STOCHASTIC ANALYSIS OF EFFECTIVE CONDUCTIVITY FOR BOUNDED, RANDOMLY HETEROGENEOUS AQUIFERS

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SUMMARY

We explore the concept of effective hydraulic conductivity for a bounded randomly heterogeneous formation under steady-state flow regime. The novelty of our study consists of establishing a tensorial nature of the effective conductivity. This occurs even for locally isotropic conductivity fields. Neuman and Orr [1] have demonstrated that stochastically averaged flow equations are non-local and non-Darcian, so that effective hydraulic conductivity does not generally exist. We derived our analytical expression for the effective conductivity tensor by localizing these equations, and assessed the accuracy of this approximation by comparing the resulting hydraulic heads and fluxes with their non-local counterparts. Our solutions are in a good agreement with both recursive non-local finite-elements results of Guadagnini and Neuman [3] and Monte Carlo simulations for mildly and strongly heterogeneous formations.

1. INTRODUCTION AND PROBLEM STATEMENT

Consider steady-state groundwater flow described by a combination of Darcy's law and conservation of mass

$$-\nabla \cdot \boldsymbol{q}(\boldsymbol{x}) + f(\boldsymbol{x}) = 0; \qquad \boldsymbol{q}(\boldsymbol{x}) = -K(\boldsymbol{x})\,\nabla h(\boldsymbol{x}); \qquad \boldsymbol{x} \in \Omega \qquad (1)$$

subject to the boundary conditions

$$h(\boldsymbol{x}) = H(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma_D; \qquad -\boldsymbol{q}(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) = Q(\boldsymbol{x}); \qquad \boldsymbol{x} \in \Gamma_N \qquad (2)$$

Here $\boldsymbol{q}(\boldsymbol{x})$ is the Darcy's flux, $K(\boldsymbol{x})$ is the hydraulic conductivity, $h(\boldsymbol{x})$ is the hydraulic head, $f(\boldsymbol{x})$ is the source term, $H(\boldsymbol{x})$ is the prescribed head on Dirichlet boundary segments Γ_D, $Q(\boldsymbol{x})$ is the prescribed flux across Neumann boundary segments Γ_N, and $\mathbf{n}(\boldsymbol{x})$ is the unit outward normal to the boundary $\Gamma = \Gamma_D \cup \Gamma_N$ of the flow domain Ω. All quantities are representative of a nonzero support volume $\omega << \Omega$ centered about $\boldsymbol{x}$, which is sufficiently large for Darcy's law to be locally valid. We treat hydraulic conductivity, $K(\boldsymbol{x})$, as a random field, so that (1) − (2) constitute a system of stochastic partial-differential equations.

Neuman and Orr [1] and Tartakovsky and Neuman [2] have developed conditional moment equations for groundwater flow in randomly heterogeneous formations under steady-state and transient conditions, respectively. Guadagnini and Neuman [3, 4] have solved the steady-state moment equations numerically by relying on recursive approximations of Tartakovsky and Neuman [2]. In their analyses, the unbiased flux estimator is obtained by taking the ensemble mean of Darcy's law (1),

$$\langle \boldsymbol{q}(\boldsymbol{x})\rangle = -\langle K(\boldsymbol{x})\rangle \, \nabla \langle h(\boldsymbol{x})\rangle + \boldsymbol{r}(\boldsymbol{x}).$$

Here $\boldsymbol{r}(\boldsymbol{x})$ is the so-called "residual" flux, whose exact form is given in [1, 2]. For practical evaluation of this term it has been found necessary to employ perturbation analysis in a small parameter σ^2, the variance of a statistically homogeneous random field of (natural) log hydraulic conductivity $Y(\boldsymbol{x}) = \ln K(\boldsymbol{x})$. This leads to perturbation expansions of the mean hydraulic head and flux, $\langle h(\boldsymbol{x})\rangle = \langle h^{(0)}(\boldsymbol{x})\rangle + \langle h^{(1)}(\boldsymbol{x})\rangle + O(\sigma^4)$ and $\langle \boldsymbol{q}(\boldsymbol{x})\rangle = \langle \boldsymbol{q}^{(0)}(\boldsymbol{x})\rangle + \langle \boldsymbol{q}^{(1)}(\boldsymbol{x})\rangle + O(\sigma^4)$. Retaining the first two terms in these expansions gives the first-order approximation of the mean Darcy's law [2 - 4],

$$\langle \boldsymbol{q}^{[1]}(\boldsymbol{x})\rangle = \langle \boldsymbol{q}^{(0)}(\boldsymbol{x})\rangle + \langle \boldsymbol{q}^{(1)}(\boldsymbol{x})\rangle$$

where

$$\langle \boldsymbol{q}^{(0)}(\boldsymbol{x})\rangle = -K_G \nabla h^{(0)}(\boldsymbol{x}); \qquad \langle \boldsymbol{q}^{(1)}(\boldsymbol{x})\rangle = -K_G \, [\nabla h^{(1)}(\boldsymbol{x}) + (\sigma^2/2)\, \nabla h^{(0)}(\boldsymbol{x})] + \boldsymbol{r}^{(1)}(\boldsymbol{x}).$$

Here $K_G = exp\langle Y\rangle$ is the geometric mean of Y, the zeroth-order mean head $h^{(0)}$ satisfies (1) – (2) where $K(\boldsymbol{x})$ is replaced by K_G, and the first-order approximation of the residual flux $\boldsymbol{r}^{(1)}(\boldsymbol{x})$ is given by

$$\boldsymbol{r}^{(1)}(\boldsymbol{x}) = \int_\Omega \boldsymbol{a}(\boldsymbol{y}, \boldsymbol{x}) \, \nabla h^{(0)}(\boldsymbol{y}) \, d\boldsymbol{y} \qquad (3)$$

where

$$\boldsymbol{a}(\boldsymbol{y}, \boldsymbol{x}) = K_G^2 \, C_Y(\boldsymbol{y}, \boldsymbol{x}) \, \nabla_x \nabla_y^{\mathrm{T}} \, G(\boldsymbol{y}, \boldsymbol{x}),$$

$C_Y(\boldsymbol{y}, \boldsymbol{x})$ is the spatial autocovariance of Y, and G is the deterministic Green's function for (1) – (2) where $K(\boldsymbol{x})$ is replaced by K_G. For the expansions of the mean hydraulic head and fluxes to remain asymptotic it is necessary that $\sigma^2 << 1$, *i.e.* the porous medium to be mildly heterogeneous. Nevertheless, numerical simulations of Guadagnini and Neuman [3, 4] have demonstrated that our first-order approximations yield remarkably accurate results for strongly non-uniform media with σ^2 as large as 4.

Since flux estimators $\boldsymbol{r}(\boldsymbol{x})$ and $\langle \boldsymbol{q}(\boldsymbol{x})\rangle$ are generally nonlocal and non-Darcian, the notion of effective conductivity looses its meaning in all but a few special cases [1, 2, 5, 6]. Tartakovsky and Neuman [5, 6] have explored a few special situations where localization of the above flux predictors is possible and have analyzed the corresponding effective conductivity. Among this flow scenarios the case of the slow-varying mean head gradient has been studied most. Under this assumption, one can approximate (3) as

$$\boldsymbol{r}^{(1)}(\boldsymbol{x}) \approx k^{(1)}(\boldsymbol{x})\, \nabla h^{(0)} \tag{4}$$

where

$$k^{(1)}(\boldsymbol{x}) = K_G^2 \int_\Omega C_Y(\boldsymbol{y}, \boldsymbol{x})\, \nabla_x\, \nabla_y^{\mathrm{T}} G(\boldsymbol{y}, \boldsymbol{x})\, d\boldsymbol{y}\,. \tag{5}$$

Then

$$<\boldsymbol{q}^{[1]}(\boldsymbol{x})> \approx -K_G\, \nabla <h^{(1)}(\boldsymbol{x})> - \boldsymbol{K}_{eff}^{[1]}\, \nabla h^{(0)}(\boldsymbol{x}); \qquad \boldsymbol{K}_{eff}^{[1]} = K_G\,[1 + (\sigma^2/2)]\,\mathbf{I} - k^{(1)}(\boldsymbol{x}) \tag{6}$$

Guadagnini and Neuman [3, 4] have shown that a localized version of the mean flow equations provides quite accurate estimates of hydraulic heads and fluxes when compared to Monte Carlo results.

For flow through infinite, statistically homogeneous porous media under mean uniform flow conditions, the mean hydraulic head gradient $\boldsymbol{J} = \nabla h^{(0)}$, and $\nabla h^{(i)} = 0$ for $i > 0$ [1]. Then $<\boldsymbol{q}^{[1]}(\boldsymbol{x})> \approx -\boldsymbol{K}_{eff}^{[1]}\, \boldsymbol{J}$, with $\boldsymbol{K}_{eff}^{[1]}$ playing the role of a *bona fide* effective hydraulic conductivity. However, Guadagnini and Neuman [3, 4] showed that for bounded domains $\nabla <h^{(1)}(\boldsymbol{x})> \neq 0$ even when $\nabla h^{(0)}$ is constant. Thus, the localization of the second order mean flow equation in the manner of (4) – (6) does not imply that $<\boldsymbol{q}^{[1]}(\boldsymbol{x})>$ is Darcian.

Guadagnini and Neuman [3, 4] have demonstrated numerically that $\nabla <h^{(1)}(\boldsymbol{x})> \ll \nabla h^{(0)}(\boldsymbol{x})$ at locations far away from singularities (*e.g.* pumping/injection wells). Then one can write $<\boldsymbol{q}^{[1]}(\boldsymbol{x})> \approx -\boldsymbol{K}_{eff}^{[1]}\, \nabla h^{(0)}(\boldsymbol{x})$. Tartakovsky and Neuman [5] have considered mean uniform flow through a box-shaped domain and evaluated numerically the component of $\boldsymbol{K}_{eff}^{[1]}$ in the direction of the mean flow. Here we explore analytically the tensorial nature of the effective parameter in (6) for a two-dimensional case.

2. EFFECTIVE CONDUCTIVITY FOR A RECTANGLE

Here we present an analytical expression for effective hydraulic conductivity tensor under two-dimensional steady-state flow through a rectangle due to a uniform mean hydraulic gradient. The rectangle is embedded within a statistically homogeneous field Y that is Gaussian and exhibits an isotropic separated exponential auto-correlation structure. The sides of the rectangle are a and b in x_1 and x_2 directions, respectively. In dimensionless coordinates $x_1^* = x_1/a$, and $x_2^* = x_2/b$, boundary conditions are taken as

$$h = H_1; \quad \text{for } x_1^* = 0; \qquad\qquad h = H_2; \quad \text{for } x_1^* = 1;$$

$$\partial\, h(\boldsymbol{x}) / \partial\, x_2^* = 0; \qquad\qquad \text{for } x_2^* = 0 \text{ and } x_2^* = 1.$$

Here H_1 and H_2 are deterministically prescribed hydraulic heads. Upon introducing $\varepsilon = b / a$, and dimensionless variables $y_1^* = y_1 / a$, $y_2^* = y_2 / b$, the auto-covariance function is written as

$$C_Y(\boldsymbol{y}^*, \boldsymbol{x}^*) = \sigma^2 \, \rho(\boldsymbol{y}^*, \boldsymbol{x}^*) ; \qquad \rho(\boldsymbol{y}^*, \boldsymbol{x}^*) = \exp\left[-\frac{|x_1^* - y_1^*|}{\lambda^*}\right] \exp\left[-\frac{\varepsilon \, |x_2^* - y_2^*|}{\lambda^*}\right] \tag{7}$$

where $\lambda^* = \lambda / a$, and λ is the correlation scale. The Green's function, $G_K = K_G \; G$, is now given by

$$G_K(\boldsymbol{y}^*, \boldsymbol{x}^*) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\gamma_n(y_2^*, x_2^*)}{n} \, \frac{\sin\,(\pi n x_1^*) \sin\,(\pi n y_1^*)}{\sinh\,(\pi n \varepsilon)}$$

$$\gamma_n(y_2^*, x_2^*) = \begin{cases} \cosh\,(\pi n \varepsilon [x_2^* - 1]) \cosh\,(\pi n \varepsilon y_2^*) & 0 \le y_2^* \le x_2^* \\ \cosh\,(\pi n \varepsilon x_2^*) \cosh\,(\pi n \varepsilon [y_2^* - 1]) & x_2^* < y_2^* \le 1 \end{cases} \tag{8}$$

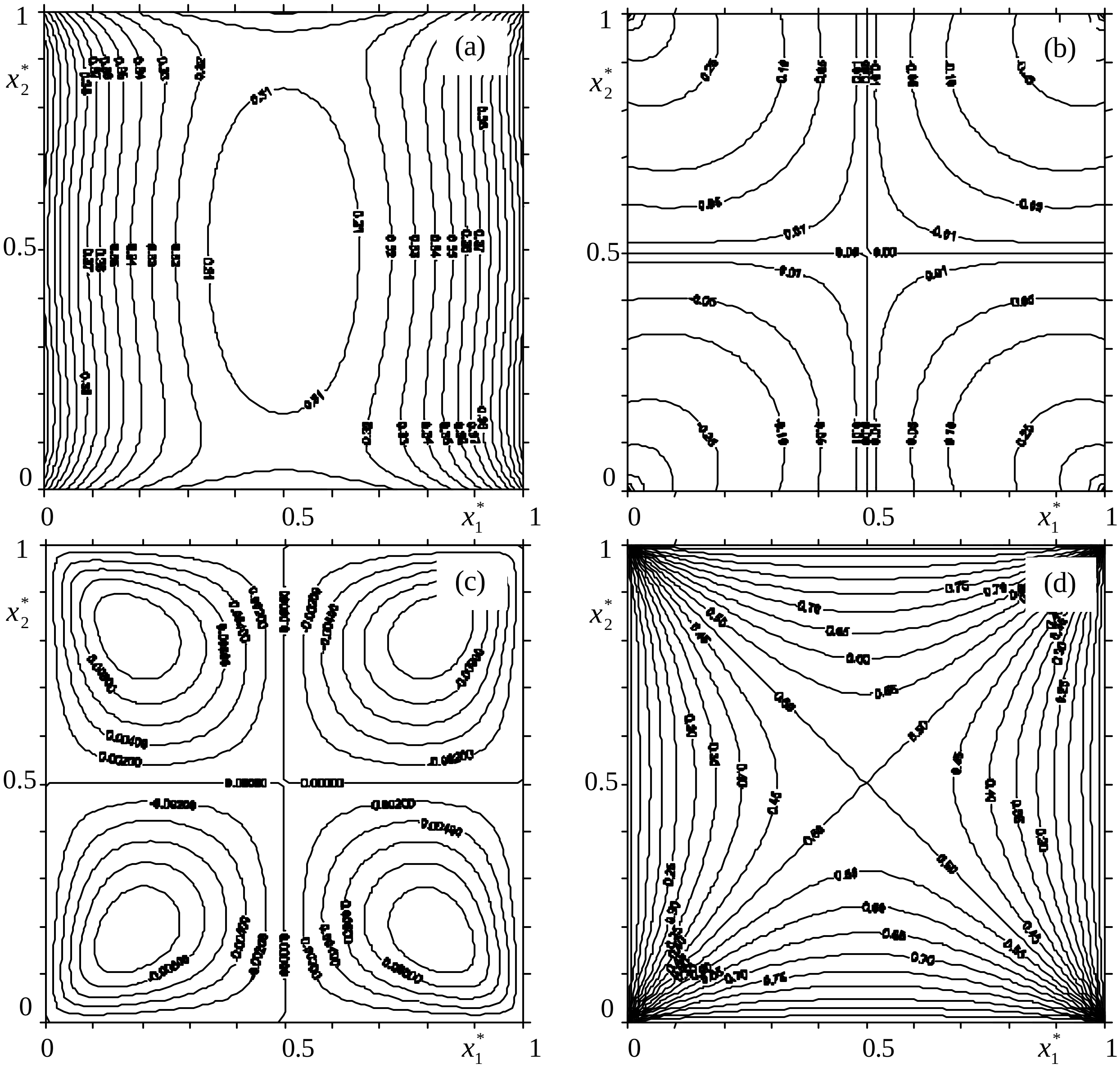

Figure 1. Contour maps of the components κ_{ij} ($\boldsymbol{x}^*$) normalized by $K_G \, \sigma^2$: (a) κ_{11}, (b) κ_{12}, (c) κ_{21}, and (d) κ_{22}. The parameters are $\varepsilon = 1$ and $\lambda^* = 0.5$.

Substituting (7) and (8) into (5) leads to the expressions for the four components κ_{ij} of the tensor $k^{(1)}$. The analytical expression for κ_{11} is given below (the superscript * is omitted), while Figure 1 depicts contour plots of all κ_{ij}, evaluated for $\varepsilon = 1$ and $\lambda^* = 0.5$.

$$\frac{\kappa_{11}}{K_G\,\sigma^2} = \lambda^2 \sum_{n=1}^{\infty} \frac{Cos(\pi n x_1)}{Sinh(\pi n \varepsilon)} \frac{1}{1+\pi^2 n^2 \lambda^2} \left\{ 2\pi n\, Cos(\pi n x_1) - \pi n \left[e^{-\frac{x_1}{\lambda}} + (-1)^n e^{\frac{x_1 - 1}{\lambda}} \right] \right\} \times$$
$$\left\{ -Cosh[\pi n \varepsilon (x_2 - 1)] \left[\frac{2}{1-\pi^2 n^2 \lambda^2} e^{-\varepsilon \frac{x_2}{\lambda}} - \frac{1}{1+\pi\, n\, \lambda} e^{\pi n\, \varepsilon\, x_2} - \frac{1}{1-\pi\, n\, \lambda} e^{-\pi\, n\, \varepsilon\, x_2} \right] + \right.$$
$$\left. Cosh[\pi n \varepsilon\, x_2] \left[-\frac{2}{1-\pi^2 n^2 \lambda^2} e^{-\varepsilon \frac{(x_2 - 1)}{\lambda}} + \frac{1}{1+\pi\, n\, \lambda} e^{\pi\, n\, \varepsilon\, (1-x_2)} + \frac{1}{1-\pi\, n\, \lambda} e^{-\pi\, n\, \varepsilon\, (1-x_2)} \right] \right\}.$$

It is clear from Figure 1 that all four components of the second-rank tensor $k^{(1)}$ are space-dependent and are symmetric with respect to the domain center. The tensor $k^{(1)}$ is non-symmetric and diagonally-dominant almost everywhere, with non-diagonal terms $\kappa_{ij} = 0$ $(i \neq j)$ at the domain center. Hence, the first-order approximation of the stochastically derived effective conductivity tensor is generally non-symmetric.

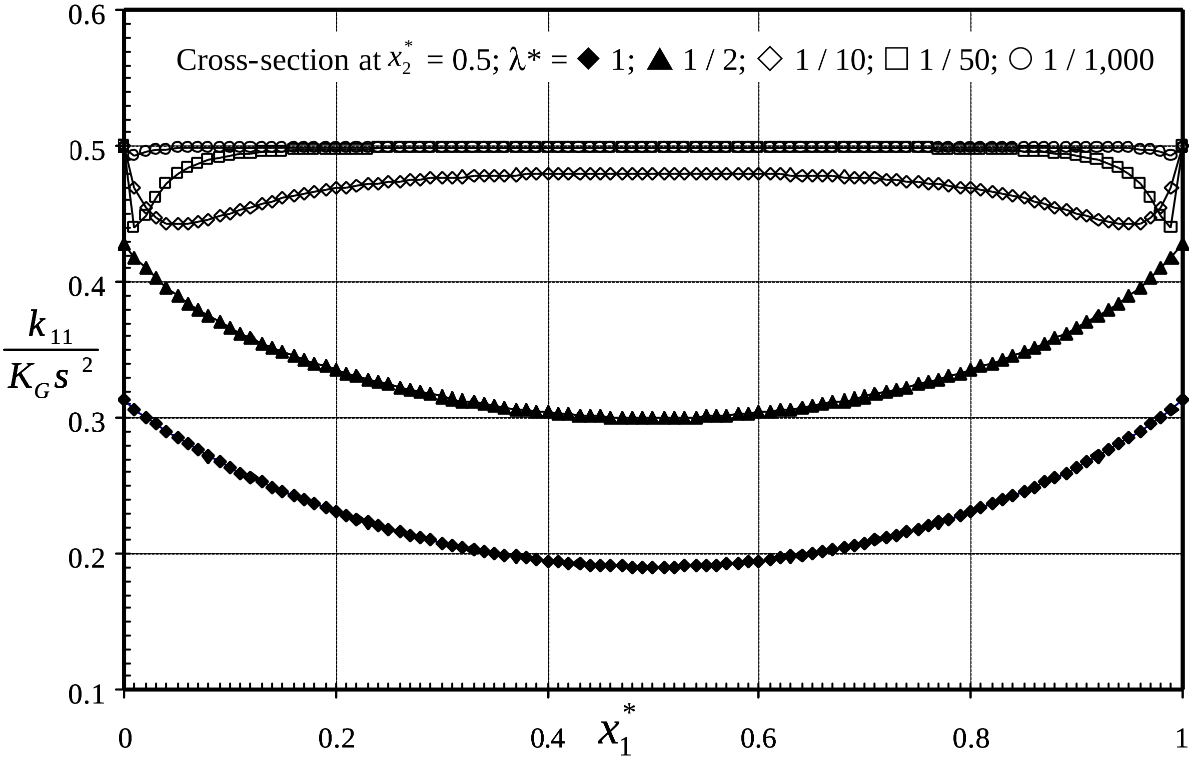

Figure 2. Longitudinal section (at x_2^* = 0.5) of $\kappa_{11}/ K_G\, \sigma^2$ for $\varepsilon = 1$ and several values of λ^*.

The boundary effects are revealed on Figure 2 which depicts the longitudinal cross-section (at x_2^* = 0.5) of $\kappa_{11} / K_G\, \sigma^2$ for $\varepsilon = 1$ and several values of λ^*. As the domain size increases (λ^* decreases), $\kappa_{11} / K_G\, \sigma^2 \rightarrow 0.5$. This is in agreement with well-established results for infinite domain [1]. Indeed, it follows from (16) that $K_{eff,11}^{[1]} \rightarrow K_G$ as $\lambda^*/a \rightarrow 0$. Although not shown here, $\kappa_{12} = \kappa_{21} = 0$ and $\kappa_{22} / K_G \sigma^2 = 0.5$ at every point in the domain, as $\lambda^*/a \to 0$. Thus anisotropy of the effective conductivity stems from the presence of boundaries. These boundary effects also cause the effective conductivity tensor to be non-symmetric.

3. CONCLUSIONS

Our work leads to the following major conclusions:

1. Stochastically averaged flux is generally non-local and non-Darcian, so that an effective hydraulic conductivity cannot be defined except in special cases. To derive an analytical expression for the effective conductivity, we adopted a first-order (in variance σ^2 of log hydraulic conductivity Y) perturbation analysis and localization of the mean flow equation.

2. First-order approximation, $\boldsymbol{K}_{eff}^{[1]}$, of the effective conductivity tensor for two-dimensional steady-state flow through a rectangle due to a uniform mean hydraulic gradient is generally non-symmetric and diagonally-dominant.

3. Anisotropy of the effective conductivity stems from the presence of boundaries. These boundary effects also cause the effective conductivity tensor to be non-symmetric. As the domain size, expressed in terms of correlation lengths of Y, increases, the diagonal terms of $\boldsymbol{K}_{eff}^{[1]}$ tend to the geometric mean of hydraulic conductivity; its cross-diagonal terms tend to zero.

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