GEOPHYSICAL RESEARCH LETTERS, VOL. 22, NO. 23, PAGES 4367-4369, DECEMBER 1, 2001

Theoretical Foundation for Conductivity Scaling

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Abstract. Scaling of conductivity with the support volume of experiments has been the subject of many recent experimental and theoretical studies. However, to date there have been few attempts to relate such scaling, or the lack thereof, to microscopic properties of porous media through theory. We demonstrate that when a pore network can be represented as a collection of hierarchical trees, scalability of the pore geometry leads to scalability of conductivity. We also derive geometrical and topological conditions under which the scaling exponent takes on specific values 1/2 and 3/4. The former is consistent with universal scaling observed by *Neuman* [1994], while the latter agrees with the allometric scaling laws derived by *West et al.* [1997].

1. Introduction

Measurements of hydraulic conductivity, K, are said to scale with $\mathcal{V}$, the volume sampled in taking measurements, when

$$K \sim \mathcal{V}^{\alpha}. \tag{1}$$

The exponent, α, is called the *scaling exponent*. *Neuman* [1990,1994] has presented evidence that conductivity statistics, specifically variograms of log conductivity, scale with $\mathcal{V}$ over a very wide range of volumes, and *DiFederico and Neuman* [1997] and *DiFederico et al.* [1999] have developed a statistical theory that explains such scaling in terms of truncated power variograms. 1/2-power scaling plays a special role in *Neuman* [1990,1994], *DiFederico and Neuman* [1997] and *DiFederico et al.* [1999] although it is motivated only in terms of statistics. The evidence for scaling in general and for specific values of the scaling exponents in particular is, furthermore, equivocal. For instance, theoretical studies of *Neuman* [1990,1994] and *Gavrilenko and Guéguen* [1998] suggest alternative values for the scaling exponent, while *Clauser* [1992] does not observe scaling in permeability of crystalline rocks at all. This raises several questions. Can scaling be derived from more basic physical principles than the statistics of conductivity? What is the physical significance, if any, of the 1/2-power scaling law in groundwater hydrology? Can the failure of data like that of *Clauser* [1992] to obey scaling be explained?

We address the first two of these questions by relating the macroscopic parameter K to underlying (microscopic) pore geometry. While scaling of K is subject to debate, power law distributions for pore-scale properties are less controversial [*Baveye et al.*, 1998]. We show microscopic scaling in *hierarchical trees* leads to macroscopic scaling of K with $\mathcal{V}$ in the sense of (1). Hierarchical trees are a type of tree-structured pore network whose pore-scale topology and geometry are defined by power law distributions. Tree-structured networks seem like a reasonable model for certain types of porous media, especially fractured systems organized around a main fracture and a few large subsidiaries. We also indicate microscopic conditions under which the scaling exponent $\alpha = 1/2$, and we contrast those with the allometric $\alpha = 3/4$ scaling of metabolism with biomass observed in physiology[*West et al.*, 1997]. The pores of a hierarchical tree yield a 1/2-power scaling law when they preserve total area and total length from one level of a tree to the next, while the allometric scaling law arises when pores are space-filling and area-preserving [*West et al.*, 1997]. We define *space-filling*, *area-preservation*, and *length-preservation* below.

So far, the relevance of hierarchical trees to groundwater hydrology has not been noted, primarily due to overly restrictive assumptions about their structure. Our goals are i) to motivate hierarchical trees as a model for flow through porous media and to extend the model to structures that are relevant to groundwater flow, ii) to point out that conductivity and other parameters of hierarchical trees scale in the sense of equation (1), and iii) to discuss circumstances under which 1/2-power scaling arises in hierarchical trees. We define hierarchical trees in the second section of this paper and show that among all regular tree-structured networks, they minimize resistance to the movement of fluid. The third section is the main part of the paper. We follow *West et al.* [1997] in deriving (1) and a general expression for α. Then we point out that pore networks which preserve length and area exhibit 1/2-power scaling. In the fourth section, we discuss hierarchical trees as models of natural flow systems.

2. Pore Network

A tree-structured pore network has a single pore, the *root*, at its bottom where fluid enters the network. Each pore in a tree-structured network receives water from exactly one parent pore and distributes it to some number of descendants. In a regular tree, every pore has the same number $n > 1$ of descendants, and n is called the tree's *branching factor*. In most porous media models it suffices to take $n = 2$. The root of a regular tree is at level 0, its direct descendants are at *level* 1, and the top, or *leaf*, pores are at level L. Hence, the number of pores in a regular tree is $N = (n^{L+1} - 1)/(n-1)$, and $N = 2^{L+1} - 1$ when $n = 2$.

A hierarchical tree is composed of $k = 0, \ldots, M$ pore classes. The tree is hierarchical in the sense that a pore in the kth class has descendants in class k or $k+1$ except for pores in class M whose descendants, if there are any, can only be in class M. Pores in the kth class are characterized by a cross-sectional area, A_k, with a corresponding perimeter, P_k, and a length, L_k. The cross-sectional area can be used to define the "hydraulic radius" $R_k = fA_k/P_k$, where f is a constant shape factor. We will suppose the pores are

Paper number 2001GL013680.
0094-8276/01/2001GL013680$05.00

cylindrical, i.e., $f = 1$ and the hydraulic radius coincides with the geometric radius, which simplifies the presentation without changing the scaling argument. Radii and lengths scale according to

$$\beta = R_{k+1}/R_k \quad \text{and} \quad \gamma = L_{k+1}/L_k \tag{2}$$

for fixed $\beta, \gamma < 1$. This means for instance that

$$R_k = \beta^{k-M} R_M \quad \text{and} \quad L_k = \gamma^{k-M} L_M \tag{3}$$

Both R_k and L_k can be random without affecting the following argument much, but we do not emphasize that point. The invariance of the smallest pores, together with Darcy's law and mass conservation, implies that [*West et al.*, 1997]

$$N_M \sim \mathcal{V}^\alpha, \tag{4}$$

where α is a constant to be determined later.

The simplest hierarchical tree has the same number of pore classes as levels, i.e., $L = M$. These trees correspond very well to the distribution systems found in physiology but are too restrictive for groundwater hydrology. To relax the model we suppose the pore space is embedded in a tree composed of subtrees. Each subtree consists of pores from only one class, say k, and all subtrees of type k have a fixed number, $L_k + 1$, of levels. We call such a tree a *k-subtree*. Although it is not essential, we will simplify the discussion by taking $n = 2$. Then the number of pores in any k-subtree is approximately 2^{L_k+1} so long as L_k is large, which we suppose throughout. The bottom-most subtree is a 0-subtree. There is only one of these. The only descendants of k-subtrees are $k+1$-subtrees. The top of the tree consists just of M-subtrees. The total number of pores of class $k > 0$ is $N_k = (2^{L_k+1} - 1)2^{L_0+\cdots+L_{k-1}+k}$. We suppose that N_k scales according to $N_k = \lambda N_{k+1} = \lambda^{M-k} N_M$ with $\lambda < 1$. This is consistent with data obtained from studies of fracture systems and soil properties [*Baveye et al.*, 1998]. It is easy to see that scaling requires $L_0 = L_1 = \ldots = L_M \equiv L$ and $\lambda = 2^{-(L+1)}$, so $N_k = 2^{(k+1)(L+1)}$. The number of levels per subtree, $L+1$, is arbitrary.

Hierarchical organization is a requirement for trees that minimize resistivity, so it is a feature of most biological networks and of those porous networks that have eroded to minimize potential. A straightforward inductive argument shows that a hierarchical tree minimizes the total resistivity of the pore space. Suppose that the branching factor $n = 2$; extensions to $n > 2$ are obvious. First, consider the tree with only two levels. Suppose without loss of generality that the pores have resistivities $\mathcal{R}_0 \le \mathcal{R}_1 \le \mathcal{R}_2$. The tree with $\mathcal{R}_0$ at the root has total resistivity

$$\mathcal{R}^0 = \mathcal{R}_0 + \frac{\mathcal{R}_1 \mathcal{R}_2}{\mathcal{R}_1 + \mathcal{R}_2},$$

which is obviously smaller than

$$\mathcal{R}^1 = \frac{\mathcal{R}_0\mathcal{R}_2 + \mathcal{R}_0\mathcal{R}_1 + \mathcal{R}_1\mathcal{R}_2}{\mathcal{R}_0 + \mathcal{R}_2},$$

the total resistivity of the tree with $\mathcal{R}_1$ at its root. Now suppose that the hierarchical tree minimizes the total resistivity of pore spaces embedded in trees with $L-2$ levels. Consider trees of height L whose first two levels consist of pores with resistivities $\mathcal{R}_0 \le \mathcal{R}_1 \le \mathcal{R}_2$. Let $\mathcal{R}_i^L$ be the L-level tree with resistivity $\mathcal{R}_i$ at the root. Then it is easy to see the $\mathcal{R}_0^L \le \mathcal{R}_1^L$. First note that

$$\mathcal{R}_1^{L-1} = \mathcal{R}_1 + \frac{\mathcal{R}_{left}^{L-2}\mathcal{R}_{right}^{L-2}}{\mathcal{R}_{left}^{L-2} + \mathcal{R}_{right}^{L-2}}$$

where $\mathcal{R}_{left}^{L-2}$ ($\mathcal{R}_{right}^{L-2}$) is the left (right) descendant of the tree rooted with a pore of resistivity $\mathcal{R}_1$. Let

$$\mathcal{R}^{L-2} \equiv \frac{\mathcal{R}_{left}^{L-2}\mathcal{R}_{right}^{L-2}}{\mathcal{R}_{left}^{L-2} + \mathcal{R}_{right}^{L-2}}$$

for convenience. Of course,

$$\mathcal{R}_0^L = \mathcal{R}_0 + \frac{\mathcal{R}_1^{L-1}\mathcal{R}_2^{L-1}}{\mathcal{R}_1^{L-1} + \mathcal{R}_2^{L-1}},$$

which we rewrite as

$$\mathcal{R}_0^L = \mathcal{R}_0 + \frac{(\mathcal{R}_1 + \mathcal{R}^{L-2})\mathcal{R}_2^{L-1}}{\mathcal{R}_1^{L-1} + \mathcal{R}_2^{L-1}}.$$

On the other hand,

$$\mathcal{R}_1^L = \mathcal{R}_1 + \frac{(\mathcal{R}_0 + \mathcal{R}^{L-2})\mathcal{R}_2^{L-1}}{\mathcal{R}_0 + \mathcal{R}^{L-2} + \mathcal{R}_2^{L-1}},$$

so $\mathcal{R}_0^L \le \mathcal{R}_1^L$ since $\mathcal{R}_1 \ge \mathcal{R}_0$.

3. Macroscopic Scaling

The simplest hierarchical trees, those with $L = M$, have been used by *West et al.* [1997] to derive the allometric scaling laws observed in biology. Representing the circulatory system of an organism of volume V by a simple tree, they showed that the total conductivity of the system satisfies (1) with $\alpha = -\ln n/\ln(\gamma\beta^2)$. If the circulatory system is area-preserving and space-filling, $\alpha = 3/4$. However, the simplest tree is too constrained by the $1-1$ correspondence between the number of levels and the number of pore classes to be of much use in hydrology.

The total volume occupied by a hierarchical tree is

$$\mathcal{V} \approx N_M V_M \frac{\lambda^M (\beta^2\gamma)^{-M}}{1 - \lambda^{-1}\beta^2\gamma} \approx V_M \frac{(\beta^2\gamma)^{-M}}{\lambda - \beta^2\gamma}$$

since $\lambda^M N_M = N_0 \approx \lambda^{-1}$. Hence,

$$(\gamma\beta^2)^{-M} \sim \mathcal{V}. \tag{5}$$

Note that $\mathcal{V} > 0$ requires $\beta^2\gamma < \lambda$.

Poiseuille's Formula yields $\mathcal{R}_k = 8\mu L_k/(\pi R_k^4)$ for the resistivity of an individual pore at the kth level to the flow of a fluid with viscosity μ. The total resistivity of the tree is $\mathcal{R} = 2(2^{L+1}-1)\sum_{k=0}^{M} 2^{-(k+1)(L+1)} R_k \approx 2R_M N_M^{-1}(\lambda - \beta^4/\gamma)^{-1}$ so long as $\beta^4/\gamma < \lambda$. Thus, $\mathcal{K} = \mathcal{R}^{-1}$ is proportional to the total number of the smallest pores,

$$\mathcal{K} \sim N_M. \tag{6}$$

This, combined with (4) and (5), yields the scaling law (1) with the scaling exponent

$$\alpha = \frac{\ln\lambda}{\ln(\gamma\beta^2)} = -\ln 2\,\frac{L+1}{\ln(\gamma\beta^2)}. \tag{7}$$

The constraint $\gamma\beta^2 < \lambda < 1$ implies $\alpha < 1$. Since $\gamma > 0$ and $\beta > 0$, there are positive constants c_γ and c_β such that $\gamma = 2^{-(L+1)c_\gamma}$, $\beta = 2^{-(L+1)c_\beta}$ and

$$\alpha = \frac{1}{c_\gamma + c_\beta}. \tag{8}$$

There are physically realizable conditions on the pore tree that lead naturally to $\alpha = 1/2$ found in hydrology and others that lead to the value $\alpha = 3/4$ found in biological networks. When the network preserves total pore area from level to level and also total pore length, $\alpha = 1/2$. A tree preserves total pore area from one level to the next if $A_k \approx A_{k+m}$ when k is large, $m > 0$, and $A_k = \pi R_k^2 N_k$ is the total area of all pores at level k. In that case, $\beta^2 \approx N_k/N_{k+1} \approx 2^{-(L+1)}$, so $c_\beta = 1$. Similarly, total pore length is preserved if, $L_k N_k \approx L_{k+1} N_{k+1}$, which implies $\gamma \approx N_k/N_{k+1} \approx 2^{-(L+1)}$ and $c_\gamma = 1$.

On the other hand, $\alpha = 3/4$ when the network preserves total pore area and is space-filling. When level k of the network fills space, each pore supplies fluid to a neighborhood that can be approximated by a sphere with volume $4\pi(L_k/2)^3/3$. Then the total volume, $\mathcal{V}$, supplied by the tree can be approximated by $V_k \approx 4\pi N_k (L_k/2)^3/3$ for k large. Hence, $V_k \approx V_{k+m}$ when k is large and $m > 0$. This implies in particular that $\gamma^3 = (L_{k+1}/L_k)^3 \approx 2^{-(L+1)}$ or $c_\gamma \approx 1/3$.

4. Discussion and Conclusions

Hierarchical trees are a type of abstract porous medium that scales according to (1) and exhibits 1/2-power scaling when the network preserves total pore length and total pore area. This contrasts with the 3/4-power scaling observed in biology [*West et al.*, 1997] where the networks are organized to deliver fluid to every point in a body. The question remains whether hierarchical trees are reasonably good models of at least some real porous media. The answer revolves around the plausibility of two sets of assumptions: i) pore lengths, pore radii and the numbers of pores obey simple scaling laws themselves and ii) the hierarchical tree structure is realistic. The properties of pores in many real porous media do exhibit power law scaling [*Baveye et al.*, 1998], so model assumption i) seems reasonable. Regarding assumption ii), hierarchy is justified when networks have evolved to maximize flow, but the actual topology of real porous networks is a matter of (difficult) observation.

When i) and ii) are met, the scaling exponent, α, can be expressed in terms of the three microscopic parameters, γ, β, and λ. In principle, any value of α is allowable so long as it is consistent with simple constraints on the relative values of γ, β, and λ. However, hierarchical trees yield $\alpha = 1/2$ scaling if the pore network preserves total pore area and length from one level to another. This raises the interesting possibility that the empirical 1/2-power law observed by *Neuman* [1990; 1994] can be attributed to sampling from networks that are actually collections of trees. By analyzing data, *Neuman* [1994] obtained $\sigma_Y^2 \sim V^\alpha$ for the variance of log conductivity Y and followed Matheron's conjecture [*Matheron*, 1967] to write $K \sim \exp(V^\alpha)$. Our rigorous analysis of hierarchical trees, on the other hand, indicates $K \sim V^\alpha$. The two would be indistinguishable based on data from experiments conducted on relatively small support volumes. A collection of independent trees will show scaling according to (1) just like an individual tree so long as the size of the smallest pores is invariant within a site. Universal scaling in the sense of *Neuman* [1994] would require a stronger assumption: the size of the smallest pores is invariant across sites.

Among the many questions outstanding at present, it is has not been shown analytically whether other topologies, e.g., a rectangular lattice, lead to simple scaling, and hence, it is not known under what conditions they might yield 1/2-power scaling.

Acknowledgments. This work was performed under the auspices of the U.S. Department of Energy (DOE): DOE/BES (Bureau of Energy Sciences) Program in the Applied Mathematical Sciences contract KC-07-01-01 and Los Alamos National Laboratory under *LDRD* 98604. This work made use of STC shared experimental facilities supported by the National Science Foundation under Agreement No. EAR-9876800.

The Editor and the authors would like to thank the reviewers of this manuscript.

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(Received June 26, 2001; accepted September 06, 2001.)