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# Stochastic modeling of heterogeneous phreatic aquifers

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**Abstract.** Phreatic flow in heterogeneous aquifers is analyzed by treating hydraulic conductivity as a random field with known statistics. A set of equations for the first and second ensemble moments of hydraulic head and phreatic surface is derived. These equations allow one to predict the behavior of phreatic aquifers, as well as to assess the uncertainty associated with such predictions. Perturbation analysis in variance of log hydraulic conductivity is employed to close the moments equations. This leads to a recursive initial boundary value problem.

## 1. Introduction

Recognition of the uncertain nature of flow and transport in natural geologic formations has spurred the development of stochastic subsurface hydrology. This approach relies on representing hydraulic conductivity of an aquifer, $K(\mathbf{x})$, as a random field with known statistics. Ensemble averaging of the stochastic flow equations provides unbiased estimations of local hydraulic head and Darcy fluxes by means of their first ensemble moments. It also provides measures of predictive uncertainty which are expressed in terms of the corresponding second ensemble moments. Stochastic analyses of flow through randomly heterogeneous porous media have been reported by *Shvidler* [1964], *Matheron* [1967], and *Freeze* [1975]. Their results have brought about a rapid growth of literature on stochastic subsurface hydrology (a review of the development of stochastic subsurface hydrology is given by *Dagan and Neuman* [1996]).

Despite notable progress in analyzing flows in confined heterogeneous aquifers, there are virtually no studies of flow with phreatic surfaces. While of considerable interest for many practical applications, "the complexity of the problem ... has defied attempts to solve it either by numerical methods or by approximate analytical ones" [*Dagan and Zeitoun*, 1998, p. 3191]. This complexity stems from a high degree of nonlinearity caused by the presence of free boundaries. *Gelhar* [1974] has studied phreatic aquifers by treating them as the lumped parameter linear reservoir, the linear Dupuit aquifer, and the linearized Laplace aquifer. *Dagan and Zeitoun* [1998] have analyzed the response of water tables to pumping by employing the Dupuit approximation and reducing heterogeneity to a perfect layering. While important as a first step in modeling heterogeneous phreatic aquifers, the above-mentioned approaches lack generality. In particular, the lumped parameter models neglect all spatial variation within a system, while the Dupuit assumption is applicable under very restrictive conditions [*Muskat*, 1946, p. 359]. Monte Carlo simulation of water tables in a heterogeneous dam has been reported by *Fenton and Griffith* [1996]. While conceptually straightforward and versatile, the Monte Carlo approach has a number of drawbacks (for detailed discussion, see *Tartakovsky et al.* [1999]).

In this paper, a deterministic alternative to Monte Carlo simulations is presented. It allows prediction of phreatic flow in randomly heterogeneous porous media without requiring any simplifying assumptions about the shape of the phreatic surface. Such a prediction is given by the solution of a stochastically averaged boundary value problem. To assess the uncertainty associated with this prediction, a set of equations for the second ensemble moments of the quantities of interest is derived. A closure for our ensemble moment equations is provided by means of a perturbation analysis in variance $\sigma_Y^2$ of log hydraulic conductivity $Y = \ln K$.



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## 2. Statement of the Problem

Consider groundwater flow in an unconfined aquifer. A typical flow region $\Omega$ can be bounded by Dirichlet segments (prescribed head boundaries and/or seepage faces) $\Gamma_D$, Neumann segments (prescribed flux boundaries) $\Gamma_N$, and phreatic surface (water table) $\Gamma_F$. Such flow is described by a combination of Darcy's law and mass conservation,

$$\mathbf{q}(\mathbf{x}, t) = -K(\mathbf{x})\nabla h(\mathbf{x}, t) \quad -\nabla \cdot \mathbf{q}(\mathbf{x}, t) + f(\mathbf{x}, t) = 0 \qquad \mathbf{x} \in \Omega(t), \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3)^T$ is the coordinate vector ($x_3$ being the vertical coordinate positive upward), (unit length (L)); $t$ is time, (unit time (T)); $\mathbf{q}(\mathbf{x}, t)$ is the flux (LT$^{-1}$); $K(\mathbf{x})$ is the hydraulic conductivity (LT$^{-1}$); $h(\mathbf{x}, t)$ is the hydraulic head (L); and $\Omega$ is the flow domain. These equations are subject to the initial and boundary conditions [e.g., *Neuman and Witherspoon*, 1971]

$$h(\mathbf{x}, 0) = h_0(\mathbf{x}) \quad \xi(x_1, x_2, 0) = \xi_0(x_1, x_2) \qquad \mathbf{x} \in \Omega(t = 0) \tag{2}$$

$$h(\mathbf{x}, t) = H(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma_D \tag{3}$$

$$\mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x}, t) = Q(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma_N \tag{4}$$

$$h(\mathbf{x}, t) = \xi(x_1, x_2, t) \quad -\mathbf{n}(\mathbf{x}) \cdot \mathbf{q}(\mathbf{x}, t) = V_n(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma_F. \tag{5}$$

Here $\xi(x_1, x_2, t)$ is the elevation of the phreatic surface (L); $h_0(\mathbf{x})$ and $\xi_0(x_1, x_2)$ are the initial head distribution and free surface elevation, respectively; $f(\mathbf{x}, t)$ is the source function (T$^{-1}$); $H(\mathbf{x}, t)$ is the prescribed hydraulic head (L); $Q(\mathbf{x}, t)$ is the prescribed flux (LT$^{-1}$); and $\mathbf{n}(\mathbf{x}) = (n_1, n_2, n_3)^T$ is the unit outward normal to the boundary $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_F$ of

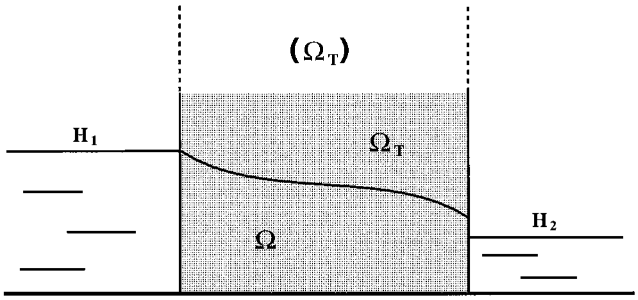


**Figure 1.** Schematic representation of phreatic flow through an earth dam. While geometry of the flow domain $\Omega$ changes with time, the total domain $\Omega_T$ is time-invariant. Here $\Omega_T$ can be either rectangle representing the dam or semi-infinite strip.

the flow domain $\Omega$. Normal velocity of the phreatic surface $V_n$ is given by [*Neuman and Witherspoon*, 1971]

$$V_n(\mathbf{x}, t) = \left(\varepsilon - S_y \frac{\partial \xi}{\partial t}\right) n_3(\mathbf{x}, t), \tag{6}$$

where $\varepsilon(x_1, x_2, \xi, t)$ is the vertical rate of infiltration at the free surface ($LT^{-1}$) and $S_y(\mathbf{x})$ is the specific yield.

A high degree of spatial variability in the hydraulic conductivity $K(\mathbf{x})$ and a lack of detailed information about its spatial distribution can be conveniently modeled by treating $K(\mathbf{x})$ as a random field. Additionally, the driving forces $\varepsilon(\mathbf{x}, t)$, $f(\mathbf{x}, t)$, $H(\mathbf{x}, t)$, and $Q(\mathbf{x}, t)$ are often uncertain, although this source of uncertainty is not considered here. The specific yield $S_y(\mathbf{x})$ is assumed to be deterministic. Because of randomness in the input parameter, (1)–(6) constitute a stochastic initial boundary value problem with free surfaces.

In what follows, it is assumed that available experimental data allow one to infer the statistics of the random field $K(\mathbf{x})$. This field does not have to be statistically homogeneous. The aim is to derive a set of deterministic equations for estimating mean dynamics of the phreatic surface $\bar{\xi}(x_1, x_2, t)$.

To facilitate stochastic averaging, we recast (1)–(6) in the form of an integral equation,

$$\begin{aligned} &-\int_\Omega \nabla_{\mathbf{y}} \cdot [\bar{K}(\mathbf{y})\nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})] h(\mathbf{y}, t)\, d\mathbf{y} \\ &= \int_\Omega \nabla_{\mathbf{y}} \cdot [K'(\mathbf{y})\nabla_{\mathbf{y}} h(\mathbf{y}, t)] G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &+ \int_\Omega f(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &+ \int_\Gamma \bar{K}(\mathbf{y})\mathbf{n} \cdot [G(\mathbf{y}, \mathbf{x})\nabla_{\mathbf{y}} h(\mathbf{y}, t) - h(\mathbf{y}, t)\nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})]\, d\mathbf{y}. \end{aligned} \tag{7}$$

Here the random field of hydraulic conductivity $K$ is represented as a sum of its mean $\bar{K}$ and zero-mean perturbations $K'$ about them, $K = \bar{K} + K'(\bar{K'} = 0)$, and $G(\mathbf{y}, \mathbf{x})$ is an arbitrary function. Defining $G(\mathbf{y}, \mathbf{x})$ as the deterministic, time-invariant Green's function which satisfies

$$\nabla_{\mathbf{y}} \cdot [\bar{K}(\mathbf{y})\nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})] + \delta(\mathbf{y} - \mathbf{x}) = 0 \qquad \mathbf{y}, \mathbf{x} \in \Omega_T \tag{8}$$

subject to the boundary conditions

$$G(\mathbf{y}, \mathbf{x}) = 0 \qquad \mathbf{y} \in \Gamma_D \tag{9}$$

$$\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x}) = 0 \qquad \mathbf{y} \in \Gamma_N \tag{10}$$

and applying Green's formula lead to

$$\begin{aligned} h(\mathbf{x}, t) = &-\int_\Omega K'(\mathbf{y})\nabla_{\mathbf{y}} h(\mathbf{y}, t) \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &+ \int_\Omega f(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} - \int_{\Gamma_N} Q(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &- \int_{\Gamma_D} H(\mathbf{y}, t)\bar{K}(\mathbf{y})\mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &+ \int_{\Gamma_F} [V_n(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x}) \\ &\qquad - \bar{K}(\mathbf{y})\xi(\mathbf{y}, t)\mathbf{n}(\mathbf{y}, t) \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})]\, d\mathbf{y}. \end{aligned} \tag{11}$$

Since the Green's function $G(\mathbf{y}, \mathbf{x})$ is defined for the entire domain $\Omega_T$ rather than just for the flow domain $\Omega$, there are no conditions on $G$ along the phreatic surface $\Gamma_F$. Specifying $G$ for the flow domain $\Omega$ would require recalculating $G$ at each time as $\Omega$ evolves, which is not computationally expedient. The definition of $\Omega_T$ is not unique, but its boundary must include those of $\Omega$. For instance, in a classical problem of phreatic flow through a rectangular earth dam, $\Omega_T$ can be either rectangle representing the dam or semi-infinite strip of the same width (Figure 1).

Integral equation (11) serves as a starting point in the stochastic analysis.

## 3. Averaged Flow Equation

In (11), representing random fields as sums of means and zero-mean perturbations, $h = \bar{h} + h'$, $\mathbf{q} = \bar{\mathbf{q}} + \mathbf{q}'$, $V_n = \bar{V}_n + V'_n$, and $\mathbf{n} = \bar{\mathbf{n}} + \mathbf{n}'$ on $\Gamma_F$, and taking the ensemble mean give

$$\begin{aligned} \bar{h}(\mathbf{x}, t) = &\int_\Omega \mathbf{r}(\mathbf{y}, t) \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} + \int_\Omega f(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &- \int_{\Gamma_N} Q(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &- \int_{\Gamma_D} H(\mathbf{y}, t)\bar{K}(\mathbf{y})\mathbf{n} \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})\, d\mathbf{y} \\ &+ \int_{\Gamma_F} \{\bar{V}_n(\mathbf{y}, t) G(\mathbf{y}, \mathbf{x}) - \bar{K}(\mathbf{y}) \\ &\cdot [\bar{\xi}(\mathbf{y}, t)\bar{\mathbf{n}}(\mathbf{y}, t) + C_{\xi\mathbf{n}}(\mathbf{y}, \mathbf{y}, t)] \cdot \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x})\}\, d\mathbf{y}, \end{aligned} \tag{12}$$

where $\mathbf{r}(\mathbf{x}, t) = -\overline{K'(\mathbf{x})\nabla h'(\mathbf{x}, t)}$ is the "residual" flux and $C_{\xi\mathbf{n}}(\mathbf{x}, \mathbf{y}, t) = \overline{\xi'(\mathbf{x}, t)\mathbf{n}'(\mathbf{y}, t)}$. To derive expressions for the residual flux for flow domains with fixed boundaries, *Neuman and Orr* [1993] and *Cushman* [1997] have used random and deterministic Green's functions, respectively.

We show in the appendix that the residual flux $\mathbf{r}(\mathbf{x}, t)$ is given implicitly by

$$\begin{aligned}\mathbf{r}(\mathbf{x}, t) = &\int_\Omega C_K(\mathbf{x}, \mathbf{y})\nabla_\mathbf{x}\nabla_\mathbf{y}^T G(\mathbf{y}, \mathbf{x})\nabla_\mathbf{y}\bar{h}(\mathbf{y}, t)\, d\mathbf{y} \\ &+ \int_\Omega \nabla_\mathbf{x}\nabla_\mathbf{y}^T G(\mathbf{y}, \mathbf{x})\overline{K'(\mathbf{x})K'(\mathbf{y})\nabla_\mathbf{y} h'(\mathbf{y})}\, d\mathbf{y} \\ &- \int_{\Gamma_F} \{C_{KV}(\mathbf{x}, \mathbf{y}, t)\nabla_\mathbf{x} G(\mathbf{y}, \mathbf{x}) - \bar{K}(\mathbf{y})\nabla_\mathbf{x}\nabla_\mathbf{y}^T G(\mathbf{y}, \mathbf{x}) \\ &\cdot [C_{K\xi}(\mathbf{x}, \mathbf{y}, t)\bar{\mathbf{n}}(\mathbf{y}, t) + \bar{\xi}(\mathbf{y}, t) C_{K\mathbf{n}}(\mathbf{x}, \mathbf{y}, t) \\ &+ \overline{K'(\mathbf{x})\xi'(\mathbf{y}, t)\mathbf{n}'(\mathbf{y}, t)}]\}\, d\mathbf{y}. \qquad (13)\end{aligned}$$

To complete this system of equations, one needs to evaluate the cross covariances $C_{\xi\mathbf{n}}(\mathbf{x}, \mathbf{y}, t)$, $C_{KV}(\mathbf{x}, \mathbf{y}, t) = \overline{K'(\mathbf{x})V_n'(\mathbf{y}, t)}$, $C_{K\xi}(\mathbf{x}, \mathbf{y}, t) = \overline{K'(\mathbf{x})\xi'(\mathbf{y}, t)}$, and $C_{K\mathbf{n}}(\mathbf{x}, \mathbf{y}, t) = \overline{K'(\mathbf{x})\mathbf{n}'(\mathbf{y}, t)}$ and to provide a closure for the third mixed moments. Such a closure is derived below by means of a perturbation analysis in the small parameter $\sigma_Y^2$, the variance of log hydraulic conductivity $Y = \ln K$.

Expanding $K$, $h$, $\mathbf{q}$, $\xi$, $G$, and $\mathbf{n}$ in powers of $Y'(\mathbf{x})$ and collecting terms of the power $\sigma_Y^2$ yields the first-order (in $\sigma_Y^2$) approximation of the residual flux,

$$\begin{aligned}\mathbf{r}^{(1)}(\mathbf{x}, t) = &K_g(\mathbf{x})\int_\Omega K_g(\mathbf{y}) C_Y(\mathbf{x}, \mathbf{y})\nabla_\mathbf{x}\nabla_\mathbf{y}^T G^{(0)}(\mathbf{y}, \mathbf{x})\nabla_\mathbf{y}\bar{h}^{(0)}(\mathbf{y}, t)\, d\mathbf{y} \\ &- \int_{\Gamma_F} \{C_{KV}^{(1)}(\mathbf{x}, \mathbf{y}, t)\nabla_\mathbf{x} G^{(0)}(\mathbf{y}, \mathbf{x}) - K_g(\mathbf{y})\nabla_\mathbf{x}\nabla_\mathbf{y}^T G^{(0)}(\mathbf{y}, \mathbf{x}) \\ &\cdot [C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t)\bar{\mathbf{n}}^{(0)}(\mathbf{y}, t) + \bar{\xi}^{(0)}(\mathbf{y}, t) C_{K\mathbf{n}}^{(1)}(\mathbf{x}, \mathbf{y}, t)]\}\, d\mathbf{y}. \qquad (14)\end{aligned}$$

Here the geometric mean $K_g = \exp(\bar{Y})$ is the zeroth-order approximation of $K$, and $C_Y$ is the correlation function of $Y$. The zeroth-order approximations, $\bar{h}^{(0)}(\mathbf{x}, t)$, $\bar{\xi}^{(0)}(x_1, x_2, t)$, and $\bar{\mathbf{n}}^{(0)}(\mathbf{x}, t)$ are the solutions of the corresponding phreatic flow problem in the medium of conductivity $K_g$. As such, they can be easily found by standard methods. The same holds for the zeroth-order approximation of the Green's function $G^{(0)}(\mathbf{y}, \mathbf{x})$.

It remains to evaluate the first-order approximations of the cross covariances $C_{KV}$, $C_{\xi\mathbf{n}}$, $C_{K\xi}$, and $C_{K\mathbf{n}}$. It follows from (6) and the derivations in the appendix that

$$\begin{aligned}C_{KV}^{(1)}(\mathbf{x}, \mathbf{y}, t) = &\varepsilon C_{Kn_3}(\mathbf{x}, \mathbf{y}, t) \\ &- S_y\left[\frac{\partial C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t)}{\partial t}\bar{n}_3^{(0)}(\mathbf{y}, t) + \frac{\partial \bar{\xi}^{(0)}(\mathbf{y}, t)}{\partial t} C_{Kn_3}^{(1)}(\mathbf{x}, \mathbf{y}, t)\right] \qquad (15)\end{aligned}$$

$$C_{K\mathbf{n}}^{(1)}(\mathbf{x}, \mathbf{y}, t) = \frac{\nabla_\mathbf{y} C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t)}{|\nabla_\mathbf{y}(\bar{\xi}^{(0)} - y_3)|} - \bar{\mathbf{n}}^{(0)}(\mathbf{y}, t)\frac{\bar{\mathbf{n}}^{(0)}(\mathbf{y}, t)\cdot\nabla_\mathbf{y} C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t)}{|\nabla_\mathbf{y}(\bar{\xi}^{(0)} - y_3)|} \qquad (16)$$

$$\begin{aligned}C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t) = &-K_g(\mathbf{x})\int_\Omega K_g(\mathbf{z}) C_Y(\mathbf{x}, \mathbf{z})\nabla_\mathbf{z} G^{(0)}(\mathbf{z}, \mathbf{y})\nabla_\mathbf{z}\bar{h}^{(0)}(\mathbf{z}, t)\, d\mathbf{z} \\ &+ \int_{\Gamma_F} \{C_{KV}^{(1)}(\mathbf{x}, \mathbf{z}, t) G^{(0)}(\mathbf{z}, \mathbf{y}) - K_g(\mathbf{z})\nabla_\mathbf{z} G^{(0)}(\mathbf{z}, \mathbf{y}) \\ &\cdot [C_{K\xi}^{(1)}(\mathbf{x}, \mathbf{z}, t)\bar{\mathbf{n}}^{(0)}(\mathbf{z}, t) + \bar{\xi}^{(0)}(\mathbf{z}, t) C_{K\mathbf{n}}^{(1)}(\mathbf{x}, \mathbf{z}, t)]\}\, d\mathbf{z} \qquad (17)\end{aligned}$$

$$C_{\xi\mathbf{n}}^{(1)}(\mathbf{x}, \mathbf{x}, t) = \frac{\nabla[\sigma_\xi^2(\mathbf{x}, t)]^{(1)}}{2|\nabla(\bar{\xi}^{(0)} - x_3)|} - \bar{\mathbf{n}}^{(0)}(\mathbf{x}, t)\frac{\bar{\mathbf{n}}^{(0)}(\mathbf{x}, t)\cdot\nabla[\sigma_\xi^2(\mathbf{x}, t)]^{(1)}}{2|\nabla(\bar{\xi}^{(0)} - x_3)|}. \qquad (18)$$

In (18), variance $\sigma_\xi^2(\mathbf{x}, t) = \overline{\xi'(\mathbf{x}, t)\xi'(\mathbf{x}, t)}$ represents a measure of uncertainty associated with our estimation of the mean position of the phreatic surface $\bar{\xi}$. It can be found by taking the limits $\mathbf{y} \to \mathbf{x}$ and $\tau \to t$ of front covariance, $C_\xi(\mathbf{x}, \mathbf{y}, t, \tau) = \overline{\xi'(\mathbf{x}, t)\xi'(\mathbf{y}, \tau)}$,

$$\begin{aligned}C_\xi^{(1)}(\mathbf{x}, \mathbf{y}, t, \tau) = &-\int_\Omega C_{K\xi}^{(1)}(\mathbf{z}, \mathbf{y}, t)\nabla_\mathbf{z}\bar{h}^{(0)}(\mathbf{z}, t)\cdot\nabla_\mathbf{z} G^{(0)}(\mathbf{z}, \mathbf{x})\, d\mathbf{z} \\ &+ \int_{\Gamma_F} \{C_{V\xi}^{(1)}(\mathbf{z}, \mathbf{y}, t, \tau) G^{(0)}(\mathbf{z}, \mathbf{x}) \\ &- K_g(\mathbf{z})\nabla_\mathbf{z} G^{(0)}(\mathbf{z}, \mathbf{x})\cdot[C_\xi^{(1)}(\mathbf{z}, \mathbf{y}, t, \tau)\bar{\mathbf{n}}^{(0)}(\mathbf{z}, t) \\ &+ C_{\xi\mathbf{n}}^{(1)}(\mathbf{y}, \mathbf{z}, \tau, t)\bar{\xi}^{(0)}(\mathbf{z}, t)]\}\, d\mathbf{z}, \qquad (19)\end{aligned}$$

where the first-order approximation of cross covariance $C_{V\xi}(\mathbf{x}, \mathbf{y}, t, \tau) = \overline{V_n'(\mathbf{x}, t)\xi'(\mathbf{y}, \tau)}$ has the form

$$\begin{aligned}C_{V\xi}^{(1)}(\mathbf{x}, \mathbf{y}, t, \tau) = &\varepsilon C_{\xi n_3}^{(1)}(\mathbf{y}, \mathbf{x}, \tau, t) \\ &- S_y\left[\frac{\partial C_\xi^{(1)}(\mathbf{x}, \mathbf{y}, t, \tau)}{\partial t}\bar{n}_3^{(0)}(\mathbf{x}, t) + \frac{\partial \bar{\xi}(\mathbf{x}, t)}{\partial t} C_{\xi n_3}(\mathbf{y}, \mathbf{x}, \tau, t)\right]. \qquad (20)\end{aligned}$$

Equations (15)–(20) constitute a closed system of integro-differential equations.

## 4. Discussion

A set of deterministic equations for estimating flow in unconfined aquifers and for assessing the corresponding confidence intervals is derived. While a system of nine integro-differential equations might seem cumbersome to solve, this number of equations is much smaller than hundreds or thousands of equations which need to be solved in Monte Carlo simulations. Moreover, the moment equations can be solved on numerical grids which are coarser than those typically required for Monte Carlo simulations.

The results are preliminary in that numerical implementation of the proposed approach is far from trivial. At this stage we only provide a numerical algorithm, while leaving its implementation for future research. The numerical solution of these equations can proceed in the following steps.

1. Using the geometric mean of hydraulic conductivity $K_g$ as an input parameter, solve a phreatic flow problem to obtain zeroth-order estimators of hydraulic head $\bar{h}^{(0)}(\mathbf{x}, t)$ and phreatic surface $\bar{\xi}^{(0)}(\mathbf{x}, t)$. This step can be accomplished by standard numerical methods, such as FREESURF II of *Neuman and Witherspoon* [1971].

2. Evaluate the residual flux $\mathbf{r}(\mathbf{x}, t)$ and the cross covariance $C_{\xi\mathbf{n}}(\mathbf{x}, t)$ in the mean phreatic flow equation through perturbation expansions in variance $\sigma_Y^2$ of log hydraulic conductivity $Y = \ln K$. Their first-order approximations, $\mathbf{r}^{(1)}(\mathbf{x}, t)$ and $C_{\xi\mathbf{n}}^{(1)}(\mathbf{x}, t)$, are given by (14) and (18), respectively. The coefficients in (14), $C_{KV}^{(1)}$, $C_{K\mathbf{n}}^{(1)}$, and $C_{K\xi}^{(1)}$, are the solutions of a system of triple integro-differential equations (15)–(17). Likewise, the coefficients in (18), $C_{\xi}^{(1)}$ and $C_{\xi V}^{(1)}$, are the solutions of the coupled integro-differential equations (19) and (20).

3. Solve the mean phreatic flow equation with the residual flux $\mathbf{r}(\mathbf{x}, t)$ (or, more precisely, $\nabla \cdot \mathbf{r}^{(1)}$) serving as a distributed source and with the cross covariance $C_{\xi\mathbf{n}}(\mathbf{x}, t)$ serving as accretion on the phreatic surface. Again, FREESURF II can be readily employed at this step. Alternatively, one can use an integral form of the mean flow equation (12) to evaluate $\bar{\xi}$. This can be accomplished by using the front-tracking numerical scheme of *Kessler et al*. [1984].

In general, (12) must be solved numerically. However, simple one-dimensional cases might be amenable to analytical treatment. In this case, the normal vector to the phreatic surface, $\mathbf{n}$, is deterministic and constant in space and time. In fact, it does not enter the picture at all. Consequently, all covariances which include $\mathbf{n}$ disappear. Moreover, the boundary integrals over $\Gamma_F$ are replaced with just a point $\xi$. An analytical solution for this problem in the absence of gravity was obtained by D. M. Tartakovsky and C. L. Winter (Stochastic analysis of free surfaces in randomly heterogeneous porous media, submitted to *SIAM Journal on Applied Mathematics*, 1999.)

## Appendix

Subtracting (12) from (11) gives an integral equation for the perturbations $h'(\mathbf{x}, t)$,

$$h'(\mathbf{x}, t) = -\int_\Omega [K'(\mathbf{y})\nabla_\mathbf{y} h(\mathbf{y}, t) + \mathbf{r}(\mathbf{y}, t)] \cdot \nabla_\mathbf{y} G(\mathbf{y}, \mathbf{x})\, d\mathbf{y}$$
$$+ \int_{\Gamma_F} \{V_n'(\mathbf{y}) G(\mathbf{y}, \mathbf{x}) - \bar{K}(\mathbf{y}) \nabla_\mathbf{y} G(\mathbf{y}, \mathbf{x}) \cdot [\xi'(\mathbf{y}, t)\mathbf{n}'(\mathbf{y}, t)$$
$$+ \xi'(\mathbf{y}, t)\bar{\mathbf{n}}(\mathbf{y}, t) + \bar{\xi}(\mathbf{y}, t)\mathbf{n}'(\mathbf{y}, t) - C_{\xi\mathbf{n}}(\mathbf{y}, \mathbf{y}, t)]\}\, d\mathbf{y}. \tag{A1}$$

Operating on (A1) with $K'(\mathbf{x})\nabla_\mathbf{x}$ and taking the ensemble mean lead to (13). Evaluating (A1) at $\mathbf{y} \in \Gamma_F$, multiplying with $K'(\mathbf{x})$, and taking the mean yield

$$C_{K\xi}(\mathbf{x}, \mathbf{y}, t) = -\int_\Omega C_K(\mathbf{x}, \mathbf{z}) \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{y}) \cdot \nabla_\mathbf{z} \bar{h}(\mathbf{z}, t)\, d\mathbf{z}$$
$$- \int_\Omega \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{y}) \cdot \overline{K'(\mathbf{x}) K'(\mathbf{z}) \nabla_\mathbf{z} h'(\mathbf{z})}\, d\mathbf{z}$$
$$+ \int_{\Gamma_F} \{C_{KV}(\mathbf{x}, \mathbf{z}, t) G(\mathbf{z}, \mathbf{y}) - \bar{K}(\mathbf{z}) \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{y})$$
$$\cdot [C_{K\xi}(\mathbf{x}, \mathbf{z}, t)\bar{\mathbf{n}}(\mathbf{z}, t) + \bar{\xi}(\mathbf{z}, t) C_{K\mathbf{n}}(\mathbf{x}, \mathbf{z}, t)$$
$$+ \overline{K'(\mathbf{x}) \xi'(\mathbf{z}, t) \mathbf{n}'(\mathbf{z}, t)}]\}\, d\mathbf{z}. \tag{A2}$$

By the same token, evaluating (A1) at $\mathbf{x} \in \Gamma_F$, multiplying with $\xi'(\mathbf{y}, \tau)$, and taking the mean yield

$$C_\xi(\mathbf{x}, \mathbf{y}, t, \tau) = -\int_\Omega [C_{K\xi}(\mathbf{z}, \mathbf{y}, \tau) \nabla_\mathbf{z} \bar{h}(\mathbf{z}, t)$$
$$+ \overline{\xi'(\mathbf{y}, \tau) K'(\mathbf{z}) \nabla_\mathbf{z} h'(\mathbf{z}, t)}] \cdot \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{x})\, d\mathbf{z}$$
$$+ \int_{\Gamma_F} \{C_{V\xi}(\mathbf{z}, \mathbf{y}, t, \tau) G(\mathbf{z}, \mathbf{x}) - \bar{K}(\mathbf{z}) \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{x})$$
$$\cdot [\overline{\xi'(\mathbf{z}, t) \xi'(\mathbf{y}, \tau) \mathbf{n}'(\mathbf{z}, t)} + C_\xi(\mathbf{z}, \mathbf{y}, t, \tau) \bar{\mathbf{n}}(\mathbf{z}, t)$$
$$+ C_{\xi\mathbf{n}}(\mathbf{y}, \mathbf{z}, \tau, t) \bar{\xi}(\mathbf{z}, t)]\}\, d\mathbf{z}. \tag{A3}$$

The first-order (in $\sigma_Y^2$) approximations of (A2) and (A3) are given by (17) and (19), respectively.

Normal unit vector $\mathbf{n}$ to the phreatic surface $\Gamma_F$ can be found as

$$\mathbf{n}(\mathbf{x}, t) = \frac{\nabla[\xi(x_1, x_2, t) - x_3]}{|\nabla[\xi(x_1, x_2, t) - x_3]|} = \frac{\nabla(\xi - x_3)}{\sqrt{[\nabla(\bar{\xi} - x_3) + \nabla \xi']^2}}$$
$$= \left(\bar{\mathbf{n}} + \frac{\nabla \xi'}{|\nabla(\bar{\xi} - x_3)|}\right) \frac{1}{\sqrt{1 + \dfrac{2\nabla(\bar{\xi} - x_3) \cdot \nabla \xi' + [\nabla \xi']^2}{[\nabla(\bar{\xi} - x_3)]^2}}}. \tag{A4}$$

A power series expansion yields [*Gradshteyn and Ryzhik*, 1994, equation (1.112(4))]

$$\mathbf{n}(\mathbf{x}, t) = \left(\bar{\mathbf{n}} + \frac{\nabla \xi'}{|\nabla(\bar{\xi} - x_3)|}\right)$$
$$\cdot \left(1 - \frac{2\nabla(\bar{\xi} - x_3) \cdot \nabla \xi' + [\nabla \xi']^2}{2[\nabla(\bar{\xi} - x_3)]^2} + \cdots\right). \tag{A5}$$

Cross covariances $C_{\xi\mathbf{n}}(\mathbf{x}, \mathbf{x}, t) = \overline{\xi'(\mathbf{x}, t)\mathbf{n}(\mathbf{x}, t)}$ and $C_{K\mathbf{n}}(\mathbf{x}, \mathbf{y}, t) = \overline{K'(\mathbf{x})\mathbf{n}(\mathbf{y}, t)}$ are now readily expressed as

$$C_{\xi\mathbf{n}}(\mathbf{x}, \mathbf{x}, t) = \frac{\nabla \sigma_\xi^2}{2|\nabla(\bar{\xi} - x_3)|} - \bar{\mathbf{n}} \frac{\bar{\mathbf{n}} \cdot \nabla \sigma_\xi^2}{2|\nabla(\bar{\xi} - x_3)|} + \cdots \tag{A6}$$

$$C_{K\mathbf{n}}(\mathbf{x}, \mathbf{y}, t) = \frac{\nabla_\mathbf{y} C_{K\xi}(\mathbf{x}, \mathbf{y}, t)}{|\nabla_\mathbf{y}(\bar{\xi} - y_3)|} - \bar{\mathbf{n}}(\mathbf{y}) \frac{\bar{\mathbf{n}}(\mathbf{y}) \cdot \nabla_\mathbf{y} C_{K\xi}(\mathbf{x}, \mathbf{y}, t)}{|\nabla_\mathbf{y}(\bar{\xi} - y_3)|} + \cdots \tag{A7}$$

First-order (in $\sigma_Y^2$) approximations of (A6) and (A7) are given by (18) and (16), respectively.

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