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Prediction of steady-state flow of real gases in randomly heterogeneous porous media

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Abstract

We consider steady-state flow of real gases through bounded, randomly heterogeneous porous media. Such flow is described by a nonlinear partial differential equation with the random coefficient (medium's permeability) and source terms subject to randomly prescribed boundary conditions. Prior to applying stochastic analysis, the problem is linearized by means of the Kirchhoff transformation which allows us to obtain the exact expressions for an effective (upscaled) gas permeability. In particular, for one-, two-, and three-dimensional mean uniform flows in infinite, statistically homogeneous and isotropic domains the resulting effective permeability is given by harmonic, geometric, and arithmetic averages, respectively. The influence of statistical anisotropy of the random permeability field and domain's boundaries on the effective gas permeability is also investigated.

Keywords: Gas flow; Porous media; Stochastic

1. Introduction

The ability of a porous medium to transmit fluids is commonly referred to as medium's permeability or hydraulic conductivity. Permeability of many natural formations (such as oil and gas reservoirs, groundwater aquifers, etc.) has been found to vary, continuously and/or discretely, over many orders of magnitude on a variety of scales. Permeability data are at best known at selected locations and depend on the scale (support volume) and mode (instrumentation and procedure) of measurement. The available data are often prone to experimental and interpretive uncertainty, and estimating permeability at points where it is not measured introduces additional errors. These errors and uncertainties render permeability random and the corresponding flow equations stochastic.

In recent decades, geostatistics has emerged as a prevailing method to deal with the uncertain nature of permeability. According to this approach, the available permeability data measured at selected locations can be viewed as a particular realization from a sampling space. The sampling space is characterized by a joint multivariate probability function or joint ensemble moments. Thus, all random fields and functions in the stochastic flow equations depend

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not only on the space coordinate $\boldsymbol{x}$, but also on the fictitious 'coordinate' $\boldsymbol{\xi}$ in the probability space. Whereas spatial moments are obtained through sampling in real space, ensemble moments require sampling in probability space. In reality, only one 'realization' of an actual reservoir or aquifer exists, and it is common to invoke ergodicity hypothesis which allows for interchanging ensemble and spatial moments. Ergodicity cannot be proven unless falling under the purview of well-established ergodic theorems such as the law of large numbers. However, it has been demonstrated by Yaglom [1] that 'in any application, non-ergodicity usually just means that the random function concerned is, in fact, an artificial union of a number of distinct ergodic stationary functions'.

Treating permeability of a porous medium as a random field leads to stochastic flow equations. Single phase flows of incompressible fluids (such as water or oil), can be described by linear diffusion equations and have been a subject of numerous stochastic analyses. Predicting flows of compressible fluids (e.g., natural gases) through randomly heterogeneous porous media is more challenging due to nonlinearity of the governing equations, and to the best of our knowledge there have been no attempts to do so stochastically.

In this paper, we employ the Kirchhoff transformation to linearize the governing equations for real gas flow. Subsequent stochastic analysis is similar to that performed for flow of incompressible fluids. Localization of the stochastically averaged flow equations, existence of upscaled (effective) permeability, and boundary and statistical anisotropy effects are investigated.

2. Mathematical formulation

Consider steady-state laminar and isothermal flow of a real gas of constant composition through randomly heterogeneous porous media, Ω. Such flow is described by the Darcy law and mass conservation

$$\boldsymbol{v}(\boldsymbol{x}) = -\frac{k(\boldsymbol{x})}{\mu(p)}\nabla p(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega \tag{1}$$

$$\nabla \cdot [\rho(p)\boldsymbol{v}(\boldsymbol{x})] + \overline{f}(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \Omega \tag{2}$$

where $\boldsymbol{v}$ is the macroscopic (Darcian) velocity, k the gas permeability of the medium, μ the gas viscosity, p the pressure, $\overline{f}$ the (random) source term, and ρ the gas density.

Equations (1) and (2) are supplemented by the equation of state

$$\rho(p) = \frac{1}{RT}\frac{p}{Z(p)} \tag{3}$$

where R is the gas constant, T is the constant temperature, and $Z(p)$ is the compressibility factor; and boundary conditions

$$p(\boldsymbol{x}) = P(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_{\mathrm{D}} \tag{4}$$

$$-\rho(p)\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = V(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_{\mathrm{N}} \tag{5}$$

where $P(\boldsymbol{x})$ and $V(\boldsymbol{x})$ are randomly prescribed head and mass flux on Dirichlet, Γ_{D}, and Neumann, Γ_{N}, boundary segments whose union forms a boundary Γ of the domain Ω, and $\boldsymbol{n}(\boldsymbol{x})$ is the unit outward normal to the boundary Γ.

While permeability k exhibits discrete and continuous variations on a multiplicity of scales, it can at best be measured at selected locations $\boldsymbol{x}_k$ inside flow domain Ω. Numerous field experiments (see, e.g., [2] and references therein) have revealed that permeability of various natural formations can be viewed as a log-normal multi-variate field. Since $k(\boldsymbol{x})$ is scale-dependent and random, so will be the pressure and fluxes. In other words, Eqs. (1)–(5) constitute a system of nonlinear stochastic differential equations.

3. Stochastic averaging

Direct stochastic averaging of Eqs. (1)–(5) would lead to the presence of ensemble means of some deterministic functions of random argument, $\langle g(p)\rangle$. It is common in stochastic analyses of unsaturated flow (which is described by a similar set of nonlinear differential equations) to retain only the leading term in the Taylor expansions of these functions, i.e., $\langle g(p)\rangle \approx g(\langle p\rangle)$. Such an approach does not seem to be quite satisfactory since the convergence of such Taylor expansions cannot be either guaranteed or verified before the solution $p(\boldsymbol{x})$ of Eqs. (1)–(5) is found. In this paper, we pursue an alternative approach which consists of linearizing Eqs. (1)–(5) prior to their averaging.

Substituting Eqs. (1) and (3) into Eqs. (2), (4) and (5), and applying the Kirchhoff transformation [3]

$$\Psi(\boldsymbol{x}) = \int_{-\infty}^{p} \frac{s}{\mu(s) Z(s)}\,\mathrm{d}s \tag{6}$$

yields

$$\nabla \cdot [k(\boldsymbol{x}) \nabla \Psi(\boldsymbol{x})] + f(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \Omega \tag{7}$$

subject to the boundary conditions

$$\Psi(\boldsymbol{x}) = H(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_{\mathrm{D}} \tag{8}$$

$$\boldsymbol{n}(\boldsymbol{x}) \cdot [k(\boldsymbol{x}) \nabla \Psi(\boldsymbol{x})] = Q(\boldsymbol{x}) \quad \boldsymbol{x} \in \Gamma_{\mathrm{N}} \tag{9}$$

where $H(\boldsymbol{x})$ is the Kirchhoff transform of $P(\boldsymbol{x})$, $Q(\boldsymbol{x}) = RTV(\boldsymbol{x})$, and $f(\boldsymbol{x}) = RT\overline{f}(\boldsymbol{x})$.

For isothermal flow of an ideal gas, $Z(p) \equiv 1$, μ is constant, and one has $2\Psi = p^2$. A similar treatment of the equations describing flow of real gases in homogeneous porous media has been carried out by Al-Hussainy et al. [4].

We represent random fields and functions A as sums of their ensemble means $\langle A\rangle$ and zero-mean perturbations $\langle A'\rangle$,

$$k(\boldsymbol{x}) = \langle k\rangle + k'(\boldsymbol{x}) \qquad \langle k'(\boldsymbol{x})\rangle \equiv 0 \tag{10}$$

$$\Psi(\boldsymbol{x}) = \langle \Psi(\boldsymbol{x})\rangle + \Psi'(\boldsymbol{x}) \qquad \langle \Psi'(\boldsymbol{x})\rangle \equiv 0. \tag{11}$$

Taking the ensemble mean of Eqs. (7)–(9) yields

$$\nabla \cdot [\langle k(\boldsymbol{x})\rangle \nabla \langle \Psi(\boldsymbol{x})\rangle - \boldsymbol{r}(\boldsymbol{x})] + \langle f(\boldsymbol{x})\rangle = 0 \quad \boldsymbol{x} \in \Omega \tag{12}$$

subject to the boundary conditions

$$\langle \Psi(\boldsymbol{x})\rangle = \langle H(\boldsymbol{x})\rangle \quad \boldsymbol{x} \in \Gamma_{\mathrm{D}} \tag{13}$$

$$\boldsymbol{n}(\boldsymbol{x}) \cdot [\langle k(\boldsymbol{x})\rangle \nabla \langle \Psi(\boldsymbol{x})\rangle - \boldsymbol{r}(\boldsymbol{x})] = \langle Q(\boldsymbol{x})\rangle \quad \boldsymbol{x} \in \Gamma_{\mathrm{N}} \tag{14}$$

where the residual flux $\boldsymbol{r}(\boldsymbol{x}) = -\langle k'(\boldsymbol{x}) \nabla \Psi'(\boldsymbol{x})\rangle$ is given explicitly [5] as the solution of an integral equation

$$\boldsymbol{r}(\boldsymbol{x}) = \int_{\Omega} \boldsymbol{A}(\boldsymbol{y}, \boldsymbol{x}) \nabla \langle \Psi(\boldsymbol{y})\rangle \,\mathrm{d}\boldsymbol{y} + \int_{\Omega} \boldsymbol{B}(\boldsymbol{y}, \boldsymbol{x}) \boldsymbol{r}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}. \tag{15}$$

Here the kernels $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by

$$\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y}) = \langle k'(\boldsymbol{x}) k'(\boldsymbol{y}) \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}}^{T} \mathcal{G}(\boldsymbol{x}, \boldsymbol{y})\rangle \tag{16}$$

$$\boldsymbol{B}(\boldsymbol{x},\boldsymbol{y}) = \langle k'(\boldsymbol{y})\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}}^{T}\mathcal{G}(\boldsymbol{x},\boldsymbol{y})\rangle \tag{17}$$

where $\mathcal{G}(\boldsymbol{x},\boldsymbol{y})$ is the random Green's function associated with boundary-value problem (7)–(9). The kernel $\boldsymbol{A}$ is a quadratic form, and the kernel $\boldsymbol{B}$ is a non-symmetric tensor. Evaluating these kernels requires some approximation since they contain mixed ensemble moments of the random Green's function $\mathcal{G}$. Nevertheless, the following important properties of the averaged gas flow equations can be discerned: (i) ensemble average of permeability $\langle k\rangle$ does not act as an upscaled (effective) permeability; and (ii) the residual flux $\boldsymbol{r}$ represents a nonlocal term (i.e., contains gradients of the mean Kirchhoff transform $\nabla\langle\Psi\rangle$ at points other than $\boldsymbol{x}$) in the averaged flow equation (12). Therefore, except for very special conditions, the averaged real gas flow equations are non-Darcian, and the upscaled (effective) permeability does not generally exist. Nonlocality of the averaged flow equations for incompressible fluids has been reported in [5–8].

Boundary-value problem (12)–(14) can be solved recursively [6] by expanding $\langle k\rangle$ and $\langle\Psi\rangle$ in asymptotic series in the variance σ_Y^2 of log-permeability $Y(\boldsymbol{x}) = \ln k(\boldsymbol{x})$. This formally limits our solution to mildly heterogeneous porous media with $\sigma_Y^2 \ll 1$. Available experimental data [2] show applicability of such an approach to a wide variety of natural formations. Expanding the relevant terms in equations (12)–(14) yields the zeroth-order approximation

$$\nabla\cdot[K_{\mathrm{G}}(\boldsymbol{x})\nabla\langle\Psi^{(0)}(\boldsymbol{x})\rangle] + \langle f(\boldsymbol{x})\rangle = 0 \quad \boldsymbol{x}\in\Omega \tag{18}$$

$$\langle\Psi^{(0)}(\boldsymbol{x})\rangle = \langle H(\boldsymbol{x})\rangle \quad \boldsymbol{x}\in\Gamma_{\mathrm{D}} \tag{19}$$

$$\boldsymbol{n}(\boldsymbol{x})\cdot\left[K_{\mathrm{G}}(\boldsymbol{x})\nabla\langle\Psi^{(0)}(\boldsymbol{x})\rangle\right] = \langle Q(\boldsymbol{x})\rangle \quad \boldsymbol{x}\in\Gamma_{\mathrm{N}} \tag{20}$$

and the first-order approximation

$$\nabla\cdot\left[K_{\mathrm{G}}(\boldsymbol{x})\left(\nabla\langle\Psi^{(1)}(\boldsymbol{x})\rangle + \frac{\sigma_Y^2(\boldsymbol{x})}{2}\nabla\langle\Psi^{(0)}(\boldsymbol{x})\rangle\right) - \boldsymbol{r}^{(1)}(\boldsymbol{x})\right] = 0 \quad \boldsymbol{x}\in\Omega \tag{21}$$

$$\langle\Psi^{(1)}(\boldsymbol{x})\rangle = 0 \quad \boldsymbol{x}\in\Gamma_{\mathrm{D}} \tag{22}$$

$$\boldsymbol{n}(\boldsymbol{x})\cdot\left[K_{\mathrm{G}}(\boldsymbol{x})\left(\nabla\langle\Psi^{(1)}(\boldsymbol{x})\rangle + \frac{\sigma_Y^2(\boldsymbol{x})}{2}\nabla\langle\Psi^{(0)}(\boldsymbol{x})\rangle\right) - \boldsymbol{r}^{(1)}(\boldsymbol{x})\right] = 0 \quad \boldsymbol{x}\in\Gamma_{\mathrm{N}} \tag{23}$$

where $K_{\mathrm{G}} = \exp(\langle k\rangle)$ is the geometric mean of permeability k;

$$\boldsymbol{r}^{(1)}(\boldsymbol{x}) = \int_{\Omega}\boldsymbol{A}^{(1)}(\boldsymbol{x},\boldsymbol{y})\nabla\langle\Psi^{(0)}(\boldsymbol{y})\rangle\,\mathrm{d}\boldsymbol{y} \tag{24}$$

$$\boldsymbol{A}^{(1)}(\boldsymbol{x},\boldsymbol{y}) = K_{\mathrm{G}}(\boldsymbol{x})K(\boldsymbol{y})C_Y(\boldsymbol{x},\boldsymbol{y})\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}}^{\mathrm{T}}\langle\mathcal{G}^{(0)}(\boldsymbol{x},\boldsymbol{y})\rangle \tag{25}$$

and $C_Y(\boldsymbol{x},\boldsymbol{y}) = \langle Y'(\boldsymbol{x})Y'(\boldsymbol{y})\rangle$ is the covariance function of Y. Advantage of using our recursive approximations is that the averaged quantities involved are relatively smooth functions defined on a coarse grid Ω. This makes it possible to use standard numerical techniques, such as finite element methods, more efficiently [9].

4. Effective permeability

Averaged equations (18)–(23) can be localized under conditions of the mean uniform flow, $\nabla\langle\Psi(\boldsymbol{x})\rangle \equiv \nabla\langle\Psi^{(0)}(\boldsymbol{x})\rangle \equiv \boldsymbol{J} = \text{constant}$ $(\nabla\langle\Psi^{(i)}(\boldsymbol{x})\rangle \equiv 0$ for $i \geq 1)$. Then it follows from Eqs. (18), (21) and (24) that

$$\boldsymbol{q}^{[1]}(\boldsymbol{x}) = -\boldsymbol{k}_{\mathrm{eff}}^{[1]}(\boldsymbol{x})\boldsymbol{J} \tag{26}$$

where $\boldsymbol{q}^{[1]} \equiv \boldsymbol{q}^{(0)} + \boldsymbol{q}^{(1)}$ is the first-order approximation of mass flux $\boldsymbol{q} = \rho \boldsymbol{v}$, and the first-order approximation of the effective permeability tensor $\boldsymbol{k}^{[1]} \equiv \boldsymbol{k}^{(0)} + \boldsymbol{k}^{(1)}$ is given by

$$\boldsymbol{k}_{\text{eff}}^{[1]}(\boldsymbol{x}) = K_{\text{G}} \left[1 + \frac{\sigma_Y^2}{2} \right] \boldsymbol{I} - \boldsymbol{D}(\boldsymbol{x}) \qquad \boldsymbol{D}(\boldsymbol{x}) = \int_{\Omega} \boldsymbol{A}^{(1)}(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \tag{27}$$

I being the identity tensor. In the context of flow of incompressible fluids, expression (27) has been studied by Paleologos et al. [10] and Tartakovsky and Neuman [11].

The flow scenario described above can be realized by treating log-permeability field Y as statistically homogeneous (with constant mean $\langle Y \rangle$ and variance σ_Y^2), and taking Ω to be a box-shaped grid-block with lateral mean no-flow boundaries and two constant head boundaries a distance L_1 apart. This results in mean uniform flow with $\boldsymbol{J} = (J_1, 0, 0)^{\text{T}}$ where $J_1 = [H_2 - H_1]/L_1$, and H_1 and H_2 are the Kirchhoff transforms of the constant heads on the Dirichlet boundaries. The mean no-flow boundaries are separated by the distances L_2 and L_3. By way of example, we assume that the log-permeability field exhibits an anisotropic exponential covariance structure

$$C_Y(\boldsymbol{z}) = \sigma_Y^2 \exp\left[-\sqrt{\sum_{i=1}^{3} \frac{z_i^2}{\lambda_i^2}} \right] \qquad z_i = y_i - x_i \tag{28}$$

where $\boldsymbol{z}$ is a displacement vector, λ_i's are principal integral scales, and the principal directions of statistical anisotropy are aligned with the box. The following analysis is due to Tartakovsky and Neuman [11].

4.1. *Infinite statistically isotropic media*

We start by considering the special case where the size of the box is large relative to any of the integral scales λ_i so that $\rho_i \equiv L_i/\lambda_i$ are very large and mathematically infinite. For statistically isotropic media, anisotropy ratios $\epsilon_i \equiv \lambda_i/\lambda_1 = 1$ $(i = 1, 2, 3)$, and taking the limit $\rho_i \to \infty$ in Eq. (27) gives (in d dimensions) $D_{11} = K_{\text{G}} \sigma_Y^2 / d$ and

$$\frac{k_{11}^{[1]}}{K_{\text{G}}} = 1 + \sigma_Y^2 \left[\frac{1}{2} - \frac{1}{d} \right] \tag{29}$$

where D_{11} and $k_{11} \equiv k_{\text{eff},11}$ are the first principal components of the tensors $\boldsymbol{D}$ and $\boldsymbol{k}_{\text{eff}}$, respectively. Hence, the effective (upscaled) permeability for gas flow is given, up to first order in σ_Y^2, by the harmonic mean for one-dimensional flow, the geometric mean for two-dimensional flow, and the arithmetic mean for three-dimensional flow. This result is the same as that obtained by Neuman and Orr [5], Dagan [12] and King [13] for flow of incompressible fluids.

4.2. *Infinite statistically anisotropic media*

In the special anisotropic case where $\epsilon_2 = \epsilon_3 \equiv \epsilon$, taking the limit $\rho_i \to \infty$ in Eq. (27) yields (in 3 dimensions)

$$\begin{aligned}
D_{11} &= \frac{\epsilon^2}{(1-\epsilon^2)^{3/2}} \left[\frac{1}{2} \ln \left(\frac{1+\sqrt{1-\epsilon^2}}{1+\sqrt{1-\epsilon^2}} \right) - \sqrt{1-\epsilon^2} \right] \quad \epsilon \le 1 \\
D_{11} &= \tfrac{1}{3} \quad \epsilon = 1 \\
D_{11} &= \frac{\epsilon^2}{(1-\epsilon^2)^{3/2}} \left[-\frac{\pi}{2} + \operatorname{arccot}\left(\sqrt{\epsilon^2 - 1} \right) + \sqrt{\epsilon^2 - 1} \right] \quad \epsilon \ge 1
\end{aligned} \tag{30}$$

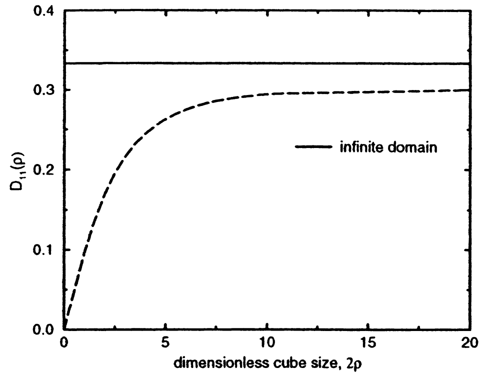

Fig. 1. Effect of box size on effective (upscaled) permeability. Horizontal line represents asymptote for $\rho \to \infty$.

From asymptotic expansions of $\ln x$ and $\operatorname{arccot} x$ one can see that $D_{11}(\epsilon)$ is continuous at $\epsilon = 1$, where it reduces to the well-established isotropic value $1/3$ of the previous section.

4.3. Boundary effects

To explore the effect of box size on the effective permeability, Eq. (27) is evaluated at the center of the box as a function of the dimensionless box size $2\rho \equiv \rho_1 = \rho_2 = \rho_3$ [11]. One can see from Fig. 1 that D_{11} increases rapidly with 2ρ when the latter is small but then tends very slowly to its asymptotic value of $1/3$.

5. Summary

We consider steady-state flow of real gases through bounded randomly heterogeneous porous media. Such flow is described by nonlinear partial differential equation with random coefficient (medium's permeability) subject to randomly prescribed driving terms (source and boundary functions). Prior to stochastic averaging of the governing equations, the problem is linearized by means of the Kirchhoff transformation. The linearized stochastic differential equations are similar to those used for describing flows of incompressible fluids in randomly heterogeneous formations. Nonlocality of the averaged equations, their localization, and effective (upscaled) permeability are investigated.

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