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PROBABILISTIC PREDICTIONS OF INFILTRATION INTO HETEROGENEOUS MEDIA WITH UNCERTAIN HYDRAULIC PARAMETERS

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Soil heterogeneity and the lack of detailed site characterization are two ubiquitous factors that render predictions of flow and transport in the vadose zone inherently uncertain. We employ the Green–Ampt model of infiltration and the Dagan–Bresler statistical parameterization of soil properties to compute probability density functions (PDFs) of infiltration rate and infiltration depth. By going beyond uncertainty quantification approaches based on mean and variance of system states, these PDF solutions enable one to evaluate probabilities of rare events that are required for probabilistic risk assessment. We investigate the temporal evolution of the PDFs of infiltration depth and corresponding infiltration rate, the relative importance of uncertainty in various hydraulic parameters and their cross-correlation, and the impact of the choice of a functional form of the hydraulic function.

KEY WORDS: *Uncertainty quantification, stochastic, infiltration rate, Green–Ampt model*

1. INTRODUCTION

Soil heterogeneity and the lack of detailed site characterization are two ubiquitous factors that hamper one's ability to predict flow and transport in the vadose zone. The continuing progress in data acquisition notwithstanding, measurements of hydraulic properties of partially saturated media remain scarce and prone to measurement and interpretive errors. Consequently, spatial distributions of hydraulic parameters (saturated and relative hydraulic conductivities, and parameters in retention curves) are typically uncertain and their statistical properties are subject to considerable debate.

Despite some reservations, e.g., [1, 2], it has become common to treat saturated hydraulic conductivity $K_s(\mathbf{x})$ as a multivariate log-normal random field whose ensemble statistics (e.g., mean, variance, and correlation length) can be inferred from spatially distributed data by means of geostatistics. No such consensus exists about statistical distributions of various hydraulic parameters entering relative hydraulic conductivity and retention curves. For example, various data analyses concluded that spatial variability of a soil parameter $\alpha_G(\mathbf{x})$ in the Gardner model of relative conductivity, which is often referred to as the reciprocal of the macroscopic capillary length, exhibits either a normal [3] or log-normal [4] distribution and is either correlated [5] or uncorrelated [3] with K_s. We defer a more detailed review of the statistical properties of both $\alpha_G(\mathbf{x})$ and parameters in the van Genuchten model of relative conductivity until Section 2. Here, it suffices to say that any approach to uncertainty quantification for flow and transport in the vadose zone must be flexible enough to accommodate arbitrary statistical distributions of soil properties.

Statistical treatment of hydraulic parameters renders corresponding flow equation stochastic. Solutions of these equations are probability density functions (PDFs) of system states (water content, pressure, and macroscopic flow velocity) and can be used not only to predict flow in heterogeneous partially saturated porous media but also to quantify

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predictive uncertainty. Rather than computing PDFs of system states, standard practice in subsurface hydrology is to compute (analytically or numerically) the first two moments of system states, and to use their ensemble means as predictors of a system's behavior and variances (or standard deviations) as a measure of predictive uncertainty. A large body of literature employing this approach to solve the stochastic Richards equation includes [6–11], to name just a few. With the exception of solutions based on the Kirchhoff transformation [12–14], such analyses require one to linearize constitutive relations in the Richards equation, introducing errors that are hard to quantify a priori. More important, none of these solutions can be used to estimate the probability of rare events, which is of crucial importance for uncertainty quantification and risk assessment [15].

The Green–Ampt model described in some detail in Section 2 (see also [16, Section 5.2]) provides an alternative description of flow in partially saturated porous media. The relative simplicity of the Green–Ampt formulation makes it easier to solve than the Richards equation, which explains its prevalence in large numerical codes—e.g., SCS developed by U.S. Environmental Protection Agency (USEPA), DR3M developed by U.S. Geological Survey (USGS), and HIRO2 developed by U.S. Department of Agriculture (USDA)—that are routinely used to predict overland and channel flows. The first analysis of the impact of soil heterogeneity and parametric uncertainty on solutions of the Green–Ampt equations was carried out by Dagan and Bresler [17]. Saturated hydraulic conductivity—the sole source of uncertainty in their analysis—was treated as a two-dimensional *random field*, $K_s(x_1, x_2)$. This enables one to model vertical infiltration with a collection of one-dimensional (in the x_3 direction) solutions each of which corresponds to a different *random variable* K_s. The Dagan–Bresler statistical model [17], whose precise formulation is provided in Section 2, was found to yield accurate predictions of infiltration into heterogeneous soils [18, 19] and has been adopted in a number of subsequent investigations, e.g., [19–24]. These and other similar analyses aimed to derive effective (ensemble averaged) infiltration equations, and some of them quantified predictive uncertainty by computing variances of system states.

Driven by the needs of probabilistic risk assessment, we focus on the derivation of single-point PDFs (rather than the first two moments) of system states describing infiltration into heterogeneous soils with uncertain hydraulic parameters. Our analysis employs the Green–Ampt model of infiltration with the Dagan–Bresler parameterization, both of which are formulated in Section 2. This Section also contains an overview of experimentally observed statistical properties of the coefficients entering the Gardner and van Genuchten expressions of relative hydraulic conductivity K_r. A general framework for derivation of PDF solutions of the Green–Ampt model is presented in Section 3. In Section 4 we investigate the temporal evolution of the PDFs of a wetting front (Setion 4.1) and corresponding infiltration rate (Section 4.2), the relative importance of uncertainty in various hydraulic parameters (Section 4.3) and their cross-correlation (Section 4.4), and the impact of the choice of a functional form of K_r (Section 4.5). Concluding remarks are presented in Section 5.

2. PROBLEM FORMULATION

Consider infiltration into a heterogeneous soil with saturated hydraulic conductivity K_s, porosity ϕ, residual water content θ_r, and relative hydraulic conductivity $K_r(\psi; \boldsymbol{\alpha})$ that varies with pressure head ψ in accordance with a constitutive model and model parameters $\boldsymbol{\alpha}$. While the subsequent analysis can be applied to any constitutive relation, we focus on the Gardner model [16, Table 2.1]

$$K_r = \mathrm{e}^{\alpha_G \psi} \tag{1}$$

and the van Genuchten model (ibid)

$$K_r = \frac{[1 - \psi_d^{mn}(1 + \psi_d^n)^{-m}]^2}{(1 + \psi_d^n)^{m/2}}, \quad \psi_d \equiv \alpha_{vG}|\psi|. \tag{2}$$

The model parameters $\boldsymbol{\alpha}$ ($\boldsymbol{\alpha} \equiv \alpha_G$ and $\{\alpha_{vG}, n, m = 1 - 1/n\}$ for the Gardner and van Genuchten models, respectively) and the rest of the hydraulic properties mentioned above vary in space and are sparsely sampled. To quantify uncertainty about values of these properties at points $\mathbf{x} = (x_1, x_2, x_3)^T$ where measurements are unavailable, we treat them as random fields. Thus, a soil parameter $\mathcal{A}(\mathbf{x}, \omega)$ varies not only in the physical domain, $\mathbf{x} \in \mathcal{D}$, but also

in the probability space $\omega \in \Omega$. A probability density function $p_{\mathcal{A}}$, which describes the latter variability, is inferred from measurements of $\mathcal{A}$ by invoking ergodicity. Experimental evidence for the selection of PDFs $p_{\mathcal{A}}$ for various soil parameters $\mathcal{A}$ is reviewed in Section 2.1, and the Dagan–Bresler statistical model used in our analysis is formulated in Section 2.2.

The overreaching aim of the present analysis is to quantify the impact of this parametric uncertainty on predictions of both the dynamics of wetting fronts and infiltration rates. Uncertainty in the former may significantly affect the accuracy and reliability of field-scale measurements of soil saturation [25], while uncertainty in the latter is of fundamental importance to flood forecasting [23].

2.1 Statistics of Soil Parameters

Saturated hydraulic conductivity. In addition to the experimental studies reviewed in [12], the data analyses reported in [4, 24], etc., support our treatment of saturated hydraulic conductivity K_s as a log-normal random field.

Gardner's constitutive parameter. The (scarce) experimental evidence reviewed in [12] suggests that α_G, the reciprocal of the macroscopic capillary length, can be treated alternatively either as a Gaussian (normal) or as a log-normal random field. While the approach described below is capable of handling both distributions, in the subsequent computational examples we will treat α_G as a log-normal field, which is a model adopted in more recent computational investigations (e.g., [4, 10]).

Van Genuchten's constitutive parameters. The van Genuchten hydraulic function (2) is a two-parameter model obtained from its more general form by setting $m = 1 - 1/n$ and $l = 1/2$ (hence, the power $m/2$ in the denominator). We employ this form because of its widespread use [16, Table 2.1], but the approach described below can be readily applied to quantify uncertainty in more general formulations with arbitrary m and l. The experimental evidence presented in [4, 26, 27] shows that the coefficient of variation of α_{vG} is much larger than that of n. These data suggest that α_{vG} can be treated as a log-normal field and the shape factor n as a deterministic constant.

Correlations between hydraulic parameters. Experimental evidence presented in [4, 12] suggests that the coefficient of variation (CV) of K_s is generally much larger than that of either α_G or α_{vG}. These parameters were found to be either perfectly correlated or uncorrelated or anticorrelated (see also [28]). Our analysis allows for an arbitrary degree of correlation between K_s and either α_G or α_{vG}.

Finally, since the difference between the full and residual saturations $\Delta\theta = \phi - \theta_r$ typically exhibits lower spatial variability than both K_s and α_G (or α_{vG}), we treat it as a deterministic constant to simplify the presentation. Our approach can be adopted to quantify uncertainty in $\Delta\theta$ and the shape factor n in the van Genuchten hydraulic function, as discussed in Section 3.

2.2 Statistical Model for Soil Parameters

Following [17], we restrict our analysis to infiltration depths that do not exceed vertical correlation lengths l_v of (random) soil parameters $\mathcal{A}(\mathbf{x}, \omega)$. Then $\mathcal{A} = \mathcal{A}(x_1, x_2, \omega)$, so that a heterogeneous soil can be represented by a collection of one-dimensional (in the vertical direction x_3) homogeneous columns of length L_3, whose uncertain hydraulic properties are modeled as random variables (rather than random fields). The restriction $l_v > L_3$ formally renders the Dagan–Bresler parameterization [17] suitable for heterogeneous topsoils, and thus can be used to model surface response to rainfall events [23, 24] and transport phenomena in topsoil [21]. Yet it was also used to derive effective properties of the whole vadose zone [4, 28]. Rubin and Or [19] provide an additional justification for the Dagan–Bresler parameterization by noting that "the determination of soil hydraulic properties through field methods. . .homogenize the properties vertically, thus eliminating the variability in the vertical direction in a practical sense."

Consider a three-dimensional flow domain $\Omega = \Omega_h \times [0, L_3]$, where Ω_h represents its horizontal extent. A discretization of Ω_h into N elements represents Ω by an assemblage of N columns of length L_3 and facilitates the complete description of a random field $\mathcal{A}(x_1, x_2, \omega)$—in the analysis below, $\mathcal{A}$ stands for K_s, α_G, and α_{vG} but can also include other hydraulic properties and the ponding pressure head ψ_0 at the soil surface $x_3 = 0$—with a joint

probability function $p_{\boldsymbol{\mathcal{A}}}(A_1, \ldots, A_N)$. Probability density functions (PDFs) of hydraulic properties of the ith column are defined as marginal distributions,

$$p_{\mathcal{A}_i}(A_i) = \int p_{\boldsymbol{\mathcal{A}}}(A_1, \ldots, A_n) \mathrm{d}A_1 \ldots \mathrm{d}A_{i-1} \mathrm{d}A_{i+1} \ldots \mathrm{d}A_N. \tag{3}$$

Since statistical properties of soil parameters $\mathcal{A}$ are inferred from spatially distributed data by invoking ergodicity, the corresponding random fields (or their fluctuations obtained by data de-trending) must be stationary so that

$$p_{\mathcal{A}_i} = p_{\mathcal{A}} \quad \text{for} \quad i = 1, \ldots, N. \tag{4}$$

Furthermore, if such soil parameters (e.g., K_s and α_G) are correlated, their statistical description requires the knowledge of a joint distribution. For multivariate Gaussian $Y_1 = \ln K_s$ and $Y_2 = \ln \alpha_G$ (or $Y_2 = \ln \alpha_{vG}$), their joint PDF is given by

$$p_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi \sigma_{Y_1} \sigma_{Y_2} \sqrt{1-\rho^2}} \exp\left[-\frac{R}{2(1-\rho^2)}\right] \tag{5a}$$

where

$$R = \frac{(y_1 - \overline{Y}_1)^2}{\sigma_{Y_1}^2} - 2\rho \frac{y_1 - \overline{Y}_1}{\sigma_{Y_1}} \frac{y_2 - \overline{Y}_2}{\sigma_{Y_2}} + \frac{(y_2 - \overline{Y}_2)^2}{\sigma_{Y_2}^2}; \tag{5b}$$

$\overline{Y}_i$ and σ_{Y_i} denote the mean and standard deviation of Y_i ($i = 1, 2$), respectively; and $-1 \le \rho \le 1$ is the linear correlation coefficient between Y_1 and Y_2. The lack of correlation between Y_1 and Y_2 corresponds to setting $\rho = 0$ in (5).

2.3 Green–Ampt Model of Infiltration

During infiltration into topsoils, the Dagan–Bresler parameterization of soil heterogeneity can be supplemented with an assumption of vertical flow. The rationale for, and implications of, neglecting the horizontal component of flow velocity can be found in [17, 19, 20] and other studies reviewed in the Introduction.

This assumption obviates the need to solve a three-dimensional flow problem, replacing the latter with a collection of N one-dimensional flow problems to be solved in homogeneous soil columns with random but constant hydraulic parameters. Such a framework was used to predict mean (ensemble averaged) flow with either the Green–Ampt model [17, 20] or the steady-state Richards equation with the Gardner hydraulic function [19]. We employ the Green–Ampt description because it enables one to handle transient flow and to employ arbitrary hydraulic functions, without resorting to linearizing approximations [29].

Let $I(t)$ denote (uncertain) cumulative infiltration due to ponding water of height ψ_0 at the soil surface $x_3 = 0$. The Green–Ampt model of infiltration approximates an S-shaped wetting front with a sharp interface $x_{\mathrm{f}}(t)$ that separates fully saturated soil (saturation ϕ) from dry soil (saturation θ_r). The latter is also known as infiltration depth. If the x_3 coordinate is positive downward, Darcy's law defines macroscopic (Darcy's) flux q as (e.g., [16, Eq. (5.1)])

$$q = -K_s \frac{\psi_{\mathrm{f}} - x_{\mathrm{f}} - \psi_0}{x_{\mathrm{f}}}. \tag{6}$$

Pressure head ψ_{f} at the infiltration depth $x_{\mathrm{f}}(t)$ is empirically set to a "capillary drive",

$$\psi_{\mathrm{f}} = -\int_{\psi_{\mathrm{in}}}^{0} K_r(\psi)\, \mathrm{d}\psi, \tag{7}$$

where ψ_{in} is the initial pressure head in the dry soil.

Mass conservation requires that $I(t) = (\omega - \theta_r)\, x_{\rm f}(t)$ and the infiltration rate $i \equiv {\rm d}I/{\rm d}t$ equals q. The first condition yields

$$i = \Delta\theta \frac{{\rm d}x_{\rm f}}{{\rm d}t}, \qquad \Delta\theta = \omega - \theta_r, \tag{8}$$

which, combined with the second condition and (6), leads to a (stochastic) ordinary differential equation for the position of the wetting front,

$$\Delta\theta \frac{{\rm d}x_{\rm f}}{{\rm d}t} = K_s \frac{\psi_0 + x_{\rm f} - \psi_{\rm f}}{x_{\rm f}}, \quad x_{\rm f}(t=0) = 0. \tag{9}$$

Our goal is to relate uncertainty in hydraulic parameters K_s and α_G (or α_{vG}) to predictive uncertainty about the infiltration depth $x_{\rm f}(t)$ and the infiltration rate $i(t)$, i.e., to express the PDFs of the latter, $p_{\rm f}(x_{\rm f}; t)$ and $p_{\rm i}(i; t)$, in terms of the PDF of the former (5).

3. PDF SOLUTIONS

To simplify the presentation, we assume that the height of ponding water, ψ_0, does not change with t during the simulation time T. Then an implicit solution of (9) takes the form

$$x_{\rm f} - (\psi_0 - \psi_{\rm f}) \ln\left(1 + \frac{x_{\rm f}}{\psi_0 - \psi_{\rm f}}\right) = \frac{K_s}{\Delta\theta} t. \tag{10}$$

For small t, (10) can be approximated by an explicit relation [16, Eq. (5.12)]

$$x_{\rm f} \approx \sqrt{\frac{2(\psi_0 - \psi_{\rm f}) K_s t}{\Delta\theta}}. \tag{11}$$

For large t, flow becomes gravity dominated, $i \sim K_s$, and [16, p. 170]

$$x_{\rm f} \approx \frac{K_s}{\Delta\theta} t. \tag{12}$$

For intermediate t, various approximations, e.g., [30] and [16, p. 170], can be used to replace the implicit solution (10) with its explicit counterparts. We will use the implicit solution (10) to avoid unnecessary approximation errors.

Several of the simplifying assumptions made above can be easily relaxed. First, since K_s and $\Delta\theta$ enter the stochastic Eq. (9) and its implicit solution (10) as the ratio $K_s^\star = K_s/\Delta\theta$, one can easily incorporate uncertainty in (randomness of) $\Delta\theta$ by replacing the PDF of K_s with the PDF of $K_s^\star$. Second, the implicit relation $F(x_{\rm f}, K_s/\Delta\theta, \boldsymbol{\alpha}; t) = 0$ given by (10) and (7) allows one to express the PDF of $x_{\rm f}$ in terms of the PDFs of *any* number of hydraulic parameters by following the procedure described below. Third, uncertainty in, and temporal variability of, the height of ponding water $\psi_0(t)$ can be dealt with by replacing (10) with an appropriate solution of (9).

3.1 PDF of Infiltration Depth

Let $G_{\rm f}(x_{\rm f}^\star) = P(x_{\rm f} \le x_{\rm f}^\star)$ denote the cumulative distribution function of $x_{\rm f}$, i.e., the probability that the random position of the wetting front $x_{\rm f}$ takes on a value not larger than $x_{\rm f}^\star$. Since (10) provides an explicit dependence of random K_s on random $x_{\rm f}$ and α (where α stands for either α_G or α_{vG}), i.e.,

$$K_s(x_{\rm f}, \alpha) = \frac{\Delta\theta}{t}\left[x_{\rm f} - (\psi_0 - \psi_{\rm f}) \ln\left(1 + \frac{x_{\rm f}}{\psi_0 - \psi_{\rm f}}\right)\right], \tag{13}$$

it follows from the definition of a cumulative distribution function that

$$G_{\rm f}(x_{\rm f}^\star) = \int_0^\infty \int_0^{K_s(x_{\rm f}^\star, \alpha)} p_{Y_1,Y_2}(K_s, \alpha) \frac{{\rm d}K_s {\rm d}\alpha}{K_s \alpha}. \tag{14}$$

The denominator in (14) reflects the transition from (5), the joint Gaussian PDF for Y_1 and Y_2, to the log-normal variables $K_s = \exp(Y_1)$ and $\alpha = \exp(Y_2)$.

The PDF of the random (uncertain) infiltration depth $p_{\mathrm{f}}(x_{\mathrm{f}}^\star; t)$ can now be obtained as

$$p_{\mathrm{f}}(x_{\mathrm{f}}^\star; t) = \frac{\mathrm{d}G_{\mathrm{f}}(x_{\mathrm{f}}^\star; t)}{\mathrm{d}x_{\mathrm{f}}^\star}. \tag{15}$$

Using Leibnitz's rule to compute the derivative of the integral in (14) and (15), we obtain

$$p_{\mathrm{f}}(x_{\mathrm{f}}^\star; t) = \int_0^\infty \frac{p_{Y_1,Y_2}[K_s(x_{\mathrm{f}}^\star, \alpha), \alpha]}{\alpha K_s(x_{\mathrm{f}}^\star, \alpha)} \frac{\partial K_s(x_{\mathrm{f}}^\star, \alpha)}{\partial x_{\mathrm{f}}^\star} \mathrm{d}\alpha. \tag{16}$$

Equation (16) holds for an arbitrary implicit solution of the Green–Ampt equation, $F(x_{\mathrm{f}}, K_s/\Delta\theta, \boldsymbol{\alpha}; t) = 0$, and hence, the PDF solution (16) is applicable to a large class of infiltration regimes that are amenable to the Green–Ampt description. For the flow regime considered in the present analysis, $K_s(x_{\mathrm{f}}^\star, \alpha)$ is given by (13), and (16) takes the form

$$p_{\mathrm{f}}(x_{\mathrm{f}}^\star; t) = \frac{\Delta\theta}{t} \int_0^\infty \frac{p_{Y_1,Y_2}[K_s(x_{\mathrm{f}}^\star, \alpha), \alpha]}{\alpha K_s(x_{\mathrm{f}}^\star, \alpha)} \frac{x_{\mathrm{f}}^\star \, \mathrm{d}\alpha}{\psi_0 - \psi_{\mathrm{f}} + x_{\mathrm{f}}^\star}. \tag{17}$$

3.2 PDF of Infiltration Rate

Let $G_{\mathrm{i}}(i^\star) = P(i \le i^\star)$ denote the cumulative distribution function of i, i.e., the probability that the random infiltration rate i takes on a value not larger than $i^\star$. Since $q = i$, Eqs. (6) and (7) define a mapping $K_s = K_s(i, \alpha)$. This enables one to compute the cumulative distribution function $G_{\mathrm{i}}(i^\star)$ as

$$G_{\mathrm{i}}(i^\star) = \int_0^\infty \int_0^{K_s(i^\star, \alpha)} p_{Y_1,Y_2}(K_s, \alpha) \frac{\mathrm{d}K_s \mathrm{d}\alpha}{K_s \alpha} \tag{18}$$

and the PDF of infiltration rate, $p_{\mathrm{i}} = \mathrm{d}G_{\mathrm{i}}/\mathrm{d}i^\star$, as

$$p_{\mathrm{i}}(i^\star; t) = \int_0^\infty \frac{p_{Y_1,Y_2}[K_s(i^\star, \alpha), \alpha]}{\alpha K_s(i^\star, \alpha)} \frac{\partial K_s(i^\star, \alpha)}{\partial i^\star} \mathrm{d}\alpha. \tag{19}$$

The derivative $\partial K_s/\partial i^\star$ is computed from (6) as the inverse of

$$\frac{\partial i^\star}{\partial K_s} = 1 + \frac{\psi_0 - \psi_{\mathrm{f}}}{x_{\mathrm{f}}} \left(1 - \frac{K_s t}{\Delta\theta} \frac{x_{\mathrm{f}} - \psi_{\mathrm{f}} + \psi_0}{x_{\mathrm{f}}^2}\right). \tag{20}$$

3.3 Dimensionless Form of PDFs

To facilitate an analysis of the effects of various sources of parametric uncertainty on the PDF $p_{\mathrm{f}}(x_{\mathrm{f}}^\star; t)$ of the uncertain (random) infiltration depth $x_{\mathrm{f}}(t)$, given by the analytical solution (17), we introduce the following dimensionless quantities. Let the averaged quantities $(\overline{\alpha})^{-1}$ and $\overline{K}_s$ represent a characteristic length scale and a characteristic value of saturated hydraulic conductivity, respectively. Then a characteristic time scale τ can be defined as

$$\tau = (\overline{\alpha}\overline{K}_s)^{-1}, \tag{21}$$

and the following dimensionless quantities can be introduced,

$$t' = \frac{t}{\tau}, \quad \psi' = \overline{\alpha}\psi, \quad \alpha' = \frac{\alpha}{\overline{\alpha}}, \quad K_s' = \frac{K_s}{\overline{K}_s}. \tag{22}$$

This leads to a PDF solution for the dimensionless infiltration depth $x_{\rm f}' = \overline{\alpha} x_{\rm f}$,

$$p_{\rm f}(x_{\rm f}'; t') = \frac{\Delta\theta}{t'} \int_0^\infty \frac{p_{Y_1', Y_2'}[K_s'(x_{\rm f}', \alpha'), \alpha']}{\alpha' K_s'(x_{\rm f}', \alpha')} \frac{x_{\rm f}' \, {\rm d}\alpha'}{\psi_0' - \psi_{\rm f}' + x_{\rm f}'}. \tag{23}$$

Likewise, the PDF of the dimensionless infiltration rate $i' = i/\overline{K}_s$ takes the form

$$p_{\rm i}(i'; t') = \int_0^\infty \frac{p_{Y_1', Y_2'}[K_s'(i', \alpha'), \alpha']}{\alpha' K_s'(i', \alpha')} \frac{\partial K_s'(i', \alpha')}{\partial i'} {\rm d}\alpha'. \tag{24}$$

In the following, we drop the primes to simplify the notation.

4. RESULTS AND DISCUSSION

In this Section, we explore the impact of various aspects of parametric uncertainty on the uncertainty in predictions of infiltration rate $i(t)$ and infiltration depth $x_{\rm f}(t)$. Specifically, we investigate the temporal evolution of the PDFs of the wetting front (Section 4.1) and the infiltration rate (Section 4.2), the relative importance of uncertainty in K_s and α_i (Section 4.3), and the effects of cross-correlation between them (Section 4.4). This is done for the Gardner hydraulic function (1), in which case (7) results in the interfacial pressure head $\psi_{\rm f} = -\alpha_G^{-1}$. In Section 4.5, we explore how the choice of a functional form of the hydraulic function, i.e., the use of the van Genuchten model (2) instead of the Gardner relation (1), affects the predictive uncertainty.

Unless explicitly noted otherwise, the simulations reported below correspond to the dimensionless initial pressure head $\psi_{\rm in} = -9999.9$, the dimensionless height of ponding water $\psi_0 = 0.1$, $\Delta\theta = 0.45$, the coefficients of variation $CV_{\ln K} \equiv \sigma_{Y_1}/\overline{Y}_1 = 3.0$ and $CV_{\ln \alpha} \equiv \sigma_{Y_2}/\overline{Y}_2 = 0.5$ with the means $\overline{Y}_1 = 0.25$ and $\overline{Y}_2 = 0.1$, and the cross-correlation coefficient $\rho = 0$. (The use of the soil data in Table 1 of [26] in conjunction with these dimensionless parameters would result in the height of ponding water $\psi_0 = 0.6$ cm.)

4.1 PDF of Wetting Front

Since the initial position of the wetting front is assumed to be known, $x_{\rm f}(t = 0) = 0$, the PDF $p_{\rm f}(x_{\rm f}; 0) = \delta(x_{\rm f})$, where $\delta(\cdot)$ denotes the Dirac delta function. As the dimensionless time becomes large ($t \to \infty$), $p_{\rm f} \sim p_{K_s}$ in accordance with (12). The PDF $p_{\rm f}(x_{\rm f}; t)$ in (23) describes the temporal evolution of predictive uncertainty between these two asymptotes, with Fig. 1 providing snapshots at dimensionless times $t = 0.01$, 0.1, and 0.5. (For the soil parameters reported in Table 1 of [26], this corresponds to dimensional times 1.5, 15, and 75 min, respectively). The uncertainty in predictions of infiltration depth increases rapidly, as witnessed by wider distributions with longer tails.

4.2 PDF of Infiltration Rate

Figure 2 provides snapshots, at dimensionless times $t = 0.01$, 0.1, and 0.5, of the temporal evolution of the PDF of infiltration rate $p_{\rm i}(i; t)$ given by (24). Both the mean infiltration rate and the corresponding predictive uncertainty decrease with time. At later times (the dimensionless time $t = 5.0$, for the parameters used in these simulations), the PDF appears to become time invariant. This is to be expected on theoretical grounds, see (12), according to which $p_{\rm i}(i'; t') \to p_K(K_s')$ as $t' \to \infty$. The reduced χ^2 test confirmed this asymptotic behavior at dimensionless time $t = 100.0$.

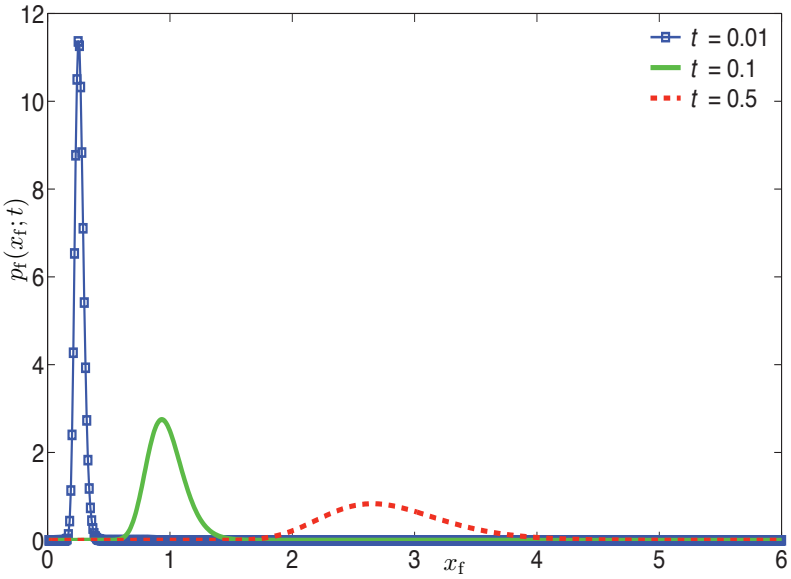

FIG. 1: Temporal evolution of the PDF of infiltration depth $p_{\mathrm{f}}(x_{\mathrm{f}};t)$.

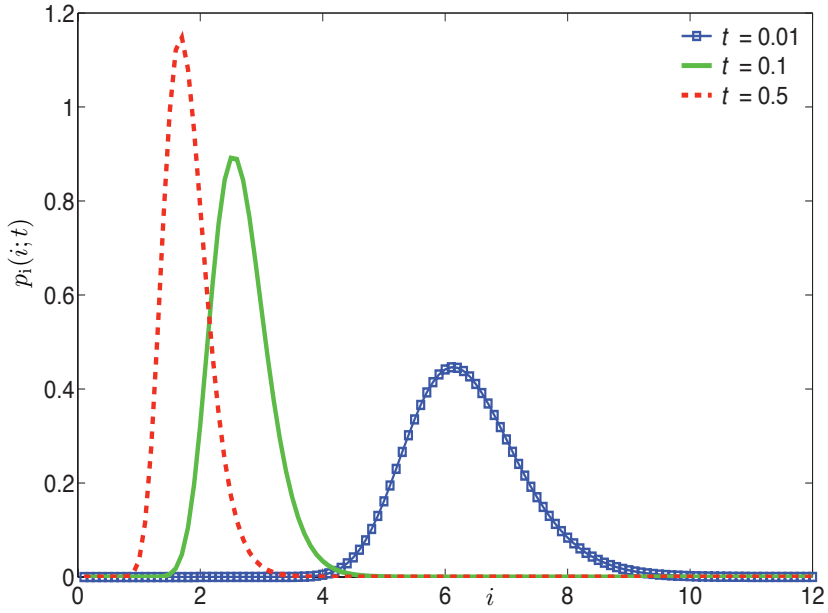

FIG. 2: Temporal evolution of the PDF of the infiltration rate $p_{\mathrm{i}}(i;t)$.

4.3 Effects of Parametric Uncertainty

The degree of uncertainty in hydraulic parameters $\ln K_s$ and $\ln \alpha_G$ is encapsulated in their coefficients of variation $CV_{\ln K}$ and CV_{α}, respectively. Figure 3 demonstrates the relative effects of these two sources of uncertainty upon the predictive uncertainty, as quantified by the infiltration depth PDF $p_{\mathrm{f}}(x_{\mathrm{f}};t)$, computed at $t = 0.1$. Uncertainty in saturated hydraulic conductivity K_s affects predictive uncertainty more than uncertainty in the Gardner parameter α_G does. Although not shown in Fig. 3, we found similar behavior at later times $t = 0.5$ and 1.0. These findings are in

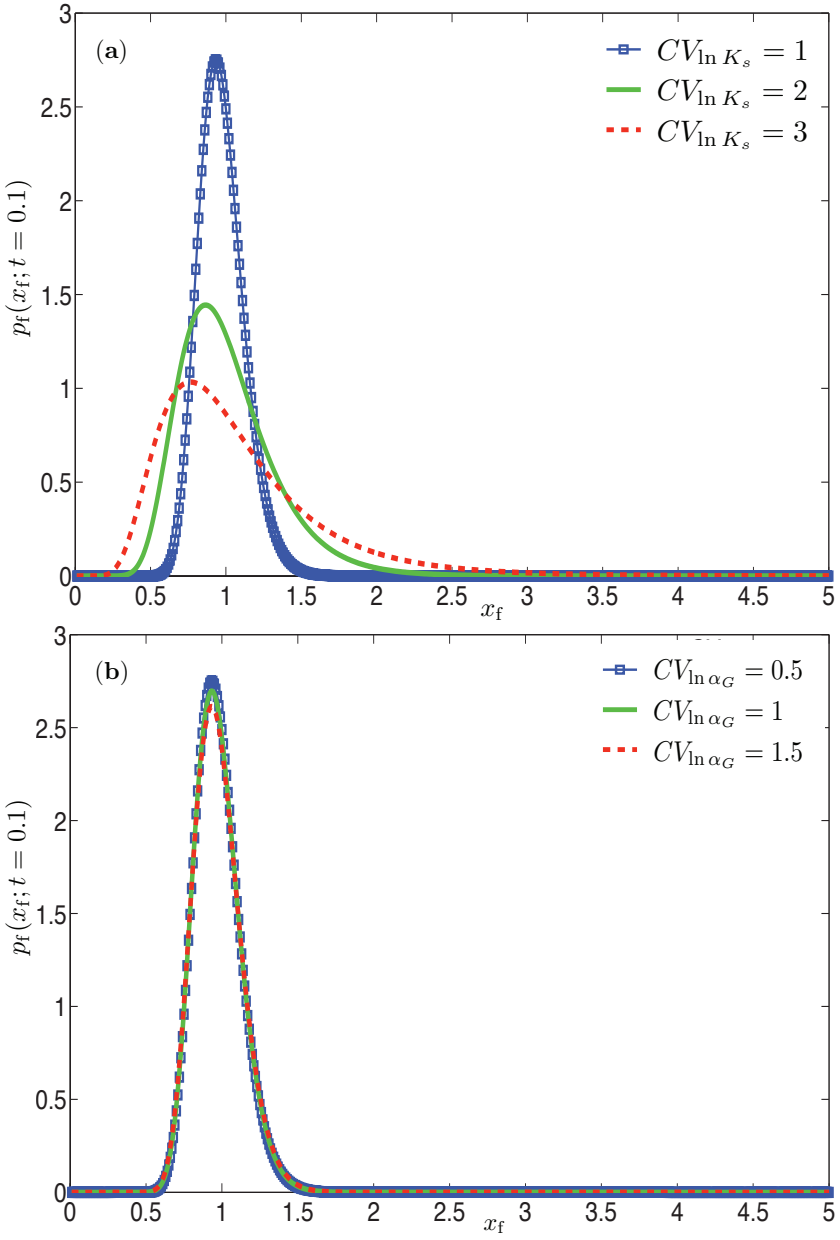

FIG. 3: The infiltration depth PDF $p_f(x_f; t = 0.1)$ for different levels of uncertainty in (a) saturated hydraulic conductivity K_s and (b) the Gardner parameter α_G.

agreement with those reported in [17, 31], wherein variances of state variables were used to conclude that uncertain saturated hydraulic conductivity K_s is the dominant factor affecting predictive uncertainty.

4.4 Effects of Cross-Correlation

The question of whether various hydraulic parameters are correlated with each other remains open, with different data sets supporting opposite conclusions (see Section 2.1). This suggests that the presence or absence of such cross-

correlations is likely to be site-specific rather than universal. The general PDF solution (23) enables us to investigate the impact of cross-correlations between saturated hydraulic conductivity K_s and the Gardner parameter α_G on predictive uncertainty. This is done by exploring the dependence of the PDF of the wetting front $p_{\mathrm{f}}(x_{\mathrm{f}}; t)$ on the correlation coefficient ρ. Figure 4 presents $p_{\mathrm{f}}(x_{\mathrm{f}}; t = 0.1)$ for $\rho = -0.99, 0.0$, and 0.99, which represent perfect anti-correlation, independence, and perfect correlation between K_s and α_G, respectively. The perfect correlation between K_s and α_G ($\rho = 0.99$) results in the minimum predictive uncertainty (the width of the distribution), while the perfect anticorrelation ($\rho = -0.99$) leads to the maximum predictive uncertainty. Predictive uncertainty resulting from the lack of correlation between K_s and α_G ($\rho = 0.0$) falls amid these two limits. The impact of cross-correlation between soil hydraulic parameters (a value of ρ) decreases with time, falling from the maximum difference of about 21% at $t = 0.01$ to about 3% at $t = 0.1$.

4.5 Effects of Selection of Hydraulic Function

Finally, we examine how the choice of a hydraulic function $K_r(\psi; \boldsymbol{\alpha})$ affects predictive uncertainty. Guided by the data analyses presented in Section 2.1, we treat α_{vG} as the only uncertain parameter in the van Genuchten hydraulic function with $n = 1.5$. To make a meaningful comparison between predictions based on the Gardner (1) and van Genuchten (2) relations, we select statistics of their respective parameters $\boldsymbol{\alpha}$ in a way that preserves the mean effective capillary drive defined by (7) [29, 32]. Specifically, we use the equivalence criteria to select the mean of $\ln \alpha_{vG}$ (-1.40, for the parameters used in these simulations) that maintains the same mean capillary drive as the Gardner model with $\overline{\ln \alpha_G} = 0.1$, and choose the variance of $\ln \alpha_{vG}$ as to maintain the original values of the coefficients of variation $CV_{\ln \alpha_{vG}} = CV_{\ln \alpha_G} = 0.5$. Figure 5 reveals that the choice between the van Genuchten and Gardner models has a significant effect on predictive uncertainty of the wetting front dynamics, although this influence diminishes with time. For example, the difference between the variances is 40% at $t = 0.01$ and 23% at $t = 0.1$.

5. CONCLUSION

We presented an approach for computing probability density functions (PDFs) of both infiltration rates and wetting fronts propagating through heterogeneous soils with uncertain (random) hydraulic parameters. Our analysis employs

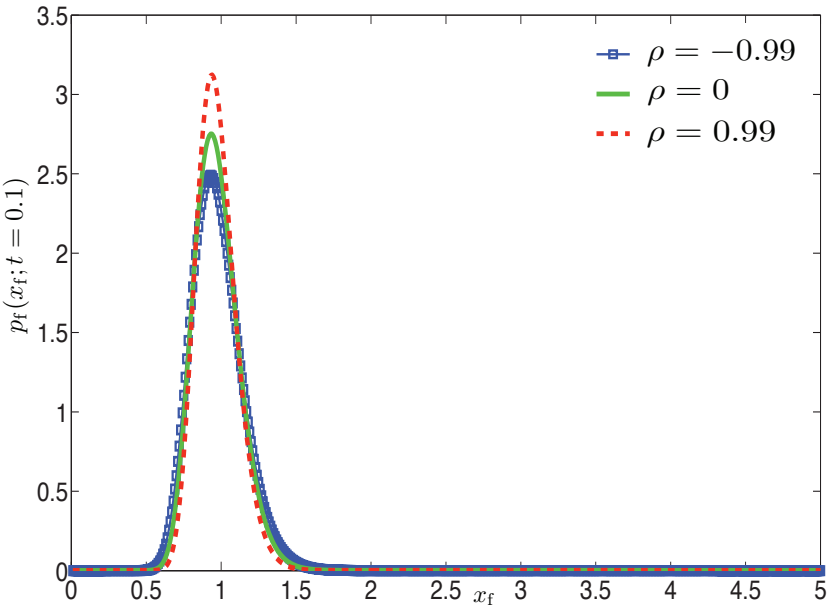

FIG. 4: The infiltration depth PDF $p_{\mathrm{f}}(x_{\mathrm{f}}; t = 0.1)$ for different levels of correlation ρ between hydraulic parameters K_s and α_G.

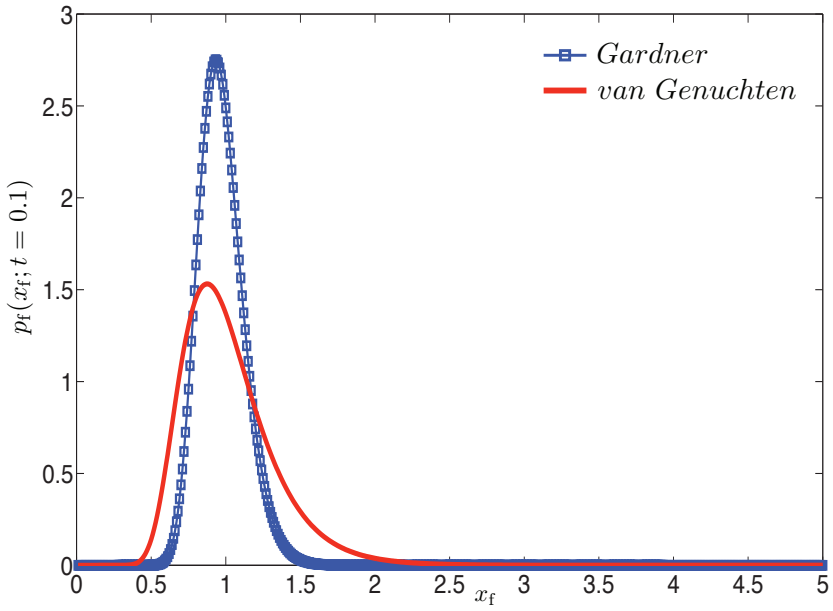

FIG. 5: The infiltration depth PDF $p_f(x_f; t = 0.1)$ resulting from use of the Gardner and van Genuchten hydraulic functions.

the Green–Ampt model of infiltration and the Dagan–Bresler statistical parameterization of soil properties. Our analysis leads to the following major conclusions.

1. The proposed approach goes beyond uncertainty quantification based on mean and variance of system states by computing their PDFs. This enables one to evaluate probabilities of rare events, which are necessary for probabilistic risk assessment.
2. Both the type and parameters of the PDF of a wetting front's depth change with time. As time increases, so does the width of the PDF, reflecting the increased predictive uncertainty.
3. Both the type and parameters of the PDF of infiltration rate change at early time. At large times, the PDF of infiltration rate coincides with the PDF of saturated hydraulic conductivity, which can serve as the lower bound of uncertainty associated with predictions of infiltration rate.
4. Predictive uncertainty is most sensitive to uncertainty in the saturated hydraulic conductivity K_s. Tripling the coefficient of variation of $\ln K_s$ significantly affects the shape of the infiltration depth PDF, while the effects of tripling the coefficient of variation of $\ln \alpha_G$ (a measure of uncertainty about the Gardner parameter α_G) are relatively insignificant.
5. The degree of correlation between the hydraulic parameters K_s and α_G has considerable influence on predictive uncertainty at early times and diminishes at later times.
6. The choice of a functional form of the hydraulic function (e.g., the Gardner model vs the van Genuchten model) has a significant effect on predictive uncertainty during early stages of infiltration. This effect diminishes with time.

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