STOCHASTIC AVERAGING AND ESTIMATE OF EFFECTIVE (UPSCALED) CONDUCTIVITY AND TRANSMISSIVITY

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SUMMARY

Numerical models for groundwater flow and transport on a regional scale require the assignment of hydraulic parameters to grid blocks of a size larger than typical scales of available field data. Most natural formations are heterogeneous with hydraulic properties varying on a multiplicity of scales. This difference in scales is often handled by using effective (upscaled) parameters. Here we investigate the problem of upscaling hydraulic conductivity and transmissivity from the small scale of measurement to the larger scale of a grid block by means of stochastic averaging and perturbation analysis. We start by deriving the statistics of a random transmissivity field from that of a conductivity field, and proceed by establishing the relationship between effective conductivity and transmissivity derived from the same data set.

1. INTRODUCTION

Models for interpreting experimental data usually assume that, at least on a measurement scale, flow takes place in a homogeneous environment. Under this assumption, hydraulic conductivity $K(\boldsymbol{x})$ represents a quantity averaged over some support volume ω_1 centered around point $\boldsymbol{x}$. If the support volume ω_1 is comparable to an aquifer thickness, B, then the trasmissivity associated with this support volume is defined as BK. However, K is usually associated with much smaller measurement volumes, and local transmissivity of heterogeneous formations is defined as a vertical average,

$$\mathrm{T} = \int_0^B K \, dx_3 \, . \tag{1}$$

Effective, or up-scaled, parameters are often used for numerical modeling of groundwater flow and transport in heterogeneous aquifers. Such models require assigning hydraulic conductivities or transmissivities to large grid-blocks, while experimental data are usually available at a much smaller scale of core or well-log measurements. These parameters can be obtained by standard inverse methods [1] or, alternatively, by stochastic averaging of the corresponding flow equations [2, 3]. In this paper we pursue the latter strategy.

We establish the relationship between effective (up-scaled) hydraulic conductivity, K_{eff}, and transmissivity, T_{eff}, for randomly heterogeneous confined formations. We show that, when T and K are defined on different support scales, the traditional definition of effective transmissivity, $T_{eff} \equiv B\,K_{eff}$, fails unless a medium is mildly heterogeneous and exhibits a lack of the vertical spatial correlation. To simplify the mathematical developments, we consider steady-state flow in the vertical cross-section (x_1, x_3). Generalizing the obtained results to transient flow scenarios can be readily accomplished by following Tartakovsky and Neuman [4].

2. TRANSMISSIVITY STATISTICS

Following standard practice in stochastic hydrogeology [5 - 7], we assume that log hydraulic conductivity $Y(\mathbf{x}) = \ln K(\mathbf{x})$ forms a statistically homogeneous Gaussian field with constant mean $\langle Y\rangle$ and variance $\sigma_Y^2 = \langle Y'(\mathbf{x})\, Y'(\mathbf{x})\rangle$, and two-point Gaussian anisotropic covariance function, $C_Y(\mathbf{x};\mathbf{y}) = \sigma_Y^2\ \rho_Y(r)$

$$\rho_Y(r) = \exp(-r^2) \qquad\qquad r^2 = \frac{(x_1-y_1)^2}{l_h^2} + \frac{(x_3-y_3)^2}{l_v^2} \tag{2}$$

where l_h and l_v are the horizontal and vertical correlation scales, respectively. Our aim is to express the transmissivity statistics in terms of the statistics of Y.

It follows from (1) that ensemble mean transmissivity, $\langle T\rangle$, is given exactly by

$$<T>= e^{<Y>} \int_0^B < e^{Y'(x)} > dx_3 = B\ K_G \exp\left(\frac{\sigma_Y^2}{2}\right)$$

where $K_G = \exp(\langle K\rangle)$ is the geometric mean of K; and the first-order (in σ_Y^2) approximation of its covariance has the form,

$$C_T(x_1,\, y_1) = K_G^2\ \sigma_Y^2 \int_0^B\!\!\int_0^B \rho_Y(r)\ dx_3\ dy_3 + O(\sigma_Y^4) \tag{3}$$

Substituting (2) into (3) yields

$$C_T(r_h) = K_G^2\ \sigma_Y^2\ \alpha(l_v,\, B)\, \rho_T(r_h)$$

where $\rho_T(r_h) = \exp(-r_h^2)$, $r_h^2 = (x_1 - y_1)^2 / l_h^2$, and

$$\frac{\alpha(l_v,\, B)}{l_v^2} = \sqrt{\pi}\ \frac{B}{l_v}\, erf\left(\frac{B}{l_v}\right) + \exp\left(-\frac{B^2}{l_v^2}\right) - 1\,.$$

In a similar manner, one has the following first-order (in σ_Y^2) approximations of mean log transmissivity $\langle Z\rangle$, geometric mean $T_G = \exp(\langle Z\rangle)$, variance σ_Z^2, and correlation function ρ_Z,

$$\langle Z\rangle = ln\,(B\,K_G) + (1-\beta)\,\frac{\sigma_Y^2}{2}$$

$$T_G = B\left[1+\frac{1-\beta}{2}\,\sigma_Y^2\right]K_G \tag{4}$$

and

$$\sigma_Z^2 = \beta\,\sigma_Y^2 \qquad \rho_Z(r_h) = \exp(-r_h^2) \qquad r_h^2 = \frac{(x_1-y_1)^2}{l_h^2} \tag{5}$$

respectively. Here

$$\beta(\lambda_v) = \lambda_v^2\,\left[\sqrt{\pi}\,\lambda_v^{-1}\,erf\,(\lambda_v^{-1}) + \exp(-\lambda_v^{-2}) - 1\right]$$

and $\lambda_v = l_v / B$.

Thus constructed random field of log transmissivity, Z, is dependent upon a number, λ_v, of the vertical correlation scales, l_v, in the total width, B, of an aquifer. This dependence manifests itself through the correction factor $\beta(\lambda_v)$. Since $\beta < 1$, it follows from (5) that the log transmissivity Z, as inferred from the given statistics of the log conductivity Y, exhibits smaller spatial variation than Y. A lack of the vertical correlation of hydraulic conductivity ($l_v = 0$) results in $\sigma_Z^2 = 0$.

It is clear that if T is defined on a support volume ω_2 through the spatial averaging (1) of K (associated with a smaller scale ω_1), log-normality of K does not imply log-normality of T. Alternatively, if transmissivity and conductivity are defined on the same support scale, then $T = BK$ and log-normality of K also implies log-normality of T.

3. EFFECTIVE TRANSMISSIVITY AND CONDUCTIVITY

Consider up-scaling of Darcy's law, $\boldsymbol{q} = -K\ \nabla h$ ($\boldsymbol{q}$ being flux and h being hydraulic head), from its support scale ω_1, to a larger grid-block, Ω, of length L and width B. Constant heads H_1 and H_2 are prescribed at the boundaries $x_1 = 0$ and $x_1 = L$, respectively; while boundaries $x_2 = 0$ and $x_2 = B$ are impermeable. Under these conditions, the first-order approximation of the effective conductivity, K_{eff}, for the two-dimensional heterogeneous grid-block, Ω, is given by [2]

$$\frac{K_{eff}(\boldsymbol{x})}{K_G} 1 + \sigma_Y^2\left[\frac{1}{2} - \kappa(\boldsymbol{x})\right] \qquad \kappa(\boldsymbol{x}) = \int_\Omega \rho_Y(r)\,\frac{\partial^2 G(\boldsymbol{x},\boldsymbol{y})}{\partial x_1\,\partial y_1}\,d\boldsymbol{y}$$

where G is the Green's function for the two-dimensional Laplace equation in Ω subject to appropriate homogeneous boundary conditions. This expression has been studied by Paleologos *et al*. [2] and Tartakovsky and Neuman [3]. These authors have demonstrated that when the size of the domain is much larger than the correlation scales of the log-hydraulic conductivity (l_h, $l_v << L, B$), one can treat the flow domain as infinite, which leads to $K_{eff} = K_G$.

For two-dimensional flow in a vertical cross-section, defining effective transmissivity, T_{eff}, in a similar manner leads to

$$\frac{T_{eff}(\boldsymbol{x})}{T_G} 1 + \sigma_Z^2 \left[\frac{1}{2} - k(\boldsymbol{x}) \right] \qquad\qquad k(\boldsymbol{x}) = \int_0^L \rho_Z(r) \frac{\partial^2 G_T(\boldsymbol{x}, \boldsymbol{y})}{\partial x_1 \, \partial y_1} \, d\boldsymbol{y} \tag{6}$$

where the one-dimensional G_T is given by

$$G_T(x_1, y_1) = -(x_1 - y_1)\,\mathrm{H}(x_1 - y_1) + \frac{L - y_1}{L} x_1 \tag{7}$$

where $\mathrm{H}(a) = 1$ when $a \geq 0$ (and = 0 otherwise) is the Heaviside function.

Substituting (5) and (7) into (6) yields

$$\frac{T_{eff}(\chi)}{T_G} = 1 - \sigma_Z^2 \, b(\chi) \qquad\qquad b(\chi) = \frac{1}{2} - \frac{\sqrt{\pi}}{2} \lambda_h \left[erf\left(\frac{\chi}{\lambda_h} \right) - erf\left(\frac{1-\chi}{\lambda_h} \right) \right]$$

where $\lambda_h = l_h \,/\, L$ and $\chi = x_1 / L$. Thus, similar to the effective hydraulic conductivity, the effective transmissivity is a function of space. Figure 1 shows the dependence of b on the normalized space coordinate χ for several values of λ_h.

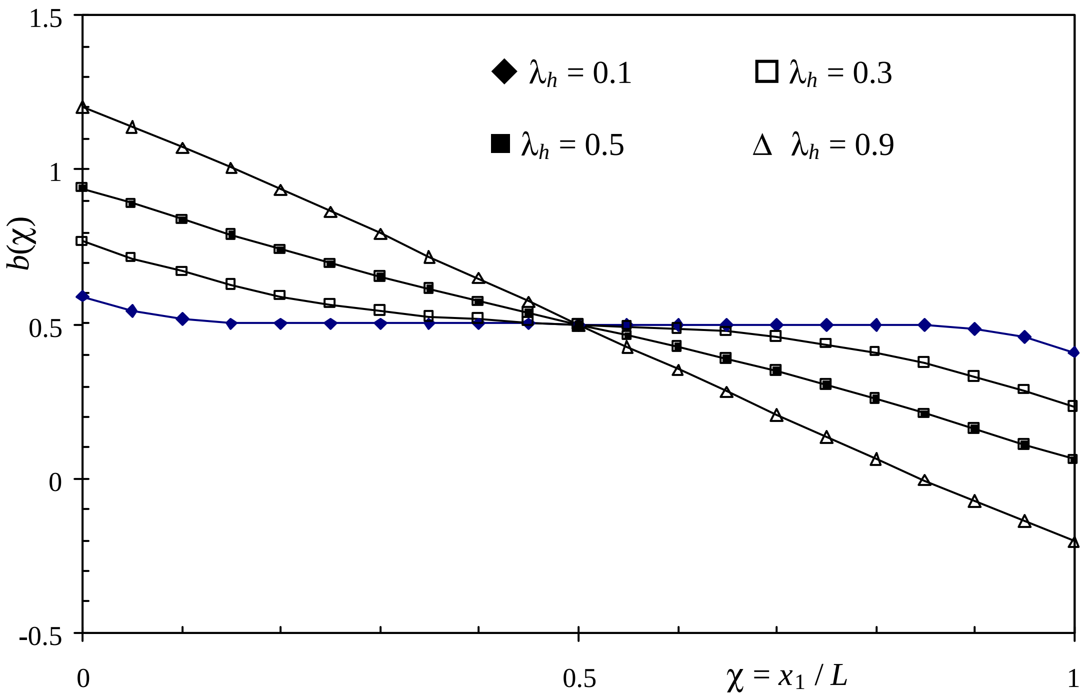

Figure 1: dependence of b on the normalized space coordinate χ for several values of λ_h.

The spatial variation of the effective transmissivity increases with λ_h. When $\lambda_h < 0.1$,

$$\frac{T_{eff}}{T_G} = 1 - \frac{\sigma_Z^2}{2} \tag{8}$$

Substituting (4) and (5) into (8) yields, up to first order in σ_Y^2,

$$\frac{T_{eff}}{B\,K_G} = 1 - \frac{1-2\,\beta}{2}\sigma_Y^2$$

Therefore, the traditional practice of setting $T_{eff} = B\,K_{eff}$ holds true if and only if aquifers are nearly homogeneous ($\sigma_Y^2 << 1$) and the vertical correlation scale of log-hydraulic conductivity is much smaller than the confined aquifer's thickness ($\lambda_v << 1$).

CONCLUSIONS

Numerical models of groundwater flow and transport often assign effective (upscaled) hydraulic parameters to grid blocks of a size larger than that of available field data. To derive such parameters, we treated hydraulic conductivity and transmissivity as log normal and stationary random fields. Starting from the premise that the hydraulic conductivity statistics are known from experimental data, we derived a relationship between the effective hydraulic conductivity K_{eff} and transmissivity T_{eff} for a grid block of a confined aquifer. Mean uniform flow conditions were imposed. This relationship demonstrates that the traditional practice of setting $T_{eff} = BK_{eff}$ (B being the aquifer thickness) is valid only under restrictive conditions of mildly heterogeneous formations with vertical correlation scale, l_v, of log hydraulic conductivity much smaller than the thickness of a confined aquifer ($l_v / B < 0.1$).

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