WATER RESOURCES RESEARCH, VOL. 35, NO. 6, PAGES 1921–1925, JUNE 1999

Extension of "Transient flow in bounded randomly heterogeneous domains, 1, Exact conditional moment equations and recursive approximations"

Daniel M. Tartakovsky

Group CIC-19, Los Alamos National Laboratory, Los Alamos, New Mexico

Shlomo P. Neuman

Department of Hydrology and Water Resources, University of Arizona, Tucson

Abstract. In a recent paper we developed exact nonlocal conditional moment equations for transient flow in bounded domains driven by random forcing terms (sources, initial head, and boundary conditions). Whereas our conditional mean equations took into account the randomness of forcing terms, our conditional second moment equations did not. We do so in this brief addendum.

1. Introduction

In a recent paper we [*Tartakovsky and Neuman*, 1998] developed exact nonlocal conditional moment equations for transient flow in bounded domains driven by random forcing terms (sources, initial head, and boundary conditions). Whereas our conditional mean equations took into account the randomness of forcing terms, our conditional second moment equations did not. We do so in this brief addendum. All symbols used in our derivation are given by *Tartakovsky and Neuman* [1998].

Before proceeding, we would like to note that Appendix A of *Tartakovsky and Neuman* [1998] contains several typographical errors [see *Tartakovsky and Neuman*, 1999].

2. Exact Second Moment Expressions

When the source function $f(\mathbf{x}, t)$, initial head distribution $H_0(\mathbf{x})$, head $H(\mathbf{x}, t)$ on Dirichlet boundary segments Γ_D and flux $Q(\mathbf{x}, t)$ across Neumann boundary segments Γ_N are random but uncorrelated with each other, our equations for the second conditional moment $C_{hc}(\mathbf{x}, \mathbf{y}, t, s) = \langle h'(\mathbf{x}, t)h'(\mathbf{y}, s)\rangle_c$ of hydraulic head $h(\mathbf{x}, t)$ take the form (see appendix for derivation)

$$\nabla_{\mathbf{x}} \cdot [\langle K(\mathbf{x})\rangle_c \nabla_{\mathbf{x}} C_{hc}(\mathbf{x}, \mathbf{y}, t, s) + u_c(\mathbf{x}, \mathbf{y}, s)\nabla_{\mathbf{x}}\langle h(\mathbf{x}, t)\rangle_c + \mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s)] + A_c(\mathbf{x}, \mathbf{y}, t, s) = S(\mathbf{x}) \frac{\partial C_{hc}(\mathbf{x}, \mathbf{y}, t, s)}{\partial t} \tag{1}$$

subject to the initial and boundary conditions

$$\begin{aligned} C_{hc}(\mathbf{x}, \mathbf{y}, 0, s) &= B_c(\mathbf{x}, \mathbf{y}, s) \qquad \mathbf{x} \in \Omega \\ C_{hc}(\mathbf{x}, \mathbf{y}, t, s) &= C_c(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{x} \in \Gamma_D \\ [\langle K(\mathbf{x})\rangle_c \nabla_{\mathbf{x}} C_{hc}(\mathbf{x}, \mathbf{y}, t, s) &+ u_c(\mathbf{x}, \mathbf{y}, s)\nabla_{\mathbf{x}}\langle h(\mathbf{x}, t)\rangle_c \\ &+ \mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s)] \cdot \mathbf{n}(\mathbf{x}) = D_c(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{x} \in \Gamma_N. \end{aligned} \tag{2}$$

Paper number 1999WR900044.
0043-1397/99/1999WR900044$09.00

Here $S(\mathbf{x})$ is a deterministically prescribed specific storage; $K(\mathbf{x})$ is a random field of hydraulic conductivities; $\langle g\rangle_c$ signifies ensemble mean of a random field g; g' is a zero mean fluctuation about $\langle g\rangle_c$;

$$u_c(\mathbf{x}, \mathbf{y}, s) = -\int_0^s \int_\Omega \mathbf{r}_c(\mathbf{z}, \tau) \cdot \nabla_{\mathbf{z}}\langle K'(\mathbf{x})G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau - \int_0^s \int_\Omega \nabla_{\mathbf{z}}\langle h(\mathbf{z}, \tau)\rangle_c \cdot \langle K'(\mathbf{x})K'(\mathbf{z})\nabla_{\mathbf{z}}G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau; \tag{3}$$

$$\begin{aligned} \mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s) &= -\int_0^s\int_\Omega \mathbf{r}_c^T(\mathbf{z}, \tau)\nabla_{\mathbf{z}}\langle K'(\mathbf{x})\nabla_{\mathbf{x}}^T h'(\mathbf{x}, t) \cdot G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \\ &- \int_0^s\int_\Omega \nabla_{\mathbf{z}}^T\langle h(\mathbf{z}, \tau)\rangle_c \langle K'(\mathbf{x})K'(\mathbf{z}) \cdot \nabla_{\mathbf{z}} G(\mathbf{z}, \mathbf{y}, s-\tau)\nabla_{\mathbf{x}}^T h'(\mathbf{x}, t)\rangle_c \, d\mathbf{z}\, d\tau \\ &+ \int_0^s\int_\Omega \langle f'(\mathbf{z}, \tau)K'(\mathbf{x})\nabla_{\mathbf{x}} h'(\mathbf{x}, t)G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \\ &+ \int_\Omega S(\mathbf{z})\langle H_0'(\mathbf{z})K'(\mathbf{x})\nabla_{\mathbf{x}} h'(\mathbf{x}, t)G(\mathbf{z}, \mathbf{y}, s)\rangle_c \, d\mathbf{z} \\ &- \int_0^s\int_{\Gamma_D} \mathbf{n}^T(\mathbf{z})\langle H'(\mathbf{z}, \tau)K'(\mathbf{x})K(\mathbf{z})\nabla_{\mathbf{z}} G(\mathbf{z}, \mathbf{y}, s-\tau) \cdot \nabla_{\mathbf{x}}^T h'(\mathbf{x}, t)\rangle_c \, d\mathbf{z}\, d\tau \\ &+ \int_0^s\int_{\Gamma_N} \langle Q'(\mathbf{z}, \tau)K'(\mathbf{x})\nabla_{\mathbf{x}} h'(\mathbf{x}, t)G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau. \end{aligned} \tag{4}$$

A_c, B_c, C_c, and D_c are forcing coefficients defined as

$$A_c(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_\Omega \langle f'(\mathbf{x}, t) f'(\mathbf{z}, \tau)\rangle \cdot \langle G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \tag{5}$$

$$B_c(\mathbf{x}, \mathbf{y}, s) = \int_\Omega S(\mathbf{z})\langle H_0'(\mathbf{x}) H_0'(\mathbf{z})\rangle \langle G(\mathbf{z}, \mathbf{y}, s)\rangle_c \, d\mathbf{z} \tag{6}$$

$$C_c(\mathbf{x}, \mathbf{y}, t, s) = -\int_0^s \int_{\Gamma_D} \langle H'(\mathbf{x}, t) H'(\mathbf{z}, \tau)\rangle \cdot \langle K(\mathbf{z}) \nabla_\mathbf{z} G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \cdot \mathbf{n}(\mathbf{z})\, d\mathbf{z}\, d\tau \tag{7}$$

$$D_c(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_{\Gamma_N} \langle Q'(\mathbf{x}, t) Q'(\mathbf{z}, \tau)\rangle \cdot \langle G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau; \tag{8}$$

$G(\mathbf{y}, \mathbf{x}, t-\tau)$ is a random Green's function introduced in Appendix A of *Tartakovsky and Neuman* [1998]; and the residual flux $\mathbf{r}(\mathbf{x}, t)$ is given explicitly by their (15) or implicitly by their (23).

When the forcing terms f, H_0, H, and Q are deterministic, $A_c = B_c = C_c = D_c \equiv 0$ and (1)–(4) reduce to (48)–(53) of *Tartakovsky and Neuman* [1998]; we note that a minus sign is missing in front of the leading integral in their (52) and (53).

It has been pointed out by *Tartakovsky and Neuman* [1998] that, for purposes of recursive approximation, it is advantageous to work with implicit rather than explicit expressions. An implicit alternative to the explicit expression for $u_c(\mathbf{x}, \mathbf{y}, s)$ in (3) is given by (B10) and (B11) of *Tartakovsky and Neuman* [1998]. Their implicit expressions (B8)–(B9) for $\mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s)$ now take the form (see appendix) of the differential equation

$$\begin{aligned}\nabla_\mathbf{y}^T[\langle K(\mathbf{y})\rangle_c \nabla_\mathbf{y} \mathbf{p}_c^T(\mathbf{x}, \mathbf{y}, t, s)] &+ \nabla_\mathbf{y}^T[\boldsymbol{\alpha}_c(\mathbf{x}, \mathbf{y}, t, s) \\ &+ \nabla_\mathbf{y}\langle h(\mathbf{y}, s)\rangle_c \boldsymbol{\beta}_c^T(\mathbf{y}, \mathbf{x}, t)] - \nabla_\mathbf{y}^T \mathbf{r}_c(\mathbf{y}, s)\mathbf{r}_c^T(\mathbf{x}, t) \\ &+ \boldsymbol{\gamma}_c^T(\mathbf{x}, \mathbf{y}, t, s) = S(\mathbf{y}) \frac{\partial \mathbf{p}_c^T(\mathbf{x}, \mathbf{y}, t, s)}{\partial s}\end{aligned} \tag{9}$$

subject to the initial and boundary conditions

$$\mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, 0) = \mathbf{E}(\mathbf{x}, \mathbf{y}, t) \qquad \mathbf{y} \in \Omega \tag{10}$$

$$\mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s) = \mathbf{F}(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{y} \in \Gamma_D \tag{11}$$

$$\begin{aligned}\mathbf{n}^T[\langle K(\mathbf{y})\rangle_c \nabla_\mathbf{y} \mathbf{p}_c^T(\mathbf{x}, \mathbf{y}, t, s) &+ \boldsymbol{\alpha}_c(\mathbf{x}, \mathbf{y}, t, s) \\ &+ \nabla_\mathbf{y}\langle h(\mathbf{y}, s)\rangle_c \boldsymbol{\beta}_c^T(\mathbf{y}, \mathbf{x}, t) - \mathbf{r}_c(\mathbf{y}, s)\mathbf{r}_c^T(\mathbf{x}, t)] \\ &= \mathbf{G}(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{y} \in \Gamma_N\end{aligned} \tag{12}$$

where $\boldsymbol{\beta}_c(\mathbf{y}, \mathbf{x}, t)$ is given by (B11) of *Tartakovsky and Neuman* [1998];

$$\begin{aligned}\boldsymbol{\alpha}_c&(\mathbf{x}, \mathbf{y}, t, s) \\ &= -\int_0^s \int_\Omega \langle K'(\mathbf{x}) K'(\mathbf{y}) \nabla_\mathbf{y} \nabla_\mathbf{z}^T G(\mathbf{z}, \mathbf{y}, s-\tau) \mathbf{r}_c(\mathbf{z}, \tau) \cdot \nabla_\mathbf{x}^T h'(\mathbf{x}, t)\rangle_c \, d\mathbf{z}\, d\tau \\ &\quad - \int_0^s \int_\Omega \langle K'(\mathbf{x}) K'(\mathbf{y}) K'(\mathbf{z}) \nabla_\mathbf{y} \nabla_\mathbf{z}^T G(\mathbf{z}, \mathbf{y}, s-\tau) \cdot \nabla_\mathbf{z}\langle h(\mathbf{z}, \tau)\rangle_c \nabla_\mathbf{x}^T h'(\mathbf{x}, t)\rangle_c \, d\mathbf{z}\, d\tau \\ &\quad + \int_0^s \int_\Omega \langle f'(\mathbf{z}, \tau) K'(\mathbf{x}) K'(\mathbf{y}) \nabla_\mathbf{x} h'(\mathbf{x}, t) \cdot \nabla_\mathbf{y}^T G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \\ &\quad + \int_\Omega S(\mathbf{z}) \langle H_0'(\mathbf{z}) K'(\mathbf{x}) K'(\mathbf{y}) \nabla_\mathbf{x} h'(\mathbf{x}, t) \cdot \nabla_\mathbf{y}^T G(\mathbf{z}, \mathbf{y}, s)\rangle_c \, d\mathbf{z} \\ &\quad - \int_0^s \int_{\Gamma_D} \langle H'(\mathbf{z}, \tau) K'(\mathbf{x}) K'(\mathbf{y}) K(\mathbf{z}) \cdot \nabla_\mathbf{y} \nabla_\mathbf{z}^T G(\mathbf{z}, \mathbf{y}, s-\tau) \mathbf{n}(\mathbf{z}) \nabla_\mathbf{x}^T h'(\mathbf{x}, t)\rangle_c \, d\mathbf{z}\, d\tau \\ &\quad + \int_0^s \int_{\Gamma_N} \langle Q'(\mathbf{z}, \tau) K'(\mathbf{x}) K'(\mathbf{y}) \nabla_\mathbf{x} h'(\mathbf{x}, t) \cdot \nabla_\mathbf{y}^T G(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau;\end{aligned} \tag{13}$$

and

$$\boldsymbol{\gamma}_c(\mathbf{x}, \mathbf{y}, t, s) = \int_0^t \int_\Omega \langle f'(\mathbf{z}, \tau) f'(\mathbf{y}, s)\rangle \langle K'(\mathbf{x}) \cdot \nabla_\mathbf{x} G(\mathbf{z}, \mathbf{x}, t-\tau)\rangle_c \, d\mathbf{z}\, d\tau \tag{14}$$

$$\mathbf{E}_c(\mathbf{x}, \mathbf{y}, t) = \int_\Omega S(\mathbf{z}) \langle H_0'(\mathbf{z}) H_0'(\mathbf{y})\rangle \langle K'(\mathbf{x}) \cdot \nabla_\mathbf{x} G(\mathbf{z}, \mathbf{x}, t-\tau)\rangle_c \, d\mathbf{z} \tag{15}$$

$$\mathbf{F}_c(\mathbf{x}, \mathbf{y}, t, s) = \int_0^t \int_{\Gamma_D} \langle H'(\mathbf{z}, \tau) H'(\mathbf{y}, s)\rangle \langle K'(\mathbf{x}) \cdot \nabla_\mathbf{x} G(\mathbf{z}, \mathbf{x}, t-\tau)\rangle_c \, d\mathbf{z} \tag{16}$$

$$\mathbf{G}_c(\mathbf{x}, \mathbf{y}, t, s) = \int_0^t \int_{\Gamma_N} \langle Q'(\mathbf{z}, \tau) Q'(\mathbf{y}, s)\rangle \langle K'(\mathbf{x}) \cdot \nabla_\mathbf{x} G(\mathbf{z}, \mathbf{x}, t-\tau)\rangle_c \, d\mathbf{z}\, d\tau. \tag{17}$$

When the forcing terms f, H_0, H, and Q are deterministic, $\boldsymbol{\gamma}_c = \mathbf{E}c = \mathbf{F}_c = \mathbf{G}_c \equiv 0$ and (9)–(13) reduce to (B8)–(B9) of *Tartakovsky and Neuman* [1998]; we note again that a minus sign is missing in front of the first integrals in their (B9) and (B11).

3. Recursive Second Moments Approximations

Recursive conditional approximations for C_{hc} were derived for the case of deterministic driving terms in Appendix D of *Tartakovsky and Neuman* [1998]. A perturbation expansion (in the measure σ_Y of the standard deviation of $Y = \ln K$) of (1)–(8) leads to the following ith-order approximation of $C_{hc}(\mathbf{x}, \mathbf{y}, t, s)$ for $i \geq 2$,

$$\nabla_{\mathbf{x}} \cdot \left[K_G(\mathbf{x}) \sum_{n=0}^{i} \frac{1}{n!} \langle Y'^n(\mathbf{x}) \rangle_c \nabla_{\mathbf{x}} C_{hc}^{(i-n)}(\mathbf{x}, \mathbf{y}, t, s) + \sum_{n=0}^{i} u_c^{(n)}(\mathbf{x}, \mathbf{y}, s) \nabla_{\mathbf{x}} \langle h^{(i-n)}(\mathbf{x}, t) \rangle_c + \mathbf{p}_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) \right] + A_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = S(\mathbf{x}) \frac{\partial C_{hc}^{(i)}(\mathbf{x}, \mathbf{y}, t, s)}{\partial t} \quad (18)$$

subject to

$$C_{hc}^{(i)}(\mathbf{x}, \mathbf{y}, 0, s) = B_c^{(i)}(\mathbf{x}, \mathbf{y}, s) \qquad \mathbf{x} \in \Omega$$

$$C_{hc}^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = C_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{x} \in \Gamma_D$$

$$\left[K_G(\mathbf{x}) \sum_{n=0}^{i} \frac{1}{n!} \langle Y'^n(\mathbf{x}) \rangle_c \nabla_{\mathbf{x}} C_{hc}^{(i-n)}(\mathbf{x}, \mathbf{y}, t, s) + \sum_{n=0}^{i} u_c^{(n)}(\mathbf{x}, \mathbf{y}, s) \nabla_{\mathbf{x}} \langle h^{(i-n)}(\mathbf{x}, t) \rangle_c + \mathbf{p}_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) \right] \cdot \mathbf{n}(\mathbf{x}) = D_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) \qquad \mathbf{x} \in \Gamma_N. \quad (19)$$

Note that there are no lower-order contributions to C_{hc}. The five leading terms in the perturbation expansion of u_c are given by (D5), (D6), (D15), and (D16) of *Tartakovsky and Neuman* [1998]. It is easily seen from (4) that the three leading terms ($i = 0, 1, 2$) in the expansion of $\mathbf{p}_c$ are zero [*Tartakovsky and Neuman*, 1998, equation (D7)]. Perturbation expansion of (9) yields the following equations for $\mathbf{p}_c^{(3)}$ and $\mathbf{p}_c^{(4)}$,

$$\nabla_{\mathbf{y}}^T [K_G(\mathbf{y}) \nabla_{\mathbf{y}} \mathbf{p}_c^{(3)T}(\mathbf{x}, \mathbf{y}, t, s)] + \nabla_{\mathbf{y}}^T [\nabla_{\mathbf{y}} \langle h^{(0)}(\mathbf{y}, s) \rangle_c \boldsymbol{\beta}_c^{(3)T}(\mathbf{y}, \mathbf{x}, t)] = S(\mathbf{y}) \frac{\partial \mathbf{p}_c^{(3)T}(\mathbf{x}, \mathbf{y}, t, s)}{\partial s} \quad (20)$$

and

$$\nabla_{\mathbf{y}}^T [K_G(\mathbf{y}) \nabla_{\mathbf{y}} \mathbf{p}_c^{(4)T}(\mathbf{x}, \mathbf{y}, t, s)] + \nabla_{\mathbf{y}}^T [\boldsymbol{\alpha}_c^{(4)}(\mathbf{x}, \mathbf{y}, t, s) + \nabla_{\mathbf{y}} \langle h^{(0)} \cdot (\mathbf{y}, s) \rangle_c \boldsymbol{\beta}_c^{(4)T}(\mathbf{y}, \mathbf{x}, t) + \nabla_{\mathbf{y}} \langle h^{(1)}(\mathbf{y}, s) \rangle_c \boldsymbol{\beta}_c^{(3)T}(\mathbf{y}, \mathbf{x}, t)] - \nabla_{\mathbf{y}}^T \mathbf{r}_c^{(2)}(\mathbf{y}, s) \mathbf{r}_c^{(2)T}(\mathbf{x}, t) + \boldsymbol{\gamma}_c^{(4)T}(\mathbf{x}, \mathbf{y}, t, s) = S(\mathbf{y}) \frac{\partial \mathbf{p}_c^{(4)T}(\mathbf{x}, \mathbf{y}, t, s)}{\partial s}. \quad (21)$$

Here from (13),

$$\boldsymbol{\alpha}_c^{(4)}(\mathbf{x}, \mathbf{y}, t, s) = -\int_0^s \int_\Omega \nabla_{\mathbf{y}} \nabla_{\mathbf{z}}^T \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau) \rangle_c \cdot \nabla_{\mathbf{z}} \langle h^{(0)}(\mathbf{z}, \tau) \rangle_c \boldsymbol{\alpha}_{1c}^{(4)T}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t)\, d\mathbf{z}\, d\tau + \int_0^s \int_\Omega \boldsymbol{\alpha}_{2c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau) \cdot \nabla_{\mathbf{y}}^T \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau) \rangle_c\, d\mathbf{z}\, d\tau + \int_\Omega S(\mathbf{z}) \boldsymbol{\alpha}_{3c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) \nabla_{\mathbf{y}}^T \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s) \rangle_c\, d\mathbf{z} - \int_0^s \int_{\Gamma_D} K_G(\mathbf{z}) \nabla_{\mathbf{y}} \nabla_{\mathbf{z}}^T \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau) \rangle_c \cdot \mathbf{n}(\mathbf{z}) \boldsymbol{\alpha}_{4c}^{(4)T}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau)\, d\mathbf{z}\, d\tau + \int_0^s \int_{\Gamma_N} \boldsymbol{\alpha}_{5c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau) \nabla_{\mathbf{y}}^T \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau) \rangle_c\, d\mathbf{z}\, d\tau, \quad (22)$$

where the vector $\boldsymbol{\alpha}_{1c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t)$ is given by (D11) of *Tartakovsky and Neuman* [1998], and the remaining kernels are vectors derived in appendix,

$$\boldsymbol{\alpha}_{2c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau) = K_G(\mathbf{x}) K_G(\mathbf{y}) \langle Y'(\mathbf{x}) Y'(\mathbf{y}) \rangle_c \cdot \int_0^t \int_\Omega C_f(\mathbf{z}, \tau; \boldsymbol{\xi}, s) \nabla_{\mathbf{x}} \langle G^{(0)}(\boldsymbol{\xi}, \mathbf{x}, t-s) \rangle_c\, d\boldsymbol{\xi}\, ds \quad (23)$$

$$\boldsymbol{\alpha}_{3c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t) = K_G(\mathbf{x}) K_G(\mathbf{y}) \langle Y'(\mathbf{x}) Y'(\mathbf{y}) \rangle_c \cdot \int_\Omega C_{H_0}(\mathbf{z}, \boldsymbol{\xi}) \nabla_{\mathbf{x}} \langle G^{(0)}(\boldsymbol{\xi}, \mathbf{x}, t) \rangle_c\, d\boldsymbol{\xi} \quad (24)$$

$$\boldsymbol{\alpha}_{4c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau) = -K_G(\mathbf{x}) K_G(\mathbf{y}) \langle Y'(\mathbf{x}) Y'(\mathbf{y}) \rangle_c \cdot \int_0^t \int_\Omega C_H(\mathbf{z}, \tau; \boldsymbol{\xi}, s) \nabla_{\mathbf{x}} \langle G^{(0)}(\boldsymbol{\xi}, \mathbf{x}, t-s) \rangle_c\, d\boldsymbol{\xi}\, ds \quad (25)$$

$$\boldsymbol{\alpha}_{5c}^{(4)}(\mathbf{x}, \mathbf{y}, \mathbf{z}, t, \tau) = K_G(\mathbf{x}) K_G(\mathbf{y}) \langle Y'(\mathbf{x}) Y'(\mathbf{y}) \rangle_c \cdot \int_0^t \int_\Omega C_Q(\mathbf{z}, \tau; \boldsymbol{\xi}, s) \nabla_{\mathbf{x}} \langle G^{(0)}(\boldsymbol{\xi}, \mathbf{x}, t-s) \rangle_c\, d\boldsymbol{\xi}\, ds. \quad (26)$$

The third- and fourth-order approximations of $\boldsymbol{\beta}_c$ in (20) and (21) are given by (D12)–(D14) of *Tartakovsky and Neuman* [1998]. It follows from (14) that the fourth-order approximation of $\boldsymbol{\gamma}_c$ in (21) is

$$\boldsymbol{\gamma}_c^{(4)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^t \int_\Omega C_f(\mathbf{z}, \tau; \mathbf{y}, s) \boldsymbol{\rho}_c^{(2)}(\mathbf{z}, \mathbf{x}, t, \tau)\, d\mathbf{z}\, d\tau. \quad (27)$$

Expanding the forcing coefficients in (5)–(8) in powers of σ_Y yields for $i \geq 2$

$$A_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_\Omega C_f(\mathbf{x}, t; \mathbf{z}, \tau)\langle G^{(i)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \tag{28}$$

$$B_c^{(i)}(\mathbf{x}, \mathbf{y}, s) = \int_\Omega S(\mathbf{z})C_{H_0}(\mathbf{x}; \mathbf{z})\langle G^{(i)}(\mathbf{z}, \mathbf{y}, s)\rangle_c \, d\mathbf{z} \tag{29}$$

$$C_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = -\int_0^s \int_{\Gamma_D} C_H(\mathbf{x}, t; \mathbf{z}, \tau) K_G(\mathbf{z}) \cdot \sum_{n=2}^{i} \left[\frac{1}{n!} \langle Y'^i(\mathbf{z})\rangle_c \nabla_\mathbf{z} \langle G^{(i-n)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c - \boldsymbol{\rho}_c^{(i)}(\mathbf{z}, \mathbf{y}, s-\tau)_c\right] \cdot \mathbf{n}(\mathbf{z})\, d\mathbf{z}\, d\tau \tag{30}$$

$$D_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_{\Gamma_N} C_Q(\mathbf{x}, t; \mathbf{z}, \tau)\langle G^{(i)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau, \tag{31}$$

where C_f, C_{H_0}, C_H, and C_Q are covariances of the driving forces f, H_0, H, and Q, respectively; and $\boldsymbol{\rho}_c^{(i)}$ is given by (47) of *Tartakovsky and Neuman* [1998]. Evaluating (28)–(31) requires computing higher than zeroth-order approximations of the ensemble mean Green's function which can be problematic. In what follows, we propose two alternative strategies for evaluating (28)–(31).

3.1. Case 1

Variances associated with C_f, C_{H_0}, C_H, and C_Q are much smaller than the variance σ_Y^2 of the log-hydraulic conductivity. Then an approximation $\langle G\rangle_c \approx \langle G^{(0)}\rangle_c$ in (5)–(8) can be used so that

$$A_c^{(2)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_\Omega C_f(\mathbf{x}, t; \mathbf{z}, \tau)\langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \tag{32}$$

$$B_c^{(2)}(\mathbf{x}, \mathbf{y}, s) = \int_\Omega S(\mathbf{z})C_{H_0}(\mathbf{x}; \mathbf{z})\langle G^{(0)}(\mathbf{z}, \mathbf{y}, s)\rangle_c \, d\mathbf{z} \tag{33}$$

$$C_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = -\int_0^s \int_{\Gamma_D} C_H(\mathbf{x}, t; \mathbf{z}, \tau) K_G(\mathbf{z}) \frac{1}{i!} \langle Y'^i(\mathbf{z})\rangle_c \cdot \nabla_\mathbf{z} \langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \cdot \mathbf{n}(\mathbf{z})\, d\mathbf{z}\, d\tau \tag{34}$$

$$D_c^{(2)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_{\Gamma_N} C_Q(\mathbf{x}, t; \mathbf{z}, \tau)\langle G^{(0)}(\mathbf{z}, \mathbf{y}, s-\tau)\rangle_c \, d\mathbf{z}\, d\tau \tag{35}$$

and $A_c^{(i)} = B_c^{(i)} = D_c^{(i)} \equiv 0$ for $i \geq 3$.

3.2. Case 2

Variances associated with C_f, C_{H_0}, C_H, and C_Q are of the same order of magnitude as the variance σ_Y^2. Then setting $\langle G\rangle_c \approx G_c$, where the latter Green's function is defined by (19)–(22) of *Tartakovsky and Neuman* [1998], yields from (28)–(31)

$$A_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_\Omega C_f(\mathbf{x}, t; \mathbf{z}, \tau) G_c^{(i)}(\mathbf{z}, \mathbf{y}, s-\tau)\, d\mathbf{z}\, d\tau \tag{36}$$

$$B_c^{(i)}(\mathbf{x}, \mathbf{y}, s) = \int_\Omega S(\mathbf{z})C_{H_0}(\mathbf{x}; \mathbf{z}) G_c^{(i)}(\mathbf{z}, \mathbf{y}, s)\, d\mathbf{z} \tag{37}$$

$$C_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = -\int_0^s \int_{\Gamma_D} C_H(\mathbf{x}, t; \mathbf{z}, \tau) K_G(\mathbf{z}) \cdot \sum_{n=0}^{i} \left[\frac{1}{n!} \langle Y'^i(\mathbf{z})\rangle_c \nabla_\mathbf{z} G_c^{(i-n)}(\mathbf{z}, \mathbf{y}, s-\tau) - \boldsymbol{\rho}_c^{(i-2)}(\mathbf{z}, \mathbf{y}, s-\tau)\right] \cdot \mathbf{n}(\mathbf{z})\, d\mathbf{z}\, d\tau \tag{38}$$

$$D_c^{(i)}(\mathbf{x}, \mathbf{y}, t, s) = \int_0^s \int_{\Gamma_N} C_Q(\mathbf{x}, t; \mathbf{z}, \tau) \cdot G_c^{(i-2)}(\mathbf{z}, \mathbf{y}, s-\tau)\, d\mathbf{z}\, d\tau. \tag{39}$$

Higher-order approximations of G_c can be readily evaluated. Moreover, since $G_c^{(0)} \equiv \langle G^{(0)}\rangle$, (36)–(39) are exact to second order in σ_Y. The second-, third-, and fourth-order approximations of the flux covariance matrix are given, as before, by (D17)–(D19) of *Tartakovsky and Neuman* [1998].

Appendix

To derive implicit expressions for the second moment of hydraulic head, we recall that the head fluctuation $h'(\mathbf{x}, t)$ satisfies [*Tartakovsky and Neuman*, 1998, equations (B1) and (B2)]

$$\nabla \cdot [K(\mathbf{x})\nabla h'(\mathbf{x}, t) + K'(\mathbf{x})\nabla\langle h(\mathbf{x}, t)\rangle_c - \langle K'(\mathbf{x})\nabla h'(\mathbf{x}, t)\rangle_c] + f'(\mathbf{x}, t) = S(\mathbf{x}) \frac{\partial h'(\mathbf{x}, t)}{\partial t} \tag{A1}$$

subject to the initial and boundary conditions

$$h'(\mathbf{x}, 0) = H_0'(\mathbf{x}) \qquad \mathbf{x} \in \Omega;$$
$$h'(\mathbf{x}, t) = H'(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma_D; \tag{A2}$$
$$[K(\mathbf{x})\nabla h'(\mathbf{x}, t) + K'(\mathbf{x})\nabla\langle h(\mathbf{x}, t)\rangle_c - \langle K'(\mathbf{x})\nabla h'(\mathbf{x}, t)\rangle_c] \cdot \mathbf{n}(\mathbf{x}) = Q'(\mathbf{x}, t) \qquad \mathbf{x} \in \Gamma_N.$$

The solution of (A1) and (A2) can be expressed in terms of the random Green's function $G(\mathbf{y}, \mathbf{x}, t-\tau)$, introduced in appendix A of *Tartakovsky and Neuman* [1998] as their (B5)

$$h'(\mathbf{x}, t) = \int_0^t \int_\Omega \langle K'(\mathbf{y})\nabla_\mathbf{y} h'(\mathbf{y}, s)\rangle_c \cdot \nabla_\mathbf{y} G(\mathbf{y}, \mathbf{x}, t-\tau)\, d\mathbf{y}\, d\tau - \int_0^t \int_\Omega K'(\mathbf{y})\nabla_\mathbf{y}\langle h(\mathbf{y}, \tau)\rangle_c \cdot \nabla_\mathbf{y} G(\mathbf{y}, \mathbf{x}, t-\tau)\, d\mathbf{y}\, d\tau + \int_0^t \int_\Omega f'(\mathbf{y}, \tau) G(\mathbf{y}, \mathbf{x}, t-\tau)\, d\mathbf{y}\, d\tau$$

$$+ \int_{\Omega} S(\mathbf{y}) H_0'(\mathbf{y}) G(\mathbf{y}, \mathbf{x}, t)\, d\mathbf{y}$$

$$- \int_0^t \int_{\Gamma_D} H'(\mathbf{y}, \tau) K(\mathbf{y}) \nabla_{\mathbf{y}} G(\mathbf{y}, \mathbf{x}, t - \tau) \cdot \mathbf{n}(\mathbf{y})\, d\mathbf{y}\, d\tau$$

$$+ \int_0^t \int_{\Gamma_N} Q'(\mathbf{y}, \tau) G(\mathbf{y}, \mathbf{x}, t - \tau)\, d\mathbf{y}\, d\tau \qquad \text{(A3)}$$

Multiplying (A1) and (A2) by $h'(\mathbf{y}, s)$ and taking conditional ensemble mean leads directly to (1). The mixed conditional moments $\mathbf{p}_c(\mathbf{x}, \mathbf{y}, t, s) = \langle K'(\mathbf{x}) \nabla_{\mathbf{x}} h'(\mathbf{x}, t) h'(\mathbf{y}, s)\rangle_c$ and $u_c(\mathbf{x}, \mathbf{y}, s) = \langle K'(\mathbf{x}) h'(\mathbf{y}, s)\rangle_c$ are obtained by multiplying (A3), written in terms of $\mathbf{y}$ and s, with $K'(\mathbf{x})$ and $K'(\mathbf{x}) \nabla_{\mathbf{x}} h'(\mathbf{x}, t)$, respectively. By the same token the mixed conditional moments $A_c(\mathbf{x}, \mathbf{y}, t, s) = \langle f'(\mathbf{x}, t) h'(\mathbf{y}, s)\rangle_c$, $B_c(\mathbf{x}, \mathbf{y}, s) = \langle H_0'(\mathbf{x}) h'(\mathbf{y}, s)\rangle_c$, $C_c(\mathbf{x}, \mathbf{y}, t, s) = \langle H'(\mathbf{x}, t) h'(\mathbf{y}, s)\rangle_c$, and $D_c(\mathbf{x}, \mathbf{y}, t, s) = \langle Q'(\mathbf{x}, t) h'(\mathbf{y}, s)\rangle_c$ are given by (5)–(8).

Rewriting (A1) and (A2) in terms of $(\mathbf{y}, s)$, postmultiplying by $K'(\mathbf{x}) \nabla_{\mathbf{x}}^T h'(\mathbf{x}, t)$, and taking conditional ensemble mean leads to (9)–(17). An explicit expression for $\boldsymbol{\alpha}_c(\mathbf{x}, \mathbf{y}, t, s) = \langle K'(\mathbf{x}) K'(\mathbf{y}) \nabla_{\mathbf{y}} h'(\mathbf{y}, s) \nabla_{\mathbf{x}}^T h'(\mathbf{x}, t)\rangle_c$ in (13) is obtained by operating with $K'(\mathbf{x}) K'(\mathbf{y}) \nabla_{\mathbf{x}} h'(\mathbf{x}, t) \nabla_{\mathbf{y}}^T$ on (A3) after replacing $(\mathbf{x}, t)$ by $(\mathbf{y}, s)$ and taking conditional ensemble mean. The driving term $\boldsymbol{\gamma}_c(\mathbf{x}, \mathbf{y}, t, s) = \langle f'(\mathbf{y}, s) K'(\mathbf{x}) \nabla_{\mathbf{x}} h'(\mathbf{x}, t)\rangle_c$ in (9) is derived by operating with $f'(\mathbf{y}, s) K'(\mathbf{x}) \nabla_{\mathbf{x}}$ on (A3) and taking conditional ensemble mean.

The vectors (23)–(26) are obtained upon operating with $K'(\mathbf{x}) K'(\mathbf{y}) f'(\mathbf{z}, \tau) \nabla_{\mathbf{x}}$, $K'(\mathbf{x}) K'(\mathbf{y}) H_0'(\mathbf{z}) \nabla_{\mathbf{x}}$, $K'(\mathbf{x}) K'(\mathbf{y}) H'(\mathbf{z}, \tau) \nabla_{\mathbf{x}}$, and $K'(\mathbf{x}) K'(\mathbf{y}) Q'(\mathbf{z}, \tau) \nabla_{\mathbf{x}}$ on (A3), respectively; taking the conditional ensemble mean; and retaining the terms of σ_Y^4 order.

Acknowledgment. We wish to thank the anonymous reviewer of this paper for pointing out the typographical errors in appendix A of *Tartakovsky and Neuman* [1998].

References

Tartakovsky, D. M., and S. P. Neuman, Transient flow in bounded randomly heterogeneous domains, 1, Exact conditional moment equations and recursive approximations, *Water Resour. Res.*, *34*(1), 1–12, 1998.

Tartakovsky, D. M., and S. P. Neuman, Correction to "Transient flow in bounded randomly heterogeneous domains, 1, Exact conditional moment equations and recursive approximations," *Water Resour. Res.*, this issue, 1999.

S. P. Neuman, Department of Hydrology and Water Resources, University of Arizona, P.O. Box 210011, Tucson, AZ 85721-0011. (neuman@hwr.arizona.edu)

D. M. Tartakovsky, Group CIC-19, Los Alamos National Laboratory, MS B256, Los Alamos, NM 87545. (dmt@lanl.gov)

(Received October 2, 1998; revised January 15, 1999; accepted February 11, 1999.)

1926