

A perturbation solution to the transient Henry problem for seawater intrusion

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Henry's formulation of seawater intrusion in coastal aquifers consists of a fully coupled system of flow and transport equations with variable density. While the Henry problem often serves as a benchmark for numerical codes, the accuracy of the existing analytical solutions is hard to gauge. We use a perturbation expansion in a small parameter, the ratio between the densities of seawater and freshwater, to derive an analytical solution for the transient Henry problem, which describes the saline intrusion caused by a sudden change in fresh groundwater discharge. This approach is effective for other variable density flow scenarios, since it allows one to decouple the flow and transport equations.

1. INTRODUCTION

The intrusion of seawater into coastal aquifers poses significant environmental and economical challenges around the world. Modeling this phenomenon is also scientifically and computationally challenging due to the complexity of governing equations. Two main approaches to describing the seawater intrusion have emerged.

The first approach assumes that a sharp interface separates the body of fresh water from the intruding saltwater. By ignoring the existence of a transitional zone this approach recasts the problem in terms of potential theory. The use of its tools, such as conformal mapping, has led to a number of analytical solutions [1–3], which provide invaluable insights into the phenomenon. There exist, however, a plethora of physical conditions for which the width of the transitional zone cannot be neglected. This is especially so since even the minute concentration of salt can make the fresh water unpotable.

The second approach, which we adopt here, accounts for the presence of the transitional zone. It is based on a combination of the Darcian, variable-density flow equations and the

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advection-dispersion equation. However, numerically solving the resulting system of fully coupled differential equations might be problematic. Indeed, even for the Henry problem, which describes one of the simplest flow configurations, various numerical techniques [4–7] lead to quantitatively different solutions [8]. The present study introduces a perturbation approach, which effectively decouples the flow and transport equations, thus making their numerical or analytical modeling more tractable.

Section 2 provides a mathematical formulation of a problem of transient seawater intrusion into a coastal aquifer. In Section 4 we propose a perturbation approach, which is applicable to a wide class of variable-density flows, and use it to obtain an analytical solution of the transient Henry problem.

2. PROBLEM FORMULATION

The intrusion of salt water into coastal aquifers is described by a coupled system of flow and transport equations. Flow is governed by Darcy's law and mass conservation,

$$\mathbf{q} = -\frac{\kappa}{\mu}(\nabla p - \rho \mathbf{g}) \quad \text{and} \quad \nabla \cdot (\rho \mathbf{q}) + n \frac{\partial \rho}{\partial t} = 0, \quad (1)$$

where the saltwater density ρ [ML^{-3}] is related to the fresh water density ρ_f by an empirical model

$$\rho = \rho_f(1 + Ec). \quad (2)$$

Here \mathbf{q} is the specific discharge [LT^{-1}], κ is the aquifer permeability [L^2], μ is the dynamic viscosity [$ML^{-1}T^{-1}$], p is the saltwater pressure, $\mathbf{g} = (0, -g)^T$ is the gravitational force, E is the empirical model parameter [L^3M^{-1}], and c is the salt concentration in salt water [$L^{-3}M$].

Employing the Boussinesque approximation,

$$\rho \approx \rho_f \quad \text{and} \quad n \frac{\partial \rho}{\partial t} \approx n \frac{d\rho_f}{dp} \frac{\partial p}{\partial t} + nE\rho_f \frac{\partial c}{\partial t}, \quad (3)$$

and introducing the freshwater hydraulic head [L] and hydraulic conductivity [LT^{-1}],

$$h = \frac{p}{\rho_f g} + z \quad \text{and} \quad K = \frac{\kappa \rho_f g}{\mu}, \quad (4)$$

transforms the flow equation (1) into

$$K \nabla^2 h + KE \frac{\partial c}{\partial z} = S_s \frac{\partial h}{\partial t} + nE \frac{\partial c}{\partial t}. \quad (5)$$

Since changes in hydraulic head propagate through the system much faster than changes in concentration, (5) can be approximated by

$$K \nabla^2 h + KE \frac{\partial c}{\partial z} = nE \frac{\partial c}{\partial t}. \quad (6)$$

The saltwater concentration c satisfies the advection-dispersion equation

$$D\Delta c - \nabla \cdot (c\mathbf{v}) = \frac{\partial c}{\partial t}, \quad (7)$$

where D is the salt dispersion coefficient [L^2T^{-1}] treated as a constant in the Henry model, and $\mathbf{v} = \mathbf{q}/n$ is the saltwater velocity.

Let c_s denote the seawater concentration, and introduce the dimensionless concentration $c_d = c/c_s$. This gives rise to a dimensionless parameter $\epsilon = Ec_s$, whose typical value, according to data in [9,8], is $\epsilon \approx 0.03$. The flow equation (6) becomes

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} = -\epsilon \frac{\partial c_d}{\partial z} + \frac{n\epsilon}{K} \frac{\partial c_d}{\partial t}. \quad (8)$$

Since

$$q_x = -\frac{\kappa}{\mu} \frac{\partial p}{\partial x} = -K \frac{\partial h}{\partial x} \quad \text{and} \quad q_z = -K \left(\frac{\partial h}{\partial z} + Ec \right), \quad (9)$$

the transport equation (7) takes the form

$$\frac{nD}{K} \Delta c_d + \frac{\partial h}{\partial x} \frac{\partial c_d}{\partial x} + \frac{\partial h}{\partial z} \frac{\partial c_d}{\partial z} + c_d \Delta h + \epsilon \frac{\partial c_d^2}{\partial z} = \frac{n}{K} \frac{\partial c_d}{\partial t}. \quad (10)$$

This head-concentration formulation of the transient Henry problem is similar to that used by Huyakorn et al. [10] in their numerical simulations.

Denoting the aquifer thickness by d and introducing the dimensionless quantities

$$x_d = \frac{x}{d}, \quad z_d = \frac{z}{d}, \quad t_d = \frac{Kt}{nd}, \quad h_d = \frac{h}{d}, \quad P = \frac{nD}{Kd} \quad (11)$$

yields a dimensionless form of the transient Henry problem,

$$\frac{\partial^2 h_d}{\partial x_d^2} + \frac{\partial^2 h_d}{\partial z_d^2} = -\epsilon \frac{\partial c_d}{\partial z_d} + \epsilon \frac{\partial c_d}{\partial t_d} \quad (12)$$

and

$$P \left(\frac{\partial^2 c_d}{\partial x_d^2} + \frac{\partial^2 c_d}{\partial z_d^2} \right) + \frac{\partial h_d}{\partial x_d} \frac{\partial c_d}{\partial x_d} + \frac{\partial h_d}{\partial z_d} \frac{\partial c_d}{\partial z_d} + c_d \Delta h_d + \epsilon \frac{\partial c_d^2}{\partial z_d} = \frac{\partial c_d}{\partial t_d}. \quad (13)$$

The two dimensionless parameters ϵ and P replace the dimensionless parameters a and b used in most analyses of the Henry problem, e.g., [9,7,8]. Our parameters ϵ and P are intrinsic characteristics of an aquifer, while a and b depend, through their dependence on the volumetric freshwater inflow rate Q , on a flow regime as well. Another advantage is that ϵ is *small*. We will use this fact to develop perturbation solutions in the following section.

3. INITIAL AND BOUNDARY CONDITIONS

We consider the seawater intrusion into a rectangular two-dimensional aquifer of length l and height d , so that $0 \leq x_d \leq \xi$ and $0 \leq z_d \leq 1$, where $\xi = l/d$ is the dimensionless length (aspect ratio). The horizontal boundaries, $z_d = 0, 1$, are impermeable to both flow and transport. Hydrostatic head is prescribed along the vertical boundaries, $x_d = 0, \xi$. Finally, a classical formulation of the Henry problem prescribes saltwater concentrations at $x_d = 0, \xi$.

For the head-concentration formulation, these conditions translate into

$$h_d(0, z_d) = H'_d \equiv H'/d, \quad h_d(\xi, z_d) = 1 + \epsilon(1 - z_d), \quad (14a)$$

$$\frac{\partial h_d(x_d, z_d = 0)}{\partial z_d} = -\epsilon c_d(x_d, 0), \quad \frac{\partial h_d(x_d, z_d = 1)}{\partial z_d} = -\epsilon c_d(x_d, 1), \quad (14b)$$

and

$$c_d(0, z_d) = 0, \quad c_d(\xi, z_d) = 1, \quad \frac{\partial c_d(x_d, z_d = 0)}{\partial z_d} = 0, \quad \frac{\partial c_d(x_d, z_d = 1)}{\partial z_d} = 0. \quad (14c)$$

Here $H' > d$ is the hydraulic head at the fresh water inlet $[L]$. We further assume that the aquifer was initially free of saltwater, so that $c_d = 0$ at $t = 0$. At time $t = 0$, hydraulic head drops from H'_d to H_d , which is low enough (e.g., $H_d = 1$) to enable the saline intrusion.

4. PERTURBATION SOLUTION

In the following, we drop the subscript d . Since ϵ is small, we expand hydraulic head and concentration in asymptotic series,

$$h = \sum_{k=0}^{\infty} \epsilon^k h_k, \quad c = \sum_{k=0}^{\infty} \epsilon^k c_k. \quad (15)$$

Substituting (15) into (12) – (14) and collecting terms of the equal powers of ϵ yields, for $k \geq 1$,

$$\frac{\partial^2 h_k}{\partial x^2} + \frac{\partial^2 h_k}{\partial z^2} = -\frac{\partial c_{k-1}}{\partial z} + \frac{\partial c_{k-1}}{\partial t} \quad (16)$$

and

$$P \left(\frac{\partial^2 c_k}{\partial x^2} + \frac{\partial^2 c_k}{\partial z^2} \right) + \sum_{i=0}^k \left(\frac{\partial h_{k-i}}{\partial x} \frac{\partial c_i}{\partial x} + \frac{\partial h_{k-i}}{\partial z} \frac{\partial c_i}{\partial z} + c_i \Delta h_{k-i} + \frac{\partial c_i c_{k-i}}{\partial z} \right) = \frac{\partial c_k}{\partial t}. \quad (17)$$

These recursive equations are subject to the boundary conditions

$$h_k(0, z) = 0, \quad h_1(\xi, z) = 1 - z, \quad h_k(\xi, z) = 0 \quad (k \geq 2), \quad (18a)$$

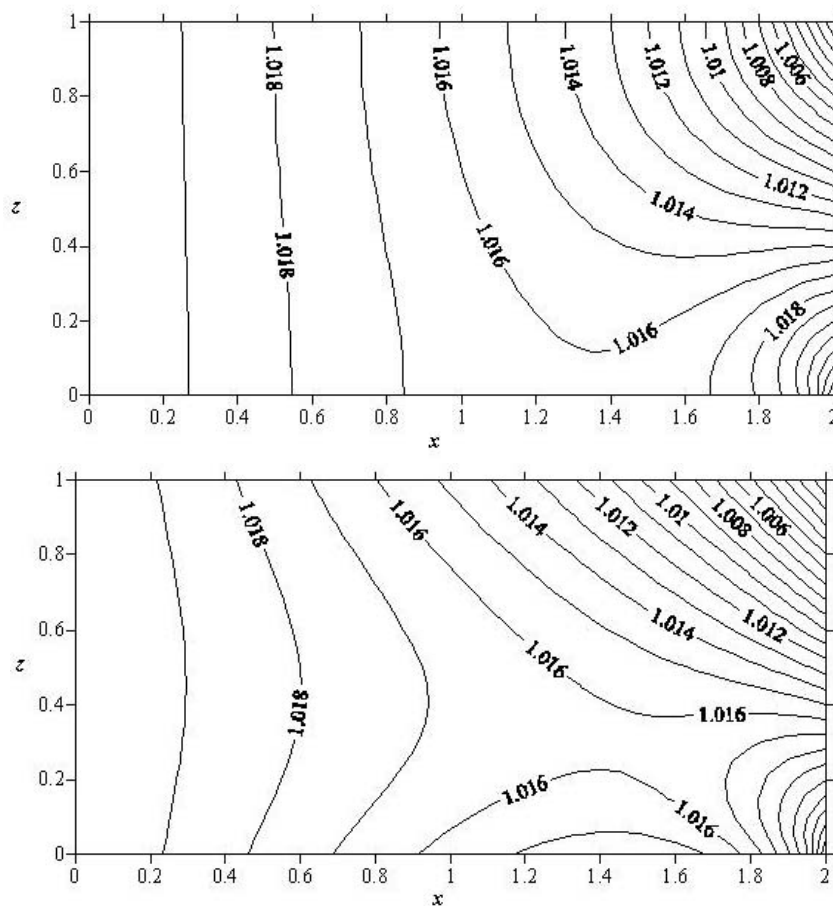


Figure 1. Spatial distributions of the dimensionless freshwater hydraulic head $h^{[1]}$ at the dimensionless times $t = 1$ (a) and $t = 100$ (b). The latter corresponds to steady state.

$$\frac{\partial h_k(x, z=0)}{\partial z} = -c_{k-1}(x, 0), \quad \frac{\partial h_k(x, z=1)}{\partial z} = -c_{k-1}(x, 1), \quad (18b)$$

and

$$c_k(0, z) = 0, \quad c_k(\xi, z) = 0, \quad \frac{\partial c_k(x, z=0)}{\partial z} = 0, \quad \frac{\partial c_k(x, z=1)}{\partial z} = 0. \quad (18c)$$

The leading terms ($k = 0$) in these expansions satisfy

$$\frac{\partial^2 h_0}{\partial x^2} + \frac{\partial^2 h_0}{\partial z^2} = 0 \quad (19)$$

and

$$P \left(\frac{\partial^2 c_0}{\partial x^2} + \frac{\partial^2 c_0}{\partial z^2} \right) + \frac{\partial h_0}{\partial x} \frac{\partial c_0}{\partial x} + \frac{\partial h_0}{\partial z} \frac{\partial c_0}{\partial z} + c_0 \Delta h_0 = \frac{\partial c_0}{\partial t}, \quad (20)$$

subject to the boundary conditions

$$h_0(0, z) = H, \quad h_0(\xi, z) = 1, \quad (21a)$$

$$\frac{\partial h_0(x, z = 0)}{\partial z} = 0, \quad \frac{\partial h_0(x, z = 1)}{\partial z} = 0, \tag{21b}$$

and

$$c_0(0, z) = 0, \quad c_0(\xi, z) = 1, \quad \frac{\partial c_0(x, z = 0)}{\partial z} = 0, \quad \frac{\partial c_0(x, z = 1)}{\partial z} = 0. \tag{21c}$$

4.1. Zeroth-order solutions

Solving (19) subject to (21) yields the zeroth-order approximation of the freshwater hydraulic head,

$$h_0 = -Jx + H, \quad \text{where} \quad J = \frac{H - 1}{\xi} > 0. \tag{22}$$

Likewise, solving the resulting equation (20) subject to (21) leads to the zeroth-order approximation of the concentration of saltwater,

$$c_0 = \frac{e^{Jx/P} - 1}{e^{J\xi/P} - 1} + \frac{2\pi P}{\xi^2} e^{-A(\xi-x)} \sum_{n=1}^{\infty} \frac{(-1)^n n}{P\pi^2 n^2 / \xi^2 - B} \sin \frac{\pi n x}{\xi} e^{(B - P\pi^2 n^2 / \xi^2)t}, \tag{23}$$

where $A = J/(2P)$ and $B = -J^2/(4P)$.

Both solutions are independent of the vertical coordinate z , and only the zeroth-order approximation of the freshwater hydraulic head varies with time.

4.2. Solution for h_1

Even though the higher-order terms in the perturbation expansion of the freshwater hydraulic head (15) are time-dependent, they satisfy the Poisson equation (16). Let $G(\mathbf{x}, \mathbf{x}')$ denote the Green's function for (16) and (18a). We show in Appendix that for $x \geq x'$ it is given by

$$G = x' - \frac{xx'}{\xi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cosh \pi n(\xi - |x - x'|) - \cosh \pi n(\xi - x - x')}{n \sinh \pi n \xi} \cos \pi n z \cos \pi n z' \tag{24a}$$

and for $x < x'$ by

$$G = x - \frac{xx'}{\xi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\cosh \pi n(\xi - |x - x'|) - \cosh \pi n(\xi - x - x')}{n \sinh \pi n \xi} \cos \pi n z \cos \pi n z'. \tag{24b}$$

Writing h_1 in terms of the Green's function G leads, after some algebraic manipulations, to the following solution

$$h_1(x, z, t) = I_1(x, t) + I_2(x, z) + I_3(x, z, t), \tag{25}$$

where

$$I_1 = \frac{2\pi P}{\xi^2} e^{-\frac{J\xi}{2P}} \left[\left(1 - \frac{x}{\xi}\right) \int_0^x e^{\frac{Jx'}{2P}} \sum_{j=1}^{\infty} (-1)^j j \sin \frac{\pi j x'}{\xi} e^{(B - \pi^2 P j^2 / \xi^2)t} x' dx' \right. \\ \left. + x \int_x^{\xi} e^{\frac{Jx'}{2P}} \sum_{j=1}^{\infty} (-1)^j j \sin \frac{\pi j x'}{\xi} e^{(B - \pi^2 P j^2 / \xi^2)t} \left(1 - \frac{x'}{\xi}\right) dx' \right], \tag{26a}$$

$$I_2 = \frac{x}{2\xi} + \frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} \frac{\sinh[\pi(2j-1)x]}{\sinh[\pi(2j-1)\xi]} \cos[\pi(2j-1)z], \quad (26b)$$

and

$$\begin{aligned} I_3 = & \frac{1}{\pi} \int_0^\xi c_0(x', t) \sum_{j=1}^{\infty} \frac{\cosh[2\pi j(\xi - x - x')]}{j \sinh(2\pi j\xi)} \cos(2\pi jz) dx' \\ & - \frac{1}{\pi} \int_0^x c_0(x', t) \sum_{j=1}^{\infty} \frac{\cosh[2\pi j(\xi - x + x')]}{j \sinh(2\pi j\xi)} \cos(2\pi jz) dx' \\ & - \frac{1}{\pi} \int_x^\xi c_0(x', t) \sum_{j=1}^{\infty} \frac{\cosh[2\pi j(\xi + x - x')]}{j \sinh(2\pi j\xi)} \cos(2\pi jz) dx'. \end{aligned} \quad (26c)$$

Figure 1 shows the spatial distributions of the first-order approximation of the freshwater hydraulic head $h^{[1]} = h_0 + \epsilon h_1$ at the dimensionless times $t = 1$ and $t = 100$. In this calculation we set $L = 200m$, $d = 100m$, and $H = 1.02m$, which results in $J = 0.01$, $A = 0.5$ and $P = 0.01$. The spatial distribution of $h^{[1]}$ at $t = 100$ corresponds to the steady state, i.e., to a solution of the standard Henry problem. This distribution qualitatively agrees with the existing numerical solutions.

5. SUMMARY

We proposed a perturbation approach for analyzing variable-density flows, which effectively decouples the flow and transport equations, thus making the derivation of their numerical or analytical solutions less problematic. The ratio between the densities of freshwater and saltwater is a small parameter used in perturbation expansions.

We used the proposed approach to obtain an analytical solution of the transient Henry problem. The first two terms in a perturbation expansion of the freshwater hydraulic head give a solution that is in qualitative agreement with the existing numerical solutions.

The accuracy and convergence of the proposed perturbation approach remains to be investigated. We also plan to apply it to more complicated flow scenario.

APPENDIX: GREEN'S FUNCTION

The Green's function for the diffusion equation subject to (18a) can be found as the product of the corresponding one-dimensional Green's functions [11],

$$G_{\text{tr}} = \frac{2}{\xi} \sum_{m=1}^{\infty} e^{-D\pi^2 m^2 t/\xi^2} \sin \frac{\pi m x}{\xi} \sin \frac{\pi m x'}{\xi} \left[1 + 2 \sum_{n=1}^{\infty} e^{-D\pi^2 n^2 t} \cos \pi n z \cos \pi n z' \right]. \quad (27)$$

Its steady-state counterpart is obtained by evaluating the limit

$$G = D \lim_{t \rightarrow \infty} \int_0^t G_{\text{tr}} d\tau, \quad (28)$$

which results in

$$G = \frac{2\xi}{\pi^2} \left[\sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{\pi mx}{\xi} \sin \frac{\pi mx'}{\xi} + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 + \xi^2 n^2} \sin \frac{\pi mx}{\xi} \sin \frac{\pi mx'}{\xi} \cos \pi n z \cos \pi n z' \right]. \quad (29)$$

Since [12, Eq. 4.3.31]

$$2 \sin \frac{\pi mx}{\xi} \sin \frac{\pi mx'}{\xi} = \cos \left(\pi m \frac{x - x'}{\xi} \right) - \cos \left(\pi m \frac{x + x'}{\xi} \right) \quad (30)$$

and [13, Eq. 1.443.3]

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad 0 \leq x \leq 2\pi, \quad (31)$$

the first sum in (29) gives

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{2}{m^2} \sin \frac{\pi mx}{\xi} \sin \frac{\pi mx'}{\xi} &= -\frac{\pi^2 |x - x'|}{2\xi} + \frac{\pi^2 (x - x')^2}{4\xi^2} + \frac{\pi^2 (x + x')}{2\xi} - \frac{\pi^2 (x + x')^2}{4\xi^2} \\ &= \pi^2 \frac{x'}{\xi} \left(1 - \frac{x}{\xi} \right) \quad \text{for } x > x' \\ &= \pi^2 \frac{x}{\xi} \left(1 - \frac{x'}{\xi} \right) \quad \text{for } x < x'. \end{aligned} \quad (32)$$

Since [13, Eq. 1.445.2]

$$\sum_{m=1}^{\infty} \frac{\cos mx}{m^2 + \alpha^2} = \frac{\pi}{2\alpha} \frac{\cosh \alpha(\pi - x)}{\sinh \alpha\pi} - \frac{1}{2\alpha^2} \quad 0 \leq x \leq 2\pi, \quad (33)$$

the second sum in (29) gives

$$4 \sum_{m=1}^{\infty} \frac{1}{m^2 + \xi^2 n^2} \sin \frac{\pi mx}{\xi} \sin \frac{\pi mx'}{\xi} = \pi \frac{\cosh \pi n(\xi - |x - x'|) - \cosh \pi n(\xi - x - x')}{n\xi \sinh \pi n\xi}. \quad (34)$$

Thus the Green's function takes the form (24).

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