore, the value to characterize the performance will be subjected to noise and finite time approximation if measurements are used to evaluate the performance specification directly. Therefore, a model-based approach will be employed to evaluate the performance of the controller C_i and to design an improved controller C_{i+1} that satisfies (2). Additionally, a model-based approach has several advantages, compared to a model free tuning.

- The design of the controller can be done for a wide class of performance objectives and robustness considerations are taken into account [38].

- Available tools for robust and model-based control design can be used.
- Design trade-offs imposed by linear feedback control [4, 8] can be taken into consideration.

Supported by the advantages mentioned above, problem 2.1 is recasted into a model-based problem formulation that can be dealt with by combining system identification and robust control design techniques.

2.2 A feasible procedure

To tackle problem 2.1 for an unknown plant P_o via a model-based approach using system identification techniques, basically two main items should be considered. Firstly a procedure to analyse the upper bound γ_i for $\|J(P_o, C_i)\|_\infty$ *a posteriori*[1] must be found. Secondly, the synthesis of a controller C_{i+1} that satisfies (2) *a priori*[2] must be formulated. To accomplish both aspects, a set of models $\mathcal{P}$ will be identified.

Basically, this set $\mathcal{P}$ will be built up from a nominal model $\hat{P}$ that approximates P_o, along with a characterization of an upper bound of a mismatch between $\hat{P}$ and P_o, such that $P_o \in \mathcal{P}$. Subsequently, system identification techniques can provide an estimate of such a set $\mathcal{P}$ on the basis of data and (additional) prior information on both the data and the plant P_o, see e.g. [13, 36]. Hence, the estimated set $\mathcal{P}$ can be used to evaluate the upper bound γ_i *a posteriori* and to synthesise a controller that satisfies (2) *a priori*. In this perspective, problem 2.1 can be tackled by including system identification techniques and considering the following problem formulation.

Problem 2.2 *Let a plant P_o and an initial controller C_i form a stable feedback connection. To evaluate $\|J(P_o, C_i)\|_\infty \leq \gamma_i$, consider the following step.*

(a) Use experimental data and prior information on both the data and the plant P_o to estimate a set of models $\mathcal{P}_i$ such that $P_o \in \mathcal{P}_i$ and determine

$$\gamma_i = \sup_{P \in \mathcal{P}_i} \|J(P, C_i)\|_\infty \tag{3}$$

Subsequently, consider the following steps.

(b) Design a controller C_{i+1} such that

$$\|J(P, C_{i+1})\|_\infty \leq \gamma_{i+1} < \gamma_i \; \forall P \in \mathcal{P}_i \tag{4}$$

(c) Use (new) experimental data and prior information on both the data and the plant P_o to estimate a set of models $\mathcal{P}_{i+1}$ such that $P_o \in \mathcal{P}_{i+1}$ and

$$\|J(P, C_{i+1})\|_\infty \leq \gamma_{i+1} \; \forall P \in \mathcal{P}_{i+1} \tag{5}$$

[1] When the controller C_i is implemented on the plant P_o

[2] Before implementing the controller C_{i+1} on the plant P_o

The formulation of problem 2.2 is a rather general set-up to generate a sequence of model-based controllers that will satisfy (2). Within this set-up, step (b) reflects the design of a robust controller in order to ensure (2). Both step (a) and (c) contain the estimation of a set of models $\mathcal{P}$. These steps will constitute an identification problem to estimate the set $\mathcal{P}$ and/or a model (in)validation problem [31] in order to guarantee $P_o \in \mathcal{P}$. However, both the identification problem and the model (in)validation problem should be control relevant. This is reflected by the fact that the quality of the models P within a set $\mathcal{P}$ is evaluated by the performance specification $\|J(P,C)\|_\infty$, where step (a) and (c) differ only in the feedback controller C being used.

Repeatingly executing the subsequent steps (b) and (c) will formulate an iterative scheme of identification and control, were (4) and (5) reflect respectively a controller and a model closed loop *validation test* in order to enforce (2). Starting from step (a), where C_i is the controller (initially) implemented on the plant P_o, (3) can be viewed as an initial closed loop *performance assessment test* to evaluate $\|J(P_o,C_i)\|_\infty$ *a posteriori*. In the robust control design of step (b), equation (4) is needed to ensure (2) *a priori*. In this way, both performance robustness and improvement of the upper bound on the closed-loop performance can be guaranteed for C_{i+1}. The performance $\|J(P_o,C_i)\|_\infty$ can be evaluated *a posteriori*, by implementation of C_{i+1} on the plant P_o and estimating a new set $\mathcal{P}_{i+1}$. If indeed (5) is satisfied, in step (b) again a new controller can be designed on the basis of $\mathcal{P}_{i+1}$.

Although the problem formulation in problem 2.2 is fairly general and somehow trivial, it does provide a monotonic non-decreasing sequence of γ_i. A similar idea was proposed also in [2], but the results were limited to a set of models $\mathcal{P}$ described by weighted open loop additive perturbations on the nominal model and a performance objective function based on a (weighted) sensitivity function of the closed loop system. Whether or not a set of models $\mathcal{P}$ described by open loop perturbations (additive as in [2] or multiplicative as in [31]) is suitable for identification and (in)validation purposes, still remains unanswered. Obviously, to provide a feasible procedure for handling problem 2.2, the choice for the structure of the set of models $\mathcal{P}$ should be addressed [33]. Summing up, the following choices will be discussed in this paper.

- The control objective function $J(P,C)$. It plays a crucial role in the closed loop validation tests and the way the controller is to be designed.

- The structure of the set of models $\mathcal{P}$. By characterizing the mismatch between a nominal model $\hat{P}$ and the plant P_o within a set $\mathcal{P}$, one should be able to evaluate the closed performance assessment test (3) and the closed loop validation test of (4) and (5) in a non-conservative way.

- Identification procedure to estimate and (in)validate a set of models $\mathcal{P}$. It should take into account the control design application of the set $\mathcal{P}$. Similar procedures are needed in step (a) and (c) by considering respectively the feedback controllers C_i and C_{i+1}.

- Robust control design method. The design of a controller on the basis of a set of models $\mathcal{P}$ in step (b) is needed to ensure (2).

The remaining part of the paper is devoted to the discussion of the items mentioned above. First, the choice of the control objective function is discussed in section 3. In section 4 the specification and the motivation of the structure of the set of models $\mathcal{P}$ chosen in this paper, will be elaborated. Subsequently, results to analyse performance robustness on the basis of the set $\mathcal{P}$ being chosen, are presented in section 5. Based on this analysis, the control design is discussed in section 6, while the identification procedure to estimate a set $\mathcal{P}$ can be found in section 7.

3 Control objective function

3.1 Feedback and stability

Let again P be used to denote either the plant P_o or a model $\hat{P}$, then a feedback connection of P and a feedback controller C is denoted with $\mathcal{T}(P,C)$ and defined as the connection structure depicted in Fig. 1. If P equals P_o and C equals the currently implemented con-

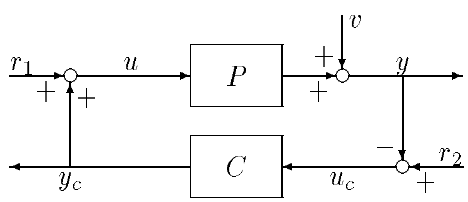

Fig. 1: Feedback connection structure $\mathcal{T}(P,C)$.

troller C_i in Fig. 1, then the signals u and y reflect respectively the inputs and outputs of the feedback controlled plant P_o. The signal v denotes an additive noise on the output y of the plant. For identification purposes, it is presumed that the noise v is uncorrelated with the external reference signals r_1, r_2 and that it can be modelled as the output of a monic stable and stably invertible noise filter H_0 having a white noise input e [20]. The signals u and y are being measured and r_1, r_2 (and consequently u_c, y_c) are possibly at our disposal.

It is assumed that the feedback connection structure $\mathcal{T}(P,C)$ is well posed, that is $\det(I+CP) \not\equiv 0$ [4]. The mapping from $col(r_2,r_1)$ onto $col(y,u)$ is given by the transfer function matrix $T(P,C)$ with

$$T(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} (I+CP)^{-1} \begin{bmatrix} C & I \end{bmatrix}, \tag{6}$$

and the signals in the closed loop system $\mathcal{T}(P_o, C_i)$ will satisfy

$$\begin{bmatrix} y \\ u \end{bmatrix} = T(P_o, C_i) \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} + \begin{bmatrix} I \\ -C_i \end{bmatrix} (I + P_o C_i)^{-1} v. \tag{7}$$

In case of an *internally* stable closed loop system $\mathcal{T}(P, C)$, all four transfer function matrices in $T(P, C)$ will be stable which implies $T(P, C) \in \mathbb{R}\mathcal{H}_\infty$ for a real rational P, where $\mathbb{R}\mathcal{H}_\infty$ denotes the set of all rational stable transfer functions.

Using the theory of fractional representations, the (possibly unstable) transfer function P will be expressed as a ratio of two stable transfer functions N and D. Following [35], P has a right coprime factorization (*rcf*) (N, D) over $\mathbb{R}\mathcal{H}_\infty$ if there exist X, Y, N and $D \in \mathbb{R}\mathcal{H}_\infty$ such that $P = ND^{-1}$ and $XN + YD = I$. In addition, a *rcf* (N, D) is normalized if it satisfies $N^*N + D^*D = I$, where * denotes the complex conjugate transpose. Dual definitions apply for left coprime factorizations (*lcf*).

3.2 Performance specification

The control objective function $J(P, C)$ used in this paper is taken to be an input/output weighted form of the transfer function matrix $T(P, C)$ defined in (6)

$$\|J(P, C)\|_\infty := \|U_2 T(P, C) U_1\|_\infty \tag{8}$$

where U_2 and U_1 are (square) weighting functions (not necessarily stable or stably invertible). Now consider problem 2.2 with the positive real number γ_i and a nominal model $\hat{P}_i$, then a controller C_{i+1} is said to satisfy the nominal performance criterion if

$$\|J(\hat{P}_i, C_{i+1})\|_\infty = \|U_2 T(\hat{P}_i, C_{i+1}) U_1\|_\infty \leq \gamma_i \tag{9}$$

In problem 2.2 the weighting functions U_1 and U_2 are assumed to be given and fixed in order to compare $J(P, C_i)$ and $J(P, C_{i+1})$ in the subsequent steps (b) and (c).

The reasoning to formulate the performance specification as in (8) is due to the fact that the feedback properties of any feedback system $\mathcal{T}(P, C)$ depend solely on the inner-loop parts C and P that create the interconnection as depicted in Fig. 1. Although it is impossible to transform any desirable control design objective into a single norm function $\|U_2 T(P, C) U_1\|_\infty$, the performance characterization (8) has wide applicability. It may include a weighted sensitivity or mixed sensitivity characterization by proper modification of the weighting functions U_2 and U_1 [27].

4 Representation of uncertainty

4.1 Structure of uncertainty set

In this paper, the uncertainty set $\mathcal{P}$, is built up from a nominal model $\hat{P}$ along with a characterization of an upper bound on the mismatch between the nominal model $\hat{P}$ and the actual plant P_o. The formulation of the set $\mathcal{P}$ used in this paper will be based on a close connection with the dual Youla parametrization. To explain this close connection, first consider this parametrization.

Lemma 4.1 *Let a controller C have a* rcf (N_c, D_c) *and consider an auxiliary model P_x with a* rcf (N_x, D_x) *such that $T(P_x, C) \in \mathbb{R}\mathcal{H}_\infty$. Then any system P satisfies $T(P, C) \in \mathbb{R}\mathcal{H}_\infty$ if and only if*

$$\exists R \in \mathbb{R}\mathcal{H}_\infty, \text{ such that } P = (N_x + D_c R)(D_x - N_c R)^{-1}$$

Proof: See [9]. □

The dual Youla parametrization of lemma 4.1 parametrizes all systems P that are internally stabilized by the controller C on the basis of an auxiliary model P_x which is known already to be stabilized by the controller C. This property is elaborated in the definition of the uncertainty set used in this paper and is given in the following definition.

Definition 4.2 *Let a nominal model $\hat{P}$ with a* rcf $(\hat{N}, \hat{D})$ *and a controller C with a* rcf (N_c, D_c) *form an internally stable feedback connection $\mathcal{T}(\hat{P}, C)$. Then the uncertainty set $\mathcal{P}$ is defined by*

$$\begin{aligned}\mathcal{P}(\hat{N}, \hat{D}, N_c, D_c, \hat{V}, \hat{W}) := \{P \mid P = (\hat{N} + D_c\Delta_R)(\hat{D} - N_c\Delta_R)^{-1} \\ \text{with } \Delta_R \in \mathbb{R}\mathcal{H}_\infty \text{ and } \|\hat{V}\Delta_R\hat{W}\|_\infty < \gamma^{-1}\}\end{aligned} \quad (10)$$

for stable and stably invertible weighting functions $\hat{V}$ and $\hat{W}$.

The set $\mathcal{P}$ essentially depends on the factorization $(\hat{N}, \hat{D})$ of the nominal model $\hat{P}$, the factorization (N_c, D_c) of the controller C and the weighting functions $\hat{W}$, $\hat{V}$. Without loss of generality, the bound on the uncertainty in (10) can also be normalized by the weighting functions $\hat{V}$ or $\hat{W}$. Hence, the set $\mathcal{P}$ does not essentially depend on the number γ, but bounding it by γ^{-1} will simplify notation considerably throughout the paper. Furthermore, the arguments of $\mathcal{P}$ will be omitted in the sequel, since the dependency mentioned above is clear from definition 4.2. For similar reasons of notational simplicity, it will be assumed that Δ_R in (10) is unstructured.

Clearly, the set $\mathcal{P}_i$ used in step (a) of problem 2.2 can be characterized by *employing* the knowledge of the stabilizing controller C_i that is implemented on the actual plant P_o. Using a *rcf* $(N_{c,i}, D_{c,i})$ of C_i and a nominal model $\hat{P}_i$ with a *rcf* $(\hat{N}_i, \hat{D}_i)$ that satisfies $T(\hat{P}_i, C_i) \in \mathbb{R}\mathcal{H}_\infty$, the set $\mathcal{P}_i$ is given by

$$\begin{aligned}\mathcal{P}_i = \{P \mid P = (\hat{N}_i + D_{c,i}\Delta_{R,i})(\hat{D}_i - N_{c,i}\Delta_{R,i})^{-1} \\ \text{with } \Delta_{R,i} \in \mathbb{R}\mathcal{H}_\infty \text{ and } \|\hat{V}_i\Delta_{R,i}\hat{W}_i\|_\infty < \gamma_i^{-1}\}\end{aligned} \quad (11)$$

for stable and stably invertible weighting functions $\hat{V}_i$ and $\hat{W}_i$. In exactly the same way, the set $\mathcal{P}_{i+1}$ of problem 2.2 can be obtained, by considering the known and stabilizing controller C_{i+1} in step (c) and a (new) nominal model $\hat{P}_{i+1}$ that satisfies $T(\hat{P}_{i+1}, C_{i+1}) \in \mathbb{R}\mathcal{H}_\infty$.

Due to the close connection with the dual Youla parametrization, the uncertainty set $\mathcal{P}_i$ in (11) contains only models that are stabilized by the currently implemented and known controller C_i, regardless of the value γ_i. This advantage, observed also by [27, pp. 139-141] or [29], is not shared by alternative uncertainty characterizations, as e.g. an open loop additive uncertainty description.

To anticipate on the results presented in the following sections, it can be noted here that the set $\mathcal{P}$ of (10) will yield an affine expression in Δ_R to evaluate $U_2 T(P, C) U_1$ used in the performance specification (8). This is a basic reason to choose the uncertainty structure (11); it can be exploited to formulate a control relevant identification problem to handle both step (a) and (c) of problem 2.2.

4.2 Representation via LFT's

Although the uncertainty set $\mathcal{P}$ is characterized fully by definition 4.2, the fairly general framework to represent any uncertainty by use of a Linear Fractional Transformation (LFT) [11] will be adopted in this paper. This LFT framework opens the possibility to rewrite the uncertainty set $\mathcal{P}$ into a standard form to which standard results can be applied for evaluating stability and performance.

To apply the LFT framework to the set $\mathcal{P}$ given in (10), the perturbation on the nominal model $\hat{P}$ is represented by an LFT with a norm bounded uncertainty $\Delta \in \mathbb{R}\mathcal{H}_\infty$ as depicted in Fig. 2. In Fig. 2, the signals u and y denote respectively the input and output of any

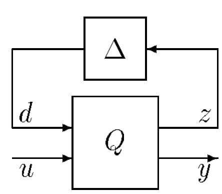

Fig. 2: LFT representation of model perturbation.

model $P \in \mathcal{P}$, while the uncertainty on the nominal model $\hat{P}$ is represented by the mapping Δ between the fictitious signals d and z. In this way, the mapping from u onto y for some $\Delta \in \mathbb{R}\mathcal{H}_\infty$ is given by the upper LFT

$$\mathcal{F}_u(Q, \Delta) := Q_{22} + Q_{21}\Delta(I - Q_{11}\Delta)^{-1}Q_{12} \tag{12}$$

provided that $(I - Q_{11}\Delta)^{-1}$ exists.

By defining $d = \hat{V}\Delta_R\hat{W}z$, it can be verified that the map from $col(d, u)$ onto $col(z, y)$ for any $P \in \mathcal{P}$ given in (10) can be represented by Fig. 3.

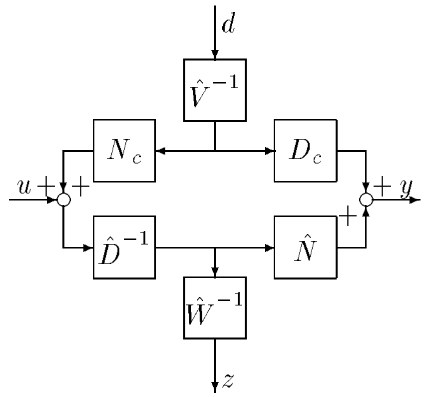

Fig. 3: Representation of Q for the set of models $\mathcal{P}$ defined in (10).

On the basis of Fig. 3 the following alternative representation of the set $\mathcal{P}$ in (10) in terms of an LFT can be obtained.

Corollary 4.3 *The set of models $\mathcal{P}$ given in* (10) *can be written as*

$$\mathcal{P} = \{P \mid P = \mathcal{F}_u(Q,\Delta) \textit{ with } \Delta \in \mathbb{R}\mathcal{H}_\infty,\ \|\Delta\|_\infty < \gamma^{-1} \textit{ and}$$

$$Q = \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] = \left[\begin{array}{c|c} \hat{W}^{-1}\hat{D}^{-1}N_c\hat{V}^{-1} & \hat{W}^{-1}\hat{D}^{-1} \\ \hline (D_c + \hat{P}N_c)\hat{V}^{-1} & \hat{P} \end{array}\right] \tag{13}$$

Proof: The entries of Q can be found by defining $\Delta = \hat{V}\Delta_R\hat{W}$ and considering the map from $col(d,u)$ onto $col(z,y)$ in Fig. 3. □

With the result of corollary 4.3, the set $\mathcal{P}_i$ of (11) as used in step (a) of problem 2.2, can be expressed in an LFT representation. This representation is obtained by using the upper bound γ_i and modifying Q in (13) using the *rcf* $(\hat{N}_i, \hat{D}_i)$, the *rcf* $(N_{c,i}, D_{c,i})$ and the weighting filters $\hat{V}_i$, $\hat{W}_i$. In a similar way, the LFT representation of the set $\mathcal{P}_{i+1}$ can be obtained.

5 Analysis of performance robustness

5.1 LFT representation

Referring to problem 2.2, the performance of a controller should be evaluated *a posteriori* in step (a) for C_i and in step (c) for C_{i+1}, while the performance of the newly designed controller C_{i+1} must be guaranteed *a priori* in step (b). Due to the model-based approach, the evaluation of the performance *a posteriori* and *a priori* can be handled relatively easily. This can be done by evaluating the worst case performance, or similarly, checking robust performance of a controller C^3 when applying C to all the models within an uncertainty set. In order to be able to check performance robustness, the performance of a controller C applied to any model $P \in \mathcal{P}_i$ of (11) is written in terms of an LFT. In this way, standard results present in literature [38] can be used to evaluate performance.

Lemma 5.1 *Consider the set $\mathcal{P}_i$ defined in* (11) *and a controller C such that the map $J(P,C) = U_2 T(P,C) U_1$ is well-posed for all $P \in \mathcal{P}_i$. Then*

$$\mathcal{P}_i = \{P \mid J(P,C) = \mathcal{F}_u(M,\Delta) \text{ with } \Delta \in \mathbb{R}\mathcal{H}_\infty,\ \|\Delta\|_\infty < \gamma_i^{-1}\}$$

where the entries of M are given by

$$\begin{aligned} M_{11} &= -\hat{W}_i^{-1}(\hat{D}_i + C\hat{N}_i)^{-1}(C - C_i)D_{c,i}\hat{V}_i^{-1} \\ M_{12} &= \hat{W}_i^{-1}(\hat{D}_i + C\hat{N}_i)^{-1}\begin{bmatrix} C & I \end{bmatrix} U_1 \\ M_{21} &= -U_2 \begin{bmatrix} -I \\ C \end{bmatrix} (I + \hat{P}_i C)^{-1}(I + \hat{P}_i C_i)D_{c,i}\hat{V}_i^{-1} \\ M_{22} &= U_2 \begin{bmatrix} \hat{N}_i \\ \hat{D}_i \end{bmatrix} (\hat{D}_i + C\hat{N}_i)^{-1}\begin{bmatrix} C & I \end{bmatrix} U_1 \end{aligned} \tag{14}$$

Proof: Consider a LFT representation of $\mathcal{P}_i$ similar to (13). Create the feedback connection of Q depicted in Fig. 2 with a controller C, where $u := r_1 + C(r_2 - y)$ and define signals $col(w_1, w_2)$ and $col(d_1, d_2)$ such that $col(r_2, r_1) = U_1 col(w_1, w_2)$ and $col(e_1, e_2) = U_2 col(y, u)$. It can be verified that the map from $col(d, w_1, w_2)$ onto $col(z, e_1, e_2)$ is given by the transfer function M in (14), whereas the map from $col(w_1, w_2)$ onto $col(e_1, e_2)$ equals the upper LFT $\mathcal{F}_u(M,\Delta)$. □

With lemma 5.1, the (worst case) performance $\|J(P,C)\|_\infty$ of a controller C applied to all models $P \in \mathcal{P}_i$ can be evaluated by

$$\|\mathcal{F}_u(M,\Delta)\|_\infty = \|M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\|_\infty \tag{15}$$

for all $\Delta \in \mathbb{R}\mathcal{H}_\infty$ with $\|\Delta\|_\infty \le \gamma_i^{-1}$. Note that the entries of the transfer function M in (14) are determined solely by the controller C, the structure and the variables used to represent the set $\mathcal{P}_i$ of (11) and the weightings U_2, U_1 of the performance specification (8). As a special entry of M, one can recognise M_{11} as the lower LFT $\mathcal{F}_l(Q,-C)$, whereas M_{22} equals the (nominal) performance specification $U_2 T(\hat{P}_i, C) U_1$.

Clearly, the controller C in (14) still needs to be specified. Referring to problem 2.2, substituting $C = C_i$ can be used for the performance assessment in step (a) *a posteriori*, while setting $C = C_{i+1}$ can be employed to check and guarantee performance of C_{i+1} *a priori* in step (b). Similar results can also be obtained for the set $\mathcal{P}_{i+1}$ used in step (c) of problem 2.2.

[3] the symbol C will be used to denote either C_i or C_{i+1}

As mentioned before, evaluating (15) can be done by applying standard results available in the literature [24, 38]. However, in order to be able to compute the (worst case) performance for the LFT given in (15) in a non-conservative way, the concept of μ or structured singular value [24] is needed.

5.2 Structured singular value

The structured singular value is a matrix function, denoted by $\mu(M)$, where M can be any (square) complex matrix. It plays a crucial role in the evaluation of performance robustness [10], which is the main reason to use it in this paper.

The definition of $\mu(\cdot)$ depends on an underlying (diagonal) structure [10, 38]. This structure, which will be denoted by $\mathbf{\Delta}$, is determined by the structure of the uncertainty set and the performance objective function being used. The structured singular value $\mu(\cdot)$ with respect to such a structure $\mathbf{\Delta}$ will be denoted by $\mu_{\mathbf{\Delta}}(\cdot)$. Using the symbol $\bar{\sigma}(\Delta)$ to denote the maximum singular value of Δ, the definition of $\mu_{\mathbf{\Delta}}(\cdot)$ adopted from [11] reads as follows.

Definition 5.2 *For a complex matrix M, the structured singular value $\mu_{\mathbf{\Delta}}(M)$ is defined by*

$$\mu_{\mathbf{\Delta}}(M) := \begin{cases} \frac{1}{\min\limits_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta)\}} & \textit{if } \exists \Delta \in \mathbf{\Delta} \ \textit{s.t. } \textit{det}(I - M\Delta) = 0 \\ 0 & \textit{if } \nexists \Delta \in \mathbf{\Delta} \ \textit{s.t. } \textit{det}(I - M\Delta) = 0 \end{cases}$$

For reasons of simplicity that become apparent in section 5.3, in this paper the structure $\mathbf{\Delta}$ used in definition 5.2 is restricted to have a diagonal form, having two unstructured uncertainty blocks Δ_1 and Δ_2 only. Now let M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{16}$$

then the blocks Δ_1 and Δ_2 are compatible in size with M_{11} and M_{22}, meaning that both $M_{11}\Delta_1$ and $M_{22}\Delta_2$ are square. In this way the structure of $\mathbf{\Delta}$ is given by

$$\mathbf{\Delta} := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \mid \Delta_1, \Delta_2 \in \mathbb{R}\mathcal{H}_\infty, \ \|\Delta_1\|_\infty < 1, \|\Delta_2\|_\infty < 1 \right\}. \tag{17}$$

In general $\mu_{\mathbf{\Delta}}(M)$ is approximated by computing upper and lower bounds. The upper bound is derived by the computation of non-negative scaling matrices D_l and D_r defined within a set $\mathcal{D}$ that commutes with the structure $\mathbf{\Delta}$. One is referred to e.g. [24] for a detailed discussion on the specification of such a set $\mathcal{D}$ of scaling matrices. Basically, the commutation of $\mathcal{D}$ with $\mathbf{\Delta}$ implies that for all $D_l, D_r \in \mathcal{D}$ and for all $\Delta \in \mathbf{\Delta}$, $D_r\Delta = \Delta D_l$ and $\mu_{\mathbf{\Delta}}(M) = \mu_{\mathbf{\Delta}}(D_l M D_r^{-1})$. This gives rise to the computation of the following upper bound.

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D_l, D_r \in \mathcal{D}} \bar{\sigma}(D_l M D_r^{-1}) \tag{18}$$

The infimization formulated in (18) can be reformulated as a convex optimization problem [24]. However, for the special cases of M and $\mathbf{\Delta}$ used in this paper, it is possible to compute $\mu_{\mathbf{\Delta}}(M)$ exactly.

Lemma 5.3 *Consider the structure* $\mathbf{\Delta}$ *given in* (17) *and* $\mu_{\mathbf{\Delta}}(M)$ *given in definition 5.2, then*

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D_l, D_r \in \mathcal{D}} \bar{\sigma}(D_l M D_r^{-1})$$

Proof: The structure $\mathbf{\Delta}$ consists of two full blocks. Application of theorem 9.1 in [24] or theorem 11.5 in [38] yields the result. □

5.3 Evaluating performance

The properties of $\mu_{\mathbf{\Delta}}(M)$ as given in definition 5.2 and the result mentioned in lemma 5.3 can now be used to study the upper LFT $\mathcal{F}_u(M, \Delta)$ of (15). In this way, the worst case performance for all stable norm bounded perturbations can be evaluated by using standard results that are available in the literature [24, 38] as formulated in the following lemma.

Lemma 5.4 *Consider stable transfer functions* $M, \Delta \in \mathbb{R}\mathcal{H}_\infty$ *where* M *is partitioned as in* (16) *and* $\mu_{\mathbf{\Delta}}(M)$ *is defined related to the structure* $\mathbf{\Delta}$ *given in* (17). *Then* $\mathcal{F}_u(M, \Delta)$ *is well-posed, BIBO stable and* $\|\mathcal{F}_u(M, \Delta)\|_\infty \leq \gamma$ *for all* Δ *with* $\|\Delta\|_\infty < \gamma^{-1}$*, if and only if*

$$\mu_{\mathbf{\Delta}}(M) \leq \gamma \tag{19}$$

Proof: By setting $\Delta = \Delta_1$ and adding a fictitious full block uncertainty $\Delta_2 \in \mathbb{R}\mathcal{H}_\infty$ with $\|\Delta_2\| < \gamma^{-1}$, the uncertainty structure (17) is obtained. Application of the main loop theorem, similar as in theorem 11.7 in [38] now proves the result. □

The result of lemma 5.4 opens the possibility to evaluate the (worst case) performance of a controller C applied to a set of models $\mathcal{P}$ in a non-conservative way. This set of models $\mathcal{P}$ can be either $\mathcal{P}_i$ as used in step (a) and (b) of problem 2.2, or a newly identified set of models $\mathcal{P}_{i+1}$ as used in step (c). The result for evaluating the performance of a controller C applied to the $\mathcal{P}_i$ of (11) is stated in the following theorem. Similar results can be derived for $\mathcal{P}_{i+1}$.

Theorem 5.5 *Consider the set* $\mathcal{P}_i$ *defined in* (11) *and a controller* C *such that* $\mathcal{T}(\hat{P}_i, C)$ *is well-posed, internally stable and satisfies* $U_2 T(\hat{P}_i, C) U_1 \in \mathbb{R}\mathcal{H}_\infty$. *Then, for all* $P \in \mathcal{P}_i$*, the* *feedback system $\mathcal{T}(P,C)$ is well-posed, internally stable and satisfies $\|U_2 T(P,C) U_1\|_\infty \le \gamma_i$ if and only if*

$$\mu_{\mathbf{\Delta}}\left(\begin{bmatrix} \hat{W}_i^{-1} & 0 \\ 0 & U_2 \end{bmatrix} T_{ext}(\hat{P}_i, C_i, C) \begin{bmatrix} -\hat{V}_i^{-1} & 0 \\ 0 & U_1 \end{bmatrix}\right) \le \gamma_i \tag{20}$$

where $T_{ext}(\hat{P}_i, C_i, C)$ is given by

$$\left[\begin{array}{c|c} Z_i(C - C_i)D_{c,i} & Z_i \begin{bmatrix} C & I \end{bmatrix} \\ \hline \left(\begin{bmatrix} \hat{N}_i \\ \hat{D}i \end{bmatrix} Z_i C + \begin{bmatrix} I \\ 0 \end{bmatrix}\right)(D_{c,i} + \hat{P}_i N_{c,i}) & \begin{bmatrix} \hat{N}_i \\ \hat{D}_i \end{bmatrix} Z_i \begin{bmatrix} C & I \end{bmatrix} \end{array}\right] \tag{21}$$

where $Z_i = (\hat{D}_i + C\hat{N}_i)^{-1} = \hat{D}_i^{-1}(I + C\hat{P}_i)^{-1}$.

Proof: Lemma 5.1 connects $\mathcal{F}_u(M, \Delta)$ with $U_2 T(P,C) U_1$ for all $P \in \mathcal{P}_i$. The expression for $T_{ext}(\hat{P}_i, C_i, C)$ can be found by use of (14) and algebraic manipulation. Applying lemma 5.4 yields the necessary and sufficient condition for $\|\mathcal{F}_u(M,\Delta)\|_\infty \le \gamma_i$ to hold for all $P \in \mathcal{P}_i$. □

Referring to problem 2.2, substituting $C = C_i$ in (20) can be used for the performance assessment in step (a) *a posteriori*. On the other hand, substitution of $C = C_{i+1}$ in (20) can be used to check and guarantee performance robustness of C_{i+1} in step (b) *a priori*. Recall from lemma 5.3 that for structure $\mathbf{\Delta}$[4] the value of (20) can be computed exactly. Similar results can be derived also for the set of models $\mathcal{P}_{i+1}$ as used in step (c) of problem 2.2.

Finally, it can be observed from (21) or (14) that substitution of $C = C_i$ yields $M_{11} = 0$. This implies that the controller C_i applied to the (identified) set of models $\mathcal{P}_i$ satisfies stability robustness [38], regardless of the value of γ_i^{-1}. This was one of the motivations already mentioned in section 4.1 to use the uncertainty set (11). Moreover, the upper LFT $\mathcal{F}_u(M,\Delta)$ modifies into

$$M_{22} + M_{21}\Delta M_{12} \tag{22}$$

which is an affine expression in Δ. The stucture of (22) will be exploited in section 7.2 to formulate a (control relevant) identification problem, by employing the knowledge of a stabilizing controller that is implemented on the (unknown) plant P_o. First, the robust control design in step (b) of problem 2.2 will be discussed.

6 Robust control design problem

In order to satisfy (4) in step (b) of problem 2.2, a controller C_{i+1} can be designed by the minimization

$$C_{i+1} = \arg\min_{C} \sup_{P \in \mathcal{P}_i} \|J(P,C)\|_\infty \tag{23}$$

[4] in the case of unstructured Δ_R

Basically, (23) is a robust performance control design, wherein the controller C_{i+1} is being designed, such that the worst case performance $J(P, C_{i+1})$ for all $P \in \mathcal{P}_i$ is being optimized.

Substitution of $C = C_{i+1}$ in theorem 5.5 yields a necessary and sufficient condition for the expression (4) to hold. Hence, the minimization (23) to synthesise a controller C_{i+1} can be replaced by the minimization

$$C_{i+1} = \arg\min_{C} \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} \hat{W}_i^{-1} & 0 \\ 0 & U_2 \end{bmatrix} T_{ext}(\hat{P}_i, C_i, C) \begin{bmatrix} -\hat{V}_i^{-1} & 0 \\ 0 & U_1 \end{bmatrix} \right) \tag{24}$$

Basically, (24) is a μ-synthesis problem that can be tackled by using the upper bound (18) and solving

$$\min_{C} \inf_{D_l, D_r \in \mathcal{D}} \left\| D_l \begin{bmatrix} \hat{W}_i^{-1} & 0 \\ 0 & U_2 \end{bmatrix} T_{ext}(\hat{P}_i, C_i, C) \begin{bmatrix} -\hat{V}_i^{-1} & 0 \\ 0 & U_1 \end{bmatrix} D_r^{-1} \right\|_{\infty} \tag{25}$$

iteratively for the scaling matrices D_l, D_r and the controller C, subjected to internal stability of the feedback connection of C and $\hat{P}_i$. This iteration is known as the D-K iteration[5] and for fixed scaling D_l, D_r with $D_l, D_r^{-1} \in \mathbb{R}\mathcal{H}_\infty$ (25) is an $\mathcal{H}_\infty$ optimization problem, for which standard solutions exists, see e.g. [38]. Although convergence of the D-K iteration is not guaranteed, several successful applications have been reported in the literature. Furthermore, it should be stressed that precise minimization of (24) is not needed. If suffices to find a controller C_{i+1} that satisfies (20), or equivalently (4).

In order to use the available standard results on $\mathcal{H}_\infty$ controller synthesis, the transfer function M of the LFT $\mathcal{F}_u(M, \Delta)$ should be represented as a lower fractional transformation $\mathcal{F}_l(G, C)$ as illustrated in Fig. 4.

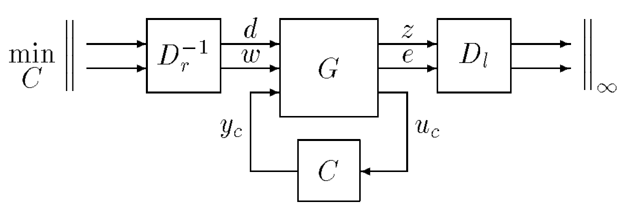

Fig. 4: Controller synthesis via $\mathcal{H}_\infty$ optimization for fixed D-scaling.

This can be done by extracting the controller C from the expression of M given in (14) and is given in the following corollary.

[5] the naming D-K iteration is widely used in the literature and is adopted here, although D-C would be more appropriate

Corollary 6.1 *Consider the map M given in* (14)*, then $M = \mathcal{F}_l(G, C)$ where G is given by*

$$G = \begin{bmatrix} \hat{W}_i^{-1} & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{c|cc|c} \hat{D}_i^{-1} N_{c,i} & \hat{D}_i^{-1} & 0 & \hat{D}_i^{-1} \\ \hline (D_{c,i} + \hat{P}_i N_{c,i}) & \hat{P}_i & 0 & \hat{P}_i \\ 0 & I & 0 & I \\ \hline -(D_{c,i} + \hat{P}_i N_{c,i}) & -\hat{P}_i & I & -\hat{P}_i \end{array}\right] \begin{bmatrix} -\hat{V}_i^{-1} & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Proof: The entries of G can be found by the map from $col(d, w, y_c)$ onto $col(z, e, u_c)$ in Fig. 4. □

Finally it can be noted that the control design discussed here is a generalization of the robust controller synthesis as presented in e.g. [3] or [22]. It can be verified from corollary 6.1 that by ignoring the map from d onto z (representing the uncertainty), G reduces to

$$\begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc|c} \hat{P}_i & 0 & \hat{P}_i \\ I & 0 & I \\ \hline -\hat{P}_i & I & -\hat{P}_i \end{array}\right] \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix}$$

and $M = \mathcal{F}_l(G, C) = U_2 T(\hat{P}_i, C) U_1$. In the special case of a *diagonal* weighting function $U = diag(U_{in}, U_{out}^{-1})$ with $U_2 = U$ and $U_1 = U^{-1}$, the controller C_{i+1} that minimizes $\|UT(\hat{P}_i, C_{i+1})U^{-1}\|_\infty$ can be found by loop shaping techniques [3, pp. 107-108]. Explicit state space formulae of the optimal controller for this special case can be found in [3] or [22].

7 Identification problem

7.1 Introduction

This section deals with the problem of estimating a set of models that appears both in step (a) and (c) of problem 2.2. Estimating the set $\mathcal{P}_i$ such that γ_i in (3) is as small as possible in step (a), could be achieved by solving

$$\min_{\mathcal{P}_i} \sup_{P \in \mathcal{P}_i} \|J(P, C_i)\|_\infty \tag{26}$$

subjected to the condition $P_o \in \mathcal{P}_i$. Similarly, a set $\mathcal{P}_{i+1}$ such that (3) will be satisfied in step (c) can be found by

$$\min_{\mathcal{P}_{i+1}} \sup_{P \in \mathcal{P}_{i+1}} \|J(P, C_{i+1})\|_\infty \tag{27}$$

again subjected to the condition $P_o \in \mathcal{P}_{i+1}$. Clearly, the identification problems of step (a) and (c) are similar and differ only in the controller being implemented on the plant P_o.

According to definition 4.2, the structure of the set of models is determined by a factorization $(\hat{N}, \hat{D})$ of a nominal model $\hat{P}$ and the weighting functions $(\hat{V}, \hat{W})$. Omitting the indices i and $i+1$ for notational convenience, a set $\mathcal{P}$ that minimizes either (26) or (27) can be solved by

$$(\hat{N},\hat{D},\hat{V},\hat{W}) = \arg \min_{N,D,V,W} \sup_{P\in\mathcal{P}} \|J(P,C)\|_\infty \tag{28}$$

subjected to both $P_o \in \mathcal{P}$ and internal stability of the feedback connection $\mathcal{T}(\hat{P},C)$. At the current state, the minimization of (28) using the variables $(\hat{N},\hat{D},\hat{V},\hat{W})$ simultaneously, cannot be solved directly. Therefore, the minimization of (28) is tackled by estimating the *rcf* $(\hat{N},\hat{D})$ and the pair $(\hat{V},\hat{W})$ separately:

- *Estimation of a nominal model*
  This involves the estimation of $\hat{P} = \hat{N}\hat{D}^{-1}$ such that (28) is being minimized using the *rcf* (N,D) only, subjected to internal stability of $\mathcal{T}(\hat{P},C)$. The pair $(\hat{V},\hat{W})$ is unknown and assumed to vary freely in order to satisfy $P_o \in \mathcal{P}$.

- *Estimation of uncertainty*
  This consists of the characterization of an upper bound on Δ_R in (11) via $(\hat{V},\hat{W})$ such that (28) is being minimized using (V,W) only, subjected to $P_o \in \mathcal{P}$. The *rcf* $(\hat{N},\hat{D})$ is fixed to the estimate obtained above.

By the separate identification of the *rcf* $(\hat{N},\hat{D})$ and the weighting functions $(\hat{V},\hat{W})$ only an upper bound on (28) can be minimized. However, it should be stressed that precise minimization of (28) is not needed. If suffices to find a set of models that passes the validation test of (3) or (5). Furthermore, (standard) tools to estimate a nominal model and to characterize uncertainty can be applied as indicated in the following two sections. Finally it can be noted that due to the separation being made, the attention can be focused on finding models of limited complexity [32]. The rationale is to avoid the computation of controllers on the basis of highly complex models as much as possible, since this will lead to high order controllers for which the computation may be badly conditioned.

7.2 Estimation of a nominal model

Following the separate identification of nominal model and uncertainty, this section deals with the estimation of a (possibly unstable) nominal model. The minimization of (28) on the basis of the *rcf* (N,D) *only*, will be used to estimate a nominal model $\hat{P}$ of limited complexity. Hence, to obtain a nominal model $\hat{P}_i$ or nominal *rcf* $(\hat{N}_i,\hat{D}_i)$ such that γ_i in (3) is as small as possible, can be achieved by

$$\min_{N,D} \sup_{P\in\mathcal{P}_i} \|J(P,C_i)\|_\infty \tag{29}$$

where $P_o \in \mathcal{P}_i$ and (N,D) is a *rcf* of limited complexity. Similar arguments can be given when estimating a nominal model $\hat{P}_{i+1}$ in step (c) of problem 2.2.

Clearly, the set $\mathcal{P}_i$ is (still) unknown and the minimization of (29) cannot be solved directly. Instead, an identification problem to estimate a *rcf* of a nominal model can be formulated by evaluating $\|J(P, C_i)\|_\infty$ only, using the following triangular inequality [27].

$$\|J(P, C_i)\|_\infty \leq \|J(P_o, C_i)\|_\infty + \|J(P, C_i) - J(P_o, C_i)\|_\infty \tag{30}$$

As $\|J(P_o, C_i)\|_\infty$ in (30) does not depend on the nominal model, the *rcf* $(\hat{N}_i, \hat{D}_i)$ of a nominal model $\hat{P}_i$ found by the minimization

$$(\hat{N}_i, \hat{D}_i) = \arg\min_{N,D} \|J(P, C_i - J(P_o, C_i)\|_\infty \tag{31}$$

can be used to formulate a control relevant identification of a nominal model.

Minimization of (31) on the basis of closed loop experiments obtained from the feedback connection of the plant P_o and the controller C_i, has been studied extensively in [7] and [34]. Following these references, access to a *rcf* of the plant P_o denoted by $(N_{o,F}, D_{o,F})$ is used to minimize (31). Such a *rcf* $(N_{o,F}, D_{o,F})$ can be obtained by considering the map from an auxiliary signal x onto $col(y, u)$. The signal x is found by an appropriate filtering of the closed loop signals

$$x = F \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} = F \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{32}$$

depicted in Fig. 1. In this way, the *rcf* of the plant P_o that can be accessed is given by

$$\begin{gathered} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N_{o,F} \\ D_{o,F} \end{bmatrix} x + \begin{bmatrix} S_{out} \\ -C_i S_{out} \end{bmatrix} v \\ \text{with } N_{o,F} := P_o S_{in} F^{-1},\ D_{o,F} := S_{in} F^{-1}. \end{gathered} \tag{33}$$

For exact details on the construction of the filter F mentioned in (32), one is referred to e.g. [7] or [34]. For a model $P_i(\theta)$ parametrized by a *rcf* $(N_i(\theta), D_i(\theta))$ with $\theta \in \Theta$, where Θ is given by

$$\Theta := \{\theta \in \mathbb{R}^n \mid (N(\theta), D(\theta)) \in \mathbb{R}\mathcal{H}_\infty\} \subset \mathbb{R}^n \tag{34}$$

the following result can be used to minimize (31).

Lemma 7.1 *Let P_o and C create an internally stable feedback system $\mathcal{T}(P_o, C)$ and let $(N_{o,F}, D_{o,F})$ be the* rcf *of P_o as given in* (33)*, where F is an appropriate filter used in* (32)*. Consider any model $P_i(\theta) = N_i(\theta) D_i(\theta)^{-1}$ then*

(i) *for all $\theta \in \Theta$ of* (34) *there exists a* rcf *$(N_i(\theta), D_i(\theta))$ of $P_i(\theta)$ such that*

$$D_i(\theta) + C_i N_i(\theta) = F^{-1}.$$

(ii) *the minimization of* (31) *equals*

$$\min_{\theta\in\Theta}\left\| U_2\left(\begin{bmatrix} N_{o,F} \\ D_{o,F}\end{bmatrix} - \begin{bmatrix} N_i(\theta) \\ D_i(\theta)\end{bmatrix}\right) F \begin{bmatrix} C_i & I \end{bmatrix} U_1 \right\|_\infty \tag{35}$$

where $(N_i(\theta), D_i(\theta))$ *is any* rcf *of* $P_i(\theta)$ *that satisfies* (i).

Proof: See [7] □

It should be noted that the minimization of (35) for a fixed filter F is not straightforward, due to the ∞-norm criterion. An alternative would be the approximation of (35) by the minimization of an 2-norm specification. This is motivated by the fact that an $\mathcal{L}_2$-norm approximation tends to $\mathcal{L}_\infty$-norm approximation provided that some smoothness conditions[6] are satisfied [5]. Similar arguments are used also in [27, pp. 158]. To minimize (35) using a $\mathcal{H}_2$-norm can be accomplished by use of the signals x and $col(y,u)$ and application of a least squares prediction error algorithm [20] employing an output error model structure, see e.g. [34]. An approach to minimize the $\mathcal{H}_\infty$-norm in (35) for a fixed filter F can be found in [6]. Similar to the procedure described in [15], frequency domain measurements are used in order to evaluate and approximate the $\mathcal{H}_\infty$ criterion.

7.3 Estimation of model uncertainty

From the control relevant identification discussed in the previous section, only a nominal model $\hat{P}_i$ is obtained. If the controller C_i internally stabilizes $\hat{P}_i$, the set $\mathcal{P}_i$ of (11) can be completed by solving

$$\min_{V,W}\sup_{P\in\mathcal{P}_i}\|J(P,C_i)\|_\infty \tag{36}$$

where $P_o \in \mathcal{P}_i$ and (V,W) are stable and stably invertible and of limited complexity. As the weighting functions in $\mathcal{P}_i$ are used only to bound $\Delta_{R,i}$ in (11), the minimization in (36) can be handled by a system identification procedure that is able to estimate the smallest (frequency dependent) upper bound on $\Delta_{R,i}$ such that $P_o \in \mathcal{P}_i$. The system identification used for this purpose will be discussed by first considering the following proposition.

Proposition 7.2 *Consider a controller* C_i *with* rcf $(N_{c,i}, D_{c,i})$, *a nominal model* $\hat{P}_i$ *with* rcf $(\hat{N}_i, \hat{D}_i)$ *and a plant* P_o. *Let the feedback connections* $\mathcal{T}(P_o, C_i)$ *and* $\mathcal{T}(\hat{P}_i, C_i)$ *be internally stable and define*

$$\begin{aligned} x &:= (\hat{D}_i + C_i\hat{N}_i)^{-1}\begin{bmatrix} C_i & I\end{bmatrix}\begin{bmatrix} y \\ u\end{bmatrix} \\ z &:= (D_{c,i} + \hat{P}_i N_{c,i})^{-1}\begin{bmatrix} I & -\hat{P}_i\end{bmatrix}\begin{bmatrix} y \\ u\end{bmatrix}\end{aligned} \tag{37}$$

[6]to verify the smoothness conditions, knowledge on the actual plant P_o is required

where $y = P_o u + v$ *and* $u = r_1 + C_i(r_2 - y)$. *Then*

$$z = \Delta_{R,i} x + D_{c,i}(I + P_o C_i)^{-1} v \tag{38}$$

where $\Delta_{R,i} \in \mathbb{R}\mathcal{H}_\infty$ *and* x *is uncorrelated with* v.

Proof: Equation (38) and the property $x \perp v$ can be verified by algebraic manipulation, see [32]. The property $\Delta_R \in \mathbb{R}\mathcal{H}_\infty$ follows from lemma 4.1. □

Proposition 7.2 gives rise to an equivalent open loop identification problem of the stable dual Youla parameter Δ_R, as also been indicated in [16] or [19]. However, compared to the approach followed in these references, the dual Youla parameter is being used here *only* to construct the set $\mathcal{P}_i$ of (11) such that $P_o \in \mathcal{P}_i$.

The system identification procedures described by [36] or [13] can be used to obtain a frequency dependent upper bound $\beta(\omega)$ on the additive error between the frequency response of the stable rational transfer function $\Delta_{R,i}(\omega)$ and a stable parametric estimate $\hat{R}_i(\omega)$. As a result, a frequency dependent upper bound $\delta(\omega)$

$$\|\Delta_{R,i}(\omega)\| \leq \delta(\omega) \text{ with probability} \geq \alpha \tag{39}$$

can be obtained, where α is a prechosen probability and $\delta(\omega) = \|\hat{R}_i(\omega)\| + \beta(\omega)$. In the multivariable case, the upper bound (39) can be obtained for each transfer function, but to avoid comprehensive notations, it is assumed that Δ_R is unstructured in (39).

On the basis of the frequency dependent information $\delta(\omega)$ obtained in (39), the linear programming algorithm presented in [26] can be used to find stable and stably invertible weighting functions $(\hat{V}_i, \hat{W}_i)$ of limited complexity. However, in order to find $(\hat{V}_i, \hat{W}_i)$ such that (36) is being minimized, additional weightings in the LPSOF algorithm of [26] need to be specified. According to lemma 5.1, evaluating $\|J(P,C)\|_\infty$ for all $P \in \mathcal{P}_i$ is equivalent to evaluating $\|\mathcal{F}_u(M,\Delta)\|_\infty$ as mentioned in (15). However, with $C = C_i$ the upper LFT $\mathcal{F}_u(M,\Delta)$ modifies into the affine expression of (22). Applying the triangular inequality

$$\|M_{22} + M_{21}\Delta M_{12}\|_\infty \leq \|M_{22}\|_\infty + \|M_{21}\Delta M_{12}\|_\infty$$

yields the weightings M_{21} and M_{12} to be used in the LPSOF algorithm. The entries M_{21} and M_{12} can be found by using (14) and $\Delta = \hat{V}_i \Delta_{R,i} \hat{W}_i$ from (11).

8 Validation tests

The performance assessment test of (3) and the closed loop validation tests given in (4) and (5) needed to handle problem 2.2, can be found by combining the result presented in the previous sections. The estimate of a *rcf* $(\hat{N}_i, \hat{D}_i)$ discussed in section 7.2 and the weighting functions $(\hat{V}_i, \hat{W}_i)$ discussed in section 7.3 are used to construct the set $\mathcal{P}_i$. The performance assessment test of (5) can now be performed by using the result mentioned in theorem 5.5 for $C = C_i$. For $C = C_{i+1}$, theorem 5.5 can also be used to perform the validation of a newly designed controller C_{i+1} in (4).

If the controller C_{i+1} passes the validation test (4), it can be implemented on the plant P_o. A new set $\mathcal{P}_{i+1}$ can be constructed by estimating a *rcf* $(\hat{N}_{i+1}, \hat{D}_{i+1})$ and weighting functions $(\hat{V}_{i+1}, \hat{W}_{i+1})$ on the basis of closed loop experiments using the newly designed controller C_{i+1}. Similarly, the validation of the set $\mathcal{P}_{i+1}$ in (3) can be verified with the result mentioned in theorem 5.5.

It should be noted, that an estimate of the weighting functions $\hat{V}$ and $\hat{W}$ is required to synthesize a robust controller, in order to form a state space realization of G given in corollary 6.1. The validation tests mentioned in problem 2.2 can be evaluated also by plotting $\mu_{\mathbf{\Delta}}(M(\omega))$ along a frequency grid ω and inspecting the peak value. In this way, the estimated upper bound $\delta(\omega)$ of (39) can be used directly, by setting $\hat{V}_i^{-1}(\omega) = \delta(w)\gamma_i$ or $\hat{W}_i^{-1}(\omega) = \delta(w)\gamma_i$.

9 Conclusions

In this paper a model-based iterative procedure of identification and robust control design is introduced wherein the robust performance is monitored during the subsequent steps of the iterative scheme. By monitoring the performance, the possibility to guarantee performance improvement in the iterative scheme is being employed.

To monitor performance, a set of models is used. The set is built up from a nominal model along with an upper bound on an allowable perturbation such that the set is guaranteed to contain the unknown plant. The nominal model is described by a stable coprime factorization having a prespecified McMillan degree, while the set of models is constructed by considering a stable perturbation in a dual Youla parametrization. This specific structure of the set is induced by the performance cost being used and can be exploited to formulate a (control relevant) identification problem to estimate the set. Currently, system identification techniques are available to estimate such a set and these techniques are utilized in this paper.

To obtain performance robustness, robust control design tools are used to synthesise controllers on the basis of a set of models being estimated. Finally, to guarantee performance improvement robustly, closed loop model and controller validation tests have been formulated in this paper, which should be verified in each step of the iterative scheme.

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